Tracking a periodic reference using nonlinear model predictive control

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Abstract—Control systems are required in diverse fields to track or reject periodic reference signals. When the reference signal is generated by a finite dimensional exosystem an interesting and important solution of the associated design problem has been provided by Francis, Byrnes and Isidori; to implement this procedure in the nonlinear case requires solution of the Francis-Byrnes-Isidori partial differential equations that define a control invariant manifold in which the tracking error is zero and a control law that maintains the state in this manifold. This note shows how model predictive control may be employed to solve the global constrained tracking problem and thereby avoid the difficult task of determining the control invariant manifold and its associated control law. The proposed approach is illustrated by the problem of landing a vertical take-off and landing (VTOL) aircraft on an oscillating platform.

I. INTRODUCTION

As emphasised in [1], controlling a dynamic system to achieve tracking (regulation) or disturbance rejection is a central problem in control theory for which an elegant theory is available if the exosystem is a finite dimensional neutrally stable system; the linear problem was shown to be solvable in [2] and this result was extended to nonlinear systems in [3]. An excellent overview of these and more recent contributions may be found in [4]. The problem is simply formulated; the system to be controlled is described by

$$
\dot{x} = f(x, u, w) \\
e = h(x, u, w) \\
\dot{w} = s(w)
$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the control, $e \in \mathbb{R}^n$ is the tracking error and $w \in \mathbb{R}^p$ is the state of the exosystem that is neutrally stable. It is assumed that $f(\cdot), h(\cdot)$ and $s(\cdot)$ are sufficiently smooth; often $s(\cdot)$ is assumed to be linear ($s(w) = Sw$) in which case the eigenvalues of $S$ lie on the imaginary axis. The tracking problem is the determination of a controller such that the state of the system (and of a dynamic controller) is bounded and such that the tracking error $e(t)$ tends to zero as $t$ tends to infinity for all initial states of the system (and controller if dynamic) and exosystem in some given set. An additional requirement, not addressed in this note, is that the closed loop system is structurally stable, i.e. boundedness of the state and asymptotic convergence of $e(t)$ to zero are preserved for all possible values of a parameter $\mu$ sufficiently small or lying in a given set. We address the requirement that the control $u$ and state $x$ satisfy the hard constraints $u \in U$ and $x \in \mathcal{X}$ where $U$ is compact and $\mathcal{X}$ is closed. In [5] an output feedback MPC scheme for nonlinear systems solving the problem of tracking and rejecting non-constant exogenous signals is proposed. The described approach guarantees local stability and requires the solution of the Francis, Byrnes and Isidori partial differential equations that are, in general, easily solvable only for constant references. The problem of offset-free tracking of unstable dynamics in linear systems is considered in [6]. In [7] a predictive control scheme for constrained linear systems to track known periodic references is proposed. The stability is guaranteed by a terminal state constraint and a constraint that enforces the artificial reference to be periodic. As pointed out in [8], the design of such a controller can be carried out in two stages. In the first stage one determines two smooth functions $\pi : w \mapsto \pi(w) \in \mathbb{R}^n$ and $c : w \mapsto c(w) \in \mathbb{R}^m$ satisfying the first order partial differential equations

$$
\frac{\partial \pi(w)}{\partial w} = f(\pi(w), c(w), w) \\
h(\pi(w), c(w), w) = 0
$$

The first equation in (2) defines a control invariant manifold $\mathcal{M} \triangleq \{(x, w) \mid x = \pi(w)\} \subset \mathbb{R}^n \times \mathbb{R}^p$; if $(x, w)$ lies in this manifold there exists a control law $c(w)$ that keeps all subsequent values of $(x, w)$ in $\mathcal{M}$. The second equation in (2) states that the tracking error $e$ is zero in this manifold. This stage of the design is an extension of the common procedure employed when $w$ satisfies $\dot{w} = 0$, i.e. determination of $(x_s, u_s)$ satisfying $f(x_s, u_s) = 0$ and $e = h(x_s, u_s) = 0$. Determination of the functions $\pi(\cdot)$ and $c(\cdot)$ is difficult.

The second stage of the design, when the state $x$ is available, is the determination of a control law $\kappa : \mathcal{X} \mapsto U$ that steers an initial state $x_0$ lying in a subset $\mathcal{X}_c \subset \mathcal{X}$ to the zero-error manifold $\mathcal{M}$ while satisfying the state and control constraints. Introducing the transverse coordinates [8] $(z, v)$ defined by

$$
z \triangleq x - \pi(w), \quad v \triangleq u - c(w)
$$

this stage of the design is the determination of a control law that steers the state $z$ of the resultant system to the origin. This stage of the design is also difficult involving, as it does, determination of a nonlinear control law.

The difficulty of determining a control law over a possibly large set $\mathcal{X}_c$ has been avoided in some of the literature, under the assumption that $f(0, 0, 0) = 0$ and $h(0, 0, 0) = 0$, by confining attention to a small neighbourhood of the origin and designing a linear dynamic controller. Alternatively, by making suitable transformations as in [9] it is sometimes possible to design control law over a large set that can be shown to be stabilising via Lyapunov type arguments. Using Taylor series expansions, Krener [10] shows how the FBI equations may be approximately solved in a neighbourhood of $(0, 0)$; the optimal cost of steering any state in this
neighbourhood to the zero error manifold is obtained using a Taylor series expansion of the solution to the associated Hamilton Jacobi Bellman equation. The patchy method of solution [11], [12] may be employed to obtain solutions of the HJB and FBI equations over larger neighbourhoods.

The approach adopted in this note is different and it is not based on local analysis. Instead of determining the functions \( \pi(\cdot) \) and \( c(\cdot) \) that define the zero-error manifold \( \mathcal{M} \) and the control law \( \kappa(\cdot) \) that steers any initial state to the zero-error manifold, we propose to use optimization to determine solutions (i.e., trajectories) corresponding to current state \((x, w)\) of the composite system. Thus we use optimization to determine the trajectories \( \pi(w(t)) \) and \( c(w(t)) \), \( t \in [0, T_w] \) where \( w(t) \) is the solution of \( \dot{w} = s(w) \) given \( w(0) = w \), the current value of \( w(\cdot) \). Then a finite horizon optimal control problem \( \mathcal{P}(x, w) \) with cost

\[
V^*(x, w, u(\cdot)) = \int_0^T \ell(x(t) - \pi(w(t)), u(t) - c(w(t))) dt
\]

control and state constraints \( u(t) \in \mathbb{U} \) and \( x(t) \in \mathbb{X} \), and terminal constraint \( x(T) = \pi(w(T)) \) is solved to yield the current control action; the cost integrand \( \ell(\cdot) \) is positive definite and quadratic and satisfies \( \ell(0, 0) = 0 \). In this way, an implementable solution to the regulator problem is obtained.

A similar approach, making use of periodic constraints, is discussed in [13] for problems admitting average asymptotic cost that is less than that of any possible steady state.

In §2 we show, firstly, how, for given initial state \( w \) of the exosystem, \( \pi(w(t)) \) and \( c(w(t)) \) may be determined and, secondly, how the current control \( u(s), s \in [t, t + \delta] \) may be determined given the current state \( x(t) = x \) of the system to be controlled. In §3 we show that the proposed MPC controller solves the tracking problem. In §4 we illustrate our solution of the tracking problem by applying our approach to the control problem, considered in [9], of landing a vertical take-off and landing (VTOL) aircraft on an oscillating platform. Finally, in §5, we present a few conclusions.

II. COMPUTATION OF \( \pi(w(\cdot)), c(w(\cdot)) \), \( \kappa(x(t)) \)

A. Computation of \( \pi(w(\cdot)), c(w(\cdot)) \)

If there exist smooth mappings \( \pi(\cdot) \) and \( c(\cdot) \) satisfying (2) and if the state \( w \) of the exogenous variable is available, then, in the absence of uncertainty, knowledge of \( \bar{x}(t) \triangleq \pi(w(t)) \) and \( \bar{u}(t) \triangleq c(w(t)), t \in \mathbb{R} \), is sufficient for control purposes; knowledge of the functions \( \pi(\cdot) \) and \( c(\cdot) \) that define the manifold \( \mathcal{M} \) is not necessary. Assuming that \( w(\cdot) \) is periodic with period \( T_w \) and (for simplicity) that system (1) with input \( u \) and output \( e \) has unique relative degree \( r \), then the periodic functions \( \bar{x}(\cdot; w_0, 0) \) and \( \bar{u}(\cdot; w_0, 0) \) are determined by solving the finite horizon optimal control problem \( \mathcal{P}(w_0) \)

\[
\begin{align*}
V^0(w_0) &= \min_{\pi(x(\cdot), w(\cdot))} \int_0^T \ell(x(t) - \pi(x(t), w(t)), u(t) - c(w(t))) dt \\
subject \ to \\
\dot{x} &= f(x, u, w) \\
\dot{w} &= \pi(x(t), w(t)), w(0) = w_0 \\
x(0) &= x(T_w)
\end{align*}
\]

with

\[
V(x, w, u(\cdot)) \triangleq \int_0^{T_w} \sum_{i=0}^r \left((d^i/dt^i)e(t)\right)^2 dt
\]

where \( e(t) \triangleq h(x(t), u(t), w(t; w_0, 0)) \). Derivatives up to order \( r \) are used in order to limit the used control energy and avoid singularity of the optimal control problem [14]. If the system (1) is detectable the cost \( |e(t)|^2 \) guarantees \( e(t) \to 0 \) but the use of all the derivatives up to order \( r \) increases numerical stability. As desired, the solution \( (\bar{x}(\cdot; w_0, 0), \bar{u}(\cdot; w_0, 0)) \) of \( \mathcal{P}(w_0) \) satisfies \( (d^i/dt^i)\bar{x}(t) = f(\bar{x}(t), \bar{u}(t), w(t)) \) and \( h(\bar{x}(t), \bar{u}(t), w(t)) = 0 \) for all \( t \geq 0 \) if \( V^0(w_0) = 0 \). Hence \( \bar{x}(\cdot; w_0, 0) = \pi(x(t; w_0, 0)) \) and \( \bar{u}(\cdot; w_0, 0) = c(w(t; w_0, 0)) \) for all \( t \geq 0 \). Because of periodicity, values of \( \bar{x}(t; w_0, 0) \) and \( \bar{u}(t; w_0, 0) \) at any \( t \geq 0 \) may be determined from their values in the interval \([0, T_w]\).

Notice that \( T \) does not need to be equal to \( T_w \) but it can be chosen separately to obtain the desired domain of attraction and performance.

Remark 1: If smooth mappings \( \pi(\cdot) \) and \( c(\cdot) \) satisfying (2) do not exist the optimization control problem (5) computes the best reachable periodic trajectory [15], [7] that is the admissible periodic trajectory giving rise to the minimum tracking error.

B. Computation at time \( t \) of \( u(s), s \in [t, t + \delta] \)

A control that ensures that \( (x(t), w(t)) \) converges to the zero error manifold \( \mathcal{M} \) may be determined by employing conventional model predictive control. If the current state of the composite system at time \( t \) is \((x, w(t; w_0, 0))\), the optimal control problem \( \mathcal{P}(x, w_0, t) \) solved on-line is defined by

\[
\begin{align*}
V^0(x, t) &= \min_{u(\cdot)} \int_t^{t + T} \ell(x(\tau) - \bar{x}(\tau), u(\tau) - \bar{u}(\tau)) d\tau \\
subject \ to \\
\dot{x} &= f(x, u, w), \quad x(t) = x \\
x(\tau) \in \mathbb{X}, \quad u(\tau) \in \mathbb{U} \\
x(t + T) &= \bar{x}(t + T; w_0, 0)
\end{align*}
\]

Where convergence to \( \mathcal{M} \), a terminal equality constraint \( x(t + T) = \bar{x}(t + T; w_0, 0) \) is employed instead of a terminal cost \( V_f(x(t + T; w_0, 0)) \) and associated terminal constraint \( x(t + T) \in X_f(t + T) \) since determination of \( V_f(\cdot) \) and \( X_f(\cdot) \) in a neighbourhood of the target trajectory \( \bar{x}(\cdot) \) is very difficult. Suppose the solution of \( \mathcal{P}(x, w_0, t) \) is \((x^0(\cdot), u^0(\cdot)) \) defined on the interval \([t, t + T]\). The control applied to the system is \( \kappa(x, w_0, t) \triangleq u^0(\cdot) \) restricted to the interval \([t, t + \delta]\); at time \( t + \delta \) the optimal control problem is solved again. Notice that the horizons \( T_w \) and \( T \) don’t need to be the same.
III. CONVERGENCE TO $\mathcal{M}$

For analysis, we note that $P(x, w_0, t)$ is equivalent to the time-invariant problem $P'(x, w)$ defined by

$$V^0(x, w) = \min_{u(\cdot)} \int_0^T \ell(x(\tau) - \bar{x}(\tau), u(\tau) - \bar{u}(\tau)) d\tau$$

where $\bar{x}(\tau) \triangleq \bar{x}(\tau; w_0), \bar{u}(\tau) \triangleq \bar{u}(\tau; w_0)$ subject to

$\dot{x} = f(x, u, w), \quad x(0) = x$

$\dot{w} = s(w), \quad w(0) = w$

$x(\tau) \in X, \quad u(\tau) \in U$

$x(T) = \bar{x}(T; w, 0)$

(7)

where $(x, w)$ is the current state of the composite system at time $t$ ($w = w(t; w_0, 0)$) so that $\bar{x}(T; w, 0) = \bar{x}(t + T; w_0, 0)$. Let the closed-loop system be described by the differential equation $(\dot{x} = f_d(x, w), \dot{w} = s(w))$ and let $(x(t; x, w, 0), w(t; w, 0))$ denote the solution of $(\dot{x} = f_d(x, w), \dot{w} = s(w))$ at time $t$ if $(x(0), w(0)) = (x, w)$. To establish convergence of the state of the composite closed loop system to the zero error manifold $\mathcal{M}$ (i.e. convergence of $x(t)$ to $\pi(w(t)))$ we require an additional assumption. Let the compact set $\mathcal{W}$ denote the set of potential values of $w(t)$ so that $w \in \mathcal{W}$ implies $w(t; w, 0) \in \mathcal{W}$ for all $t \geq 0$. Also, let $X_T(x, w) \triangleq \{ (x, w) \in X \times \mathcal{W} \mid \pi(w(t; w, 0)) = x(T; x, w, 0) \}$ denote the set of initial states $(x, w)$ for which a solution to $P'(x, w)$ exists.

**Assumption 1:** In addition to our prior assumptions on $f(\cdot), \ell(\cdot), X, U$ and $\mathcal{W}$ we assume that $\pi(w(t))$ lies in the interior of $X$ and $c(w(t))$ lies in the interior of $U$ for all $t \geq 0$ for every solution of $\dot{w} = s(w)$ satisfying $w(0) = w \in \mathcal{W}$.

**Theorem 1:** Under the above assumptions, every solution of $\dot{x} = f_d(x, w), \dot{w} = s(w)$ with initial state $(x, w) \in X_T(x, w)$ remains in $X_T(x, w)$ and converges to $\mathcal{M}$.

**Proof:** First of all we show that feasibility of $P'(x_0, w_0)$ is guaranteed for all times if $P'()$ is feasible at the first time instant. Let assume that $u^0(\tau; x_0, w_0), \tau \in [0, T]$, is the optimal solution at time $t_0$ to $P'(x_0, w_0)$ where $(x_0, w_0) \triangleq (x(t_0), w(t_0))$. At time $t_0 + \delta$ the input function $\bar{u}(\tau; x_0, \tau_0, w(t_0) + \delta; w_0, t_0) = u^0(\tau + \delta; x_0, w_0), \tau \in [0, T - \delta], \tau \in [T - \delta, T]$ is a feasible solution to $P'(x(t_0 + \delta; x_0, w_0, t_0), w(t_0 + \delta; w_0, t_0))$. Thus the invariance of $X_T(x, w)$ is ensured. The convergence of $x(t; x, w, 0)$ to the manifold $\mathcal{M}$ is proved showing the convergence of the closed loop trajectory to $\bar{x}(t; x, w, 0)$. The optimal value function $V^0(x, w)$ is selected as a candidate Lyapunov function. It satisfies $V^0(x, w) \geq 0$ and $V^0(x, w) = 0$ if and only if $x = x_0 \triangleq \bar{x}(0; x, w, 0)$. For all $(x, w) \in X_T(x, w)$ there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$\alpha_1(||x - \bar{x}_0||) \leq V^0(x, w) \leq \alpha_2(||x - \bar{x}_0||)$

At time $t_0$ given $x$ and $w$ the optimization problem $P'(x, w)$ is solved and the control $u^0(\tau; x, w), \tau \in [0, \delta]$ is applied to the system. After the sampling time $\delta$ the optimization problem $P'(\cdot)$ is solved again with $u_\delta \triangleq u(t_0 + \delta; w, t_0)$ and $x_\delta \triangleq x(t_0 + \delta; x, w, t_0)$. The following relation holds

$V^0(x_\delta, w_\delta) \leq V^*(x_\delta, w_\delta, \bar{u}(\tau; x_\delta, w_\delta)) = V^0(x, w) - \int_0^\delta (\ell(x^0(\tau) - \bar{x}(\tau), u^0(\tau) - \bar{u}(\tau)) d\tau$

where $\bar{u}(\tau; x_\delta, w_\delta)$ is the feasible input defined in (8) and the functions $u^0(\tau)$ are $u^0(\tau)$ are the optimal solution to $P'(x, w)$. Since $\int_0^\delta (\ell(x^0(\tau) - \bar{x}(\tau), u^0(\tau) - \bar{u}(\tau)) d\tau > 0$ for all $x$ such that $\|x - \bar{x}_0\| > 0, 0 \leq V^0(x, w) < V^0(x, w)$ is satisfied for all $x$ such that $\|x - \bar{x}(0; w, 0)\| > 0$ and convergence of $x(t; x, w, 0)$ to $\bar{x}(t; x, w, 0)$ is obtained by standard Lyapunov arguments.

IV. ILLUSTRATIVE EXAMPLE

We illustrate the effectiveness of the proposed approach by its application to the problem of landing an autonomous vehicle on an oscillating platform [9]. The system is described by

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\sin(\theta_1)u_1 + 2\cos(\theta_1)\sin(\alpha)u_2$

$\dot{y}_1 = y_2$

$\dot{y}_2 = \cos(\theta_1)u_1 + 2\sin(\theta_1)\sin(\alpha)u_2 - g$

$\dot{\theta}_1 = \theta_2$

$\dot{\theta}_2 = (2lM/J)\cos(\alpha)u_2$

where $x_1, y_1$ and $\theta_1$ denote, respectively, the horizontal and vertical position of the center of mass and the roll angle with respect to the horizon of the aircraft; $u_1$ and $u_2$ are normalised control commands specifying, respectively, the ‘upwards’ engine thrust and the rolling moment generated on the wings. The parameters are $M = 5 \times 10^4$Kg, $J = 1.25 \times 10^4$Kg m$^2$, $\alpha = 4\pi/180$rad/s, $l = 5$m and $g$ is the gravitational acceleration. The tracking error $e$ is given by

$e = \begin{bmatrix} y_1 - x_1 \\ \theta_2 \end{bmatrix} - H$

(10)

where $H = 15$m is a vertical offset to prevent undesired crashes during the approaching manoeuvre to the deck; the signal $r(w) = Rw$ is generated by the exogenous system $w = Sw$. The constraint $e_2 = 0$ specifies the requirement that the height of the aircraft needs to be synchronised with the oscillating deck in high seas during the landing. The vertical motion of the deck has been modeled as superposition of two sinusoidal function with frequency $\omega_1 = 1$rad/s and $\omega_2 = 1.6$rad/s.

$\dot{w} = \begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{bmatrix} w$

(11)

$\dot{r}(w) = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} w$

In addition there are two conventional requirements $e_1 = x_1 = 0$ and $e_3 = \theta_1 = 0$. The system is subject to input and state constraints

$X = \{ x \in \mathbb{R}^6 \mid x_5 \in [-1.5, 1.5], x_6 \in [-1, 1] \}$

$U = \{ u \in \mathbb{R}^2 \mid u_1 \geq 0 \}$

5098
where \( x \triangleq [x_1, x_2, y_1, y_2, \theta_1, \theta_2] \). The reference trajectories \( \bar{x}(t), \bar{u}(t) \) were computed solving the optimal control problem \( \mathcal{P} \) with \( w(0) = [2, 2.2, 1, 2.2] \) and \( T_w = 10\pi \). Notice that the relative degree \( r \) of the system (9) with output (10) is 2 for all components of the output. The cost of the optimization problem \( \mathcal{P} \) contains the derivatives of all output components up to degree two. The proposed MPC uses the terminal constraint \( x(t + T) = \bar{x}(t + T) \) and parameters \( Q = I_6, R = I_2 \) and \( T = 10\pi \). The sampling time used for the controller is \( T_s = T/2000 \) seconds. The discrete-time model is implicitly obtained via the optimization process using the nonlinear optimization code IPOPT [16] together with the toolbox ICLOCS [17].

Figures 1, 2, 3, 4, 5, 6, 7 and 8 illustrate the behavior of the controlled system starting from the initial conditions \( x(0) = [10, 10, 100, 10, 1, 1] \). The selected initial condition \( x(0) \) is considerably far from the desired error manifold and the constraints on the states and the inputs become active during the transient. Figures 1, 2, 3, 4, 5, 6, 7 and 8 show how the state variables \( x_i \) and input variables \( u_i \) (black solid) effectively track the reference trajectory \( \bar{x}_i \) and \( \bar{u}_i \) (red dashed) satisfying the constraints.

V. Conclusions

The proposed method of solving the tracking problem has both advantages and disadvantages. On the positive side it employs a well tried methodology for handling nonlinearity and hard constraints and the obtained solution is not just valid locally but in the whole region of feasibility of the optimization problem. The simplicity of the solution is appealing; it has the MPC property of replacing offline determination of complex functions such as \( \pi(\cdot), c(\cdot) \) and \( \kappa(\cdot) \) by repeated on-line solution of a standard finite horizon optimal control problems to compute the current control action. On the negative side the on-line computation that is necessary may require more time than available for controlling fast dynamics. The solution proposed here requires accessibility of the state \((x, u)\); an observer has to be added if this is not the case. To ensure robustness of the controlled system against parameter variation, modification of the controller, such as addition of an internal model, is required.

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