

Can a matter-dominated model with constant bulk viscosity drive the accelerated expansion of the universe?

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Abstract. We test a cosmological model which the only component is a pressureless fluid with a constant bulk viscosity as an explanation for the present accelerated expansion of the universe. We classify all the possible scenarios for the universe predicted by the model according to their past, present and future evolution and we test its viability performing a Bayesian statistical analysis using the SCP “Union” data set (307 SNe Ia), imposing the second law of thermodynamics on the dimensionless constant bulk viscous coefficient $\tilde{\zeta}$ and comparing the predicted age of the universe by the model with the constraints coming from the oldest globular clusters.

The best estimated values found for $\tilde{\zeta}$ and the Hubble constant H_0 are: $\tilde{\zeta} = 1.922 \pm 0.089$ and $H_0 = 69.62 \pm 0.59$ (km/s)Mpc $^{-1}$ with a $\chi^2_{\min} = 314$ ($\chi^2_{\text{d.o.f}} = 1.031$). The age of the universe is found to be 14.95 ± 0.42 Gyr. We see that the estimated value of H_0 as well as of $\chi^2_{\text{d.o.f}}$ are very similar to those obtained from Λ CDM model using the same SNe Ia data set. The estimated age of the universe is in agreement with the constraints coming from the oldest globular clusters. Moreover, the estimated value of $\tilde{\zeta}$ is positive in agreement with the second law of thermodynamics (SLT).

On the other hand, we perform different forms of marginalization over the parameter H_0 in order to study the sensibility of the results to the way how H_0 is marginalized. We found that it is almost negligible the dependence between the best estimated values of the free parameters of this model and the way how H_0 is marginalized in the present work.

Therefore, this simple model might be a viable candidate to explain the present acceleration in the expansion of the universe.

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1. Introduction.

In the last ten years the observations of type Ia supernovae (SNe Ia) have suggested a possible late time accelerated expansion of the universe (see for instance [1]-[7] and references therein). This discovery has additional support from the cosmic microwave background [8] and the large scale structure [9] observations.

Several models have been proposed to explain this recent acceleration, one of them is the assumption of the existence of the so-called *dark energy* as responsible of such acceleration, i.e., a new unknown component that must be $\sim 70\%$ of the total content of matter and energy in the universe [1]-[7]. The leading dark energy candidates are a cosmological constant and a slowly varying rolling scalar field (quintessence models) [10]-[14]. However, a cosmological constant model faces several strong problems, one of them is the huge discrepancy between its predicted and observed value (of about 120 orders of magnitude) [15]-[18], another one is the so-called the “*cosmic coincidence problem*”, i.e., why are we living in a moment where the matter density in the universe is of the same order than the dark energy density? [19]-[21].

On the other hand, it has been known since several years ago before of the discovery of the present acceleration in the expansion of the universe that a bulk viscous fluid can produce an accelerating cosmology (although it was originally proposed in the context of an inflationary period in the early universe) without the need of any cosmological constant or dark energy component [22]-[35] although some authors do not agree with this conclusion [36].

So, it is natural to think of the bulk viscous pressure as one of the possible mechanism that can accelerate the universe today (see for instance [37]-[46], where in some of these works the bulk viscous coefficient is assumed a priori without being derived from known physics or known particle properties). However, this idea faces the problem of that it is necessary to propose a viable mechanism for the origin of the bulk viscosity and in this sense some proposals have been already suggested [47]-[50].

Nowadays, with the observational data sets of SNe Ia [1]-[7] and some other cosmological observations like the shift parameter R of the cosmic microwave background radiation (CMB) given by the Wilkinson Microwave Anisotropy Probe (WMAP) observations [51], and the baryon acoustic oscillation (BAO) measurement from the Sloan Digital Sky Survey (SDSS) [52] it is possible to *test* cosmological models. So, in the present work we study and test bulk viscous matter-dominated cosmologies, i.e., scenarios which the only component of the universe is a pressureless fluid with bulk viscosity. The pressureless fluid characterizes to both baryon and dark matter components. The idea of this model is to explain the present acceleration of the universe using the bulk viscous pressure of the fluid without the need of any dark energy.

As it was mentioned above, the explicit form of the bulk viscosity has to be assumed a priori or obtained from a known physical mechanism. In the present work we choose the first possibility. So, we assume a *constant* bulk viscous coefficient that it is the simplest parametrization for the bulk viscosity and we estimate its value using SNe Ia

observations. The theoretical solution of this model has been also analyzed in [22], [53]-[56] and other parametrizations for the bulk viscous coefficient ζ have been also proposed and studied in [37]-[45] and [57]-[60]. There is also other general approach to the bulk viscous cosmologies called “fluids with inhomogeneous equation of state” (see [61, 65]).

In section 2 we present the generalities of bulk viscous fluids in General Relativity (GR), in section 3 we apply this formalism to a bulk viscous matter-dominated universe model where we find the explicit expression for the Hubble parameter in function of the redshift. In section 4 we analyze the behavior of the scale factor for the possible scenarios that the model predicts for the universe according to the value of the bulk viscous coefficient. In section 5 we study the behavior of the deceleration parameter q , the curvature scalar R , the total matter density ρ_m and the age of the universe and in section 6 we briefly review the second law of thermodynamics (SLT). Section 7 presents the SNe Ia test to constrain the model and compute the best estimated values for the bulk viscous coefficient and the Hubble constant. After that, section 8 presents the best estimated values and the probability distribution functions for the bulk viscous coefficient by marginalizing over the Hubble constant assuming three different priors and finally in section 9 we present our conclusions. The appendices A, B and C detail the marginalization methods used in the present work.

2. Theory of relativistic bulk viscous fluids.

The origin of the bulk viscosity in a physical system is due to its deviations from the local thermodynamic equilibrium (for a review in the theory of relativistic dissipative fluids see [66]). In a cosmological fluid, the bulk viscosity arises when the fluid expands (or contracts) too fast so that the system does not have enough time to restore the local thermodynamic equilibrium and then it appears an *effective pressure* restoring the system to its local thermodynamic equilibrium. The bulk viscosity can be seen as a measurement of this effective pressure. When the fluid reaches again the thermal equilibrium then the bulk viscosity pressure ceases [49], [67]–[69].

In an *accelerated* expanding universe it is very possible that the expansion process is actually a collection of states out of thermal equilibrium in a small fraction of time. So, it is natural to assume the existence of a bulk viscous coefficient in a more realistic description of the accelerated universe today.

We use the Weinberg formalism [70]–[72] for the imperfect fluids. So, in the present work we consider a bulk viscous fluid as source of matter in the Einstein fields equations $G_{\mu\nu} = 8\pi GT_{\mu\nu}$, where G is the Newton gravitational constant.

On the other hand, the energy-momentum tensor of an imperfect fluid with a first-order deviation from the thermodynamic equilibrium has the form [49, 70, 71]:

$$T_{\mu\nu} = \rho u_\mu u_\nu + (g_{\mu\nu} + u_\mu u_\nu)P^* \quad (1)$$

where

$$P^* \equiv P - \zeta \nabla^\nu u_\nu \quad (2)$$

In the equations (1) and (2) the four-velocity vector u_ν is of an observer who measures the *effective* pressure P^* , P and ρ are the pressure and density of the fluid respectively. The term ζ is the bulk viscous coefficient that arises in the fluid which is out of the local thermodynamic equilibrium.

It can be seen that the energy-momentum tensor (1) is similar to that of a perfect fluid but with an effective pressure P^* composed by the usual pressure P of the fluid plus the pressure due to its bulk viscosity $P_{\text{visc}} \equiv -\zeta \nabla^\nu u_\nu$. This viscous pressure, P_{visc} , can be seen as a “measurement” of the pressure to restore the local thermodynamic equilibrium [49], [67]–[69]. The conservation equation for the viscous fluid is

$$u^\nu \nabla_\nu \rho + (\rho + P^*) \nabla^\nu u_\nu = 0 \quad (3)$$

The effective pressure (2) was originally proposed by Eckart [74] in 1940 for a relativistic dissipative process in the context of thermodynamics systems out of local equilibrium, and subsequently Landau & Lifshitz presented an equivalent formulation [75].

However, the Eckart theory has problems in several aspects. One of them is that all the equilibrium states in this theory are unstable [76]. Another one is that signals can propagate through the fluids with superluminal velocities [77, 78].

In 1979, Israel-Stewart [79, 80] developed a more consistent and general theory that avoids these problems, and from which the Eckart theory is the first-order limit of the Israel-Stewart theory when the relaxation time goes to zero. Nevertheless, the Eckart theory is simpler to deal with than the Israel-Stewart theory.

Despite of the inherent problems of the Eckart theory, and due to that it is simpler than the Israel-Stewart theory, it has been widely used recently by several authors to model bulk viscous dark fluids as responsible of the recent observed acceleration of the universe assuming that the approximation is valid for this purpose (see, for instance [37]–[41], [44, 45, 49, 56, 60]), i.e., in these papers assume a vanishing relaxation time, so that, in this limit the Eckart theory is a good approximation to the Israel-Stewart theory. In this context, it is convenient to mention that Hiscock *et al* [36] showed that flat Friedmann-Robertson-Walker cosmological models containing a bulk viscous Boltzmann gas expand more rapidly using the Eckart theory than the Israel-Stewart one, and suggesting that inflationary acceleration driven by bulk viscosity could be an effect of applying a pathological theory such as the Eckart theory. These results suggest that in this context the use of the Israel-Stewart causal theory would not produce a recent accelerating epoch as the Eckart theory would. However, posterior studies have suggested that this conclusion could not be true because consistent inflationary solutions have been found using the Israel-Stewart theory [34, 35].

It is important to point out that there exists other more general formulation for irreversible processes than the Israel-Stewart theory developed by D. Pavón *et al.* where the temperature is not necessary that of the thermodynamic equilibrium, see [81] for

details. In the present work we assume the Eckart theory and we constrain the model described above using SNe Ia observations.

3. Cosmological model of bulk viscous matter-dominated universes.

We study a cosmological model in a flat universe where the only component is a pressureless fluid with constant bulk viscosity as an explanation for the present accelerated expansion of the universe. The pressureless fluid characterizes both the baryon and dark matter components.

Note that this model does not have the Cosmic Coincidence problem and that this fluid represents an unified description of the dark sector plus the baryon component in a similar way than the Chaplygin gas model (see for instance [44, 82] and references therein). In this approach the present acceleration of the universe is driven by the bulk viscous pressure of the fluid instead of a dark energy component. The present work differs from the approach of Colistete *et al* [44] in that they propose two fluids, one of them is a bulk viscous fluid representing in an unified way the dark sector with a bulk viscous coefficient ζ proportional to a power of the energy density ρ (i.e., $\zeta = \tilde{\zeta} \rho^\nu$, with $\tilde{\zeta}, \nu$ constants), and the other one is a pressureless fluid representing the baryon component. Nevertheless, most of their analysis and test using cosmological observations is done using the ansatz $\nu = 0$.

Since we work with a pressureless fluid ($P = 0$) then $P^* = P_{\text{visc}} \equiv -\zeta \nabla^\nu u_\nu$, where ζ is the bulk viscous coefficient of the matter fluid.

We consider a spatially flat geometry for the Friedmann-Robertson-Walker (FRW) cosmology as favored by WMAP [83]

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2) \quad (4)$$

where the function $a(t)$ is the *scale factor*. On the other hand, we spread the conservation equation (3) in all its components

$$\dot{\rho}_m + (\rho_m - 3H\zeta)3H = 0 \quad (5)$$

where $H \equiv \dot{a}/a$ is the *Hubble parameter*, ρ_m is the total matter density, the dot means time derivative and $\nabla^\nu u_\nu = 3H$.

The conservation equation (5) can be written in terms of the scale factor as

$$a \frac{d\rho_m}{da} = 3(3H\zeta - \rho_m) \quad (6)$$

This equation is valid for any parametrization of ζ , in particular we assume the ansatz $\zeta = \text{constant}$ that it is perhaps the simplest parametrization for the bulk viscous coefficient that can be proposed. Its value is going to be estimated from the SNe Ia observations.

On the other hand, the first Friedmann equation for a flat universe is

$$H^2 = \frac{8\pi G}{3} \rho_m \quad (7)$$

So, substituting (7) into (6) we obtain

$$a \frac{d\rho_m}{da} + 3\rho_m - \gamma\rho_m^{1/2} = 0, \quad \text{where} \quad \gamma \equiv 9 \left(\frac{8\pi G}{3} \right)^{1/2} \zeta \quad (8)$$

Changing of variable from the scale factor to the redshift z using the relationship[‡] $a = 1/(1+z)$, we obtain the ordinary differential equation (ODE):

$$(1+z) \frac{d\rho_m}{dz} - 3\rho_m + \gamma\rho_m^{1/2} = 0 \quad (9)$$

The exact solution of this ODE is:

$$\rho_m(z) = \left[\frac{\gamma}{3} + \left(\rho_{m0}^{1/2} - \frac{\gamma}{3} \right) (1+z)^{3/2} \right]^2 \quad (10)$$

where ρ_{m0} is the matter density evaluated today. Substituting this solution into (7) we obtain

$$H^2(z) = H_0^2 \left[\frac{\tilde{\zeta}}{3} + \left(\Omega_{m0}^{1/2} - \frac{\tilde{\zeta}}{3} \right) (1+z)^{3/2} \right]^2 \quad (11)$$

where H_0 is the Hubble constant and where we have defined the dimensionless bulk viscous coefficient $\tilde{\zeta}$, the matter density parameter Ω_{m0} and the critical density today ρ_{crit}^0 as:

$$\tilde{\zeta} \equiv \frac{24\pi G}{H_0} \zeta, \quad \Omega_{m0} \equiv \frac{\rho_{m0}}{\rho_{\text{crit}}^0}, \quad \rho_{\text{crit}}^0 \equiv \frac{3H_0^2}{8\pi G} \quad (12)$$

In this model the bulk viscous matter is the only component of the universe implying that the first Friedmann equation (7) evaluated today is $\Omega_{m0} = 1$. With this, the expression (11) finally becomes

$$H(z) = \frac{H_0}{3} \left[\tilde{\zeta} + (3 - \tilde{\zeta}) (1+z)^{3/2} \right] \quad (13)$$

4. Classification and evolution of bulk viscous matter-dominated models.

In this section we classify all the possibilities for the universe predicted by this bulk viscous matter-dominated model using different values of the constant bulk viscous coefficient. Note that we have one different model for each value of $\tilde{\zeta}$, so actually we have a collection of models depending of the value of $\tilde{\zeta}$.

We analyze the theoretical behavior of the scale factor in terms of the cosmic time. We begin expressing (13) in terms of the scale factor

$$H(a) \equiv \frac{\dot{a}}{a} = \frac{H_0}{3} \left(\frac{\tilde{\zeta} a^{3/2} + 3 - \tilde{\zeta}}{a^{3/2}} \right) \quad (14)$$

Integrating (14) yields

$$H_0(t - t_0) = H_0 \int_{t_0}^t dt' = 3 \int_1^a \frac{a'^{1/2} da'}{\tilde{\zeta} a'^{3/2} + 3 - \tilde{\zeta}} \quad (15)$$

where t labels the *cosmic time* and t_0 the *cosmic time today*.

[‡] We assume that the value of the scale factor evaluated *today* is equal to one.

4.1. Case $\tilde{\zeta} = 0$.

When $\tilde{\zeta} = 0$ we recover the usual matter-dominated universe (with null bulk viscosity) with a scale factor coming from the integration of (15) like:

$$a(t) = \left(\frac{3}{2} H_0 (t - t_0) + 1 \right)^{2/3}, \quad \tilde{\zeta} = 0 \quad (16)$$

The first and second derivatives of the scale factor with respect to $x \equiv H_0(t - t_0)$ are

$$\frac{da}{dx} = \left(\frac{2}{3x + 2} \right)^{1/3} \quad (17)$$

and

$$\frac{d^2a}{dx^2} = -\frac{2^{1/3}}{(3x + 2)^{4/3}} \quad (18)$$

The plot of functions (16)–(18) are shown in figures 1–3 respectively (the long dashed lines).

This case predicts an *eternal decelerated expanding* universe (see figures 1–7). In this case the curvature scalar and the matter density are $R = (3H_0^2)a^{-3}$ and $\rho_m = (3H_0^2/8\pi G)a^{-3}$ respectively, that both diverge when the scale factor goes to zero (see sections 5.2 and 5.3 for details). When we have these two conditions ($R(a \rightarrow 0) \rightarrow \infty$ and $\rho_m(a \rightarrow 0) \rightarrow \infty$) in the past of the universe we say that there was a *Big-Bang*. The elapsed time between the Big-Bang time till today is

$$t_B = t_0 - \frac{2}{3H_0} \quad (19)$$

where the subscript ‘B’ stands for “Big-Bang”. On the other hand, when $a \rightarrow \infty$ then R and ρ_m decrease to zero.

4.2. Case $\tilde{\zeta} \neq 0$.

For $\tilde{\zeta} \neq 0$ we do the change of variable $y \equiv a^{3/2}$ in expression (15) yielding:

$$H_0(t - t_0) = \frac{2}{\tilde{\zeta}} \int_1^y \frac{\tilde{\zeta} dy'}{\tilde{\zeta} y' + 3 - \tilde{\zeta}} = \frac{2}{3\tilde{\zeta}} \ln \left| \tilde{\zeta} a^{3/2} + 3 - \tilde{\zeta} \right| \quad (20)$$

This last expression can be rewritten as

$$3 \exp \left[\frac{\tilde{\zeta}}{2} H_0 (t - t_0) \right] = \left| \tilde{\zeta} a^{3/2} + 3 - \tilde{\zeta} \right| \quad (21)$$

When $\tilde{\zeta} a^{3/2} + 3 - \tilde{\zeta} \geq 0$ we can remove the absolute value bar of the right hand side term and then we obtain

$$a(t) = \left[\frac{3 \exp \left(\frac{1}{2} \tilde{\zeta} H_0 (t - t_0) \right) - 3 + \tilde{\zeta}}{\tilde{\zeta}} \right]^{2/3}, \quad \tilde{\zeta} \neq 0 \quad (22)$$

The first and second derivative of the scale factor with respect to $x \equiv H_0(t - t_0)$ are

$$\frac{da}{dx} = \left[\frac{\tilde{\zeta} \exp\left(\frac{3}{2}\tilde{\zeta}x\right)}{3 \exp\left(\frac{3}{2}\tilde{\zeta}x\right) + \tilde{\zeta} - 3} \right]^{1/3} \quad (23)$$

and

$$\frac{d^2a}{dx^2} = \frac{\tilde{\zeta}^{4/3} \exp\left(\frac{1}{2}\tilde{\zeta}x\right) \left[2 \exp\left(\frac{1}{2}\tilde{\zeta}x\right) + \tilde{\zeta} - 3\right]}{2 \left[3 \exp\left(\frac{1}{2}\tilde{\zeta}x\right) + \tilde{\zeta} - 3\right]^{4/3}} \quad (24)$$

The behavior of the expressions (22)–(24) are shown in figures 1–3 respectively.

The case $\tilde{\zeta}a^{3/2} + 3 - \tilde{\zeta} < 0$ does not correspond to a physical case of our interest because this case predicts an eternal *contraction* of the universe ($H(t) < 0$ for any value of the cosmic time t) that is in contradiction with the observations (see the expression (14) to note that $\tilde{\zeta}a^{3/2} + 3 - \tilde{\zeta} < 0$ implies $H(t) < 0$).

In order to study carefully the expression (22) for different values of $\tilde{\zeta}$ we take four cases:

- $0 < \tilde{\zeta} < 3$
- $\tilde{\zeta} = 3$
- $\tilde{\zeta} > 3$
- $\tilde{\zeta} < 0$

4.2.1. Case $0 < \tilde{\zeta} < 3$. From the expression (22) we can see that when $t \rightarrow \infty$ then the scale factor tends to have the form like that of the *de Sitter universe*, i.e.,

$$a(t) \propto e^{(\tilde{\zeta}/3)H_0(t-t_0)} \quad (25)$$

On the other hand, for any value of $\tilde{\zeta}$ in this interval all the models predict an universe having a Big-Bang in the past in the cosmic time:

$$t_B = t_0 + \frac{2}{\tilde{\zeta}H_0} \ln \left(1 - \frac{\tilde{\zeta}}{3}\right) \quad (26)$$

So, the universe begins with a Big-Bang followed by an *eternal expansion* (there is not any recollapse epoch, see figures 1 and 2) and this expansion begins with a decelerated epoch followed by an *eternal accelerated* one (see figures 2 and 3).

The *transition* between the decelerated–accelerated expansion epochs depends on the value of $\tilde{\zeta}$. We compute the value of the scale factor where the transition happens. For that, we derive to \dot{a} with respect to a using the expression (14):

$$\frac{d\dot{a}}{da} = \frac{H_0}{3} \left(\tilde{\zeta} + \frac{\tilde{\zeta} - 3}{2a^{3/2}} \right) \quad (27)$$

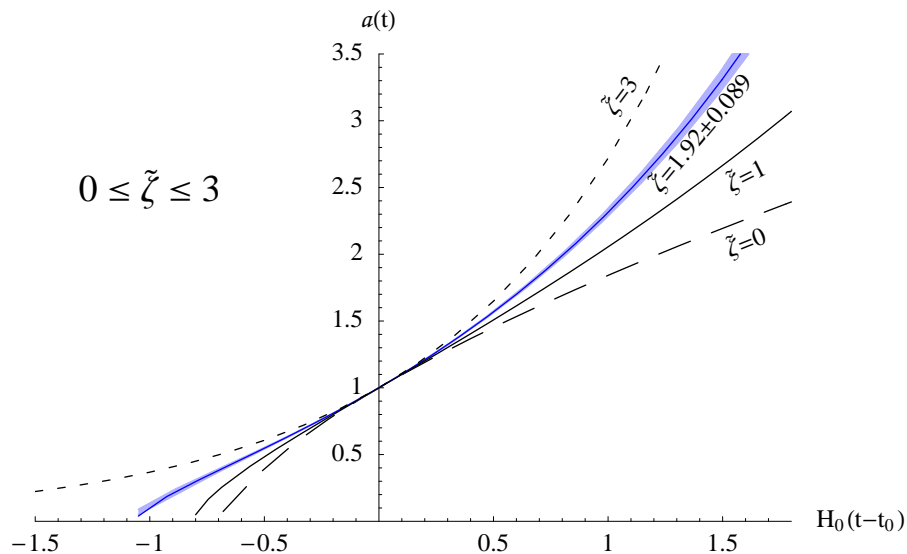


Figure 1. Plot of $a(t, \tilde{\zeta})$ for different values of $\tilde{\zeta}$ in the interval $0 \leq \tilde{\zeta} \leq 3$ (see expressions (16) and (22)). H_0 is the Hubble constant, t is the cosmic time and t_0 is its present value. The long dashed line corresponds to $\tilde{\zeta} = 0$ (a flat matter-dominated universe with null bulk viscosity, see expression (16)). The short dashed line corresponds to $\tilde{\zeta} = 3$ (the *de Sitter* universe). The blue line with the blue band correspond to a model with $\tilde{\zeta} = 1.922 \pm 0.089$. This is the best estimated value of $\tilde{\zeta}$ coming from the SCP “Union” SNe Ia data set analysis (see section 7). The band corresponds to the error at the 68.3% confidence level.

Then, we make equal to zero to expression (27) in order to obtain the value of the scale factor “ a_t ” where the transition happens, obtaining

$$a_t = \left(\frac{3 - \tilde{\zeta}}{2\tilde{\zeta}} \right)^{2/3} \quad (28)$$

where the subscript “t” stands for “transition”. Writing the expression (28) in terms of the redshift z yields

$$z_t = \left(\frac{2\tilde{\zeta}}{3 - \tilde{\zeta}} \right)^{2/3} - 1 \quad (29)$$

From the expression (28) we can see that for values of $\tilde{\zeta}$ in the interval $0 < \tilde{\zeta} < 1$ the transition between the decelerated epoch to the accelerated one takes place in the future ($a_t > 1$). When $\tilde{\zeta} \rightarrow 0$, the value of a_t tends to infinity in the future. When $\tilde{\zeta} = 1$ then the transition takes place today ($a_t = 1$), when $1 < \tilde{\zeta} < 3$ the transition takes place in the past of the universe ($0 < a_t < 1$) and when the value of $\tilde{\zeta}$ is closer to 3, the transition is closer to the Big-Bang time (see figures 2–4 and 7).

The curvature scalar R is always positive and it has a very large value when the scale factor is very small (at the Big-Bang time) and it decreases forever tending to its minimum value $R = \frac{4}{3}H_0^2\tilde{\zeta}$ when $a \rightarrow \infty$ (see section 5.2 and figure 8) what is the

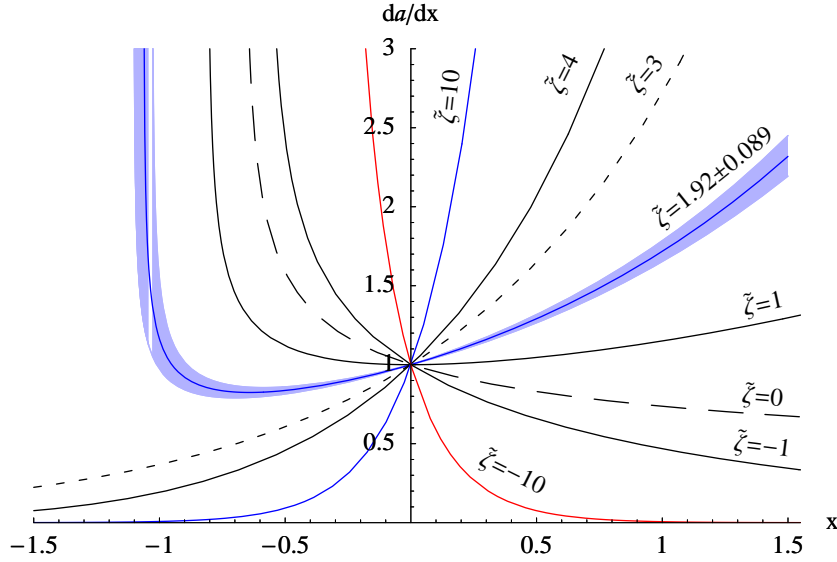


Figure 2. Plot of the derivative of the scale factor with respect to $x \equiv H_0(t - t_0)$ for different values of $\tilde{\zeta}$ (see expression (23)). The long dashed line corresponds to $\tilde{\zeta} = 0$ (a flat matter-dominated universe with null bulk viscosity, see expression (17)) and the short dashed line corresponds to $\tilde{\zeta} = 3$ (the *de Sitter* universe). The blue line with the blue band correspond to a model with $\tilde{\zeta} = 1.922 \pm 0.089$. This is the best estimated value of $\tilde{\zeta}$ coming from the SCP “Union” SNe Ia data set analysis (see section 7). The band corresponds to the error at the 68.3% confidence level.

de Sitter universe (consistent with the expression (25)). In the same way, the matter density ρ_m is also very large when the scale factor is very small (at the Big-Bang time) and it decreases forever tending to its minimum value $\rho_m = (H_0^2/24\pi G)\tilde{\zeta}^2$ when $a \rightarrow \infty$ (see section 5.3).

The possible value for $\tilde{\zeta}$ at least at the 99.9% confidence level estimated from the SCP “Union” SNe Ia data set is in the interval $1 < \tilde{\zeta} < 3$. So, assuming the best estimated value $\tilde{\zeta} = 1.922 \pm 0.089$ (see table 1), the value of the scale factor at the transition time between deceleration–acceleration epochs is (see expression (28)): $a_t = 0.42 \pm 0.03$ ($z_t = 1.33 \pm 0.2$). Note that the transition comes directly from the model, it is not necessary to assume any particular ansatz to estimate or induce it.

4.2.2. Case $\tilde{\zeta} = 3$. In the expression (22) we put $\tilde{\zeta} = 3$ to obtain

$$a(t) = e^{H_0(t-t_0)} \quad (30)$$

This case corresponds to the *de Sitter* universe. Figures 1–5 show the behavior of expression (30) (the short-dashed lines). The scale factor becomes zero just when $H_0(t - t_0) \rightarrow -\infty$ and when $H_0(t - t_0) \rightarrow \infty$ the scale factor increases to infinity.

In this case the model predicts an universe in an *eternal accelerated expansion*. The curvature scalar and the matter density are constants with values $R = 12H_0^2$ and $\rho_m = 3H_0^2/8\pi G$ respectively. See sections 5.2 and 5.3 for details.

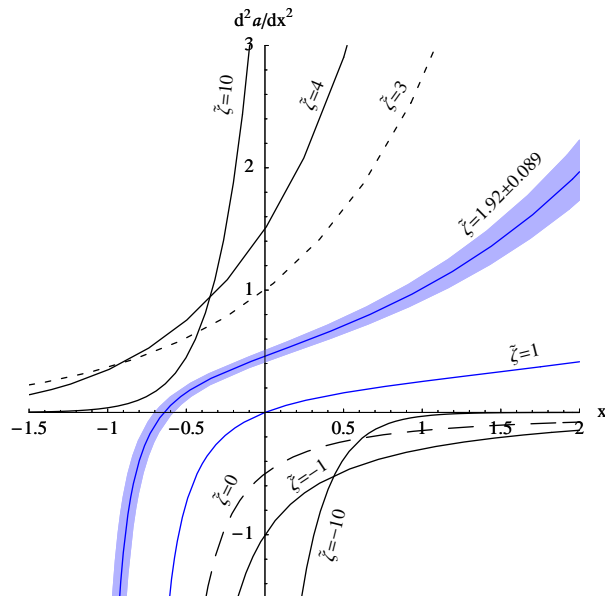


Figure 3. Plot of the *second* derivative of the scale factor with respect to $x \equiv H_0(t-t_0)$ for different values of $\tilde{\zeta}$ (see expression (24)). The long dashed line corresponds to $\tilde{\zeta} = 0$ (a flat matter-dominated universe with null bulk viscosity, see expression (18)) and the short dashed line corresponds to $\tilde{\zeta} = 3$ (the *de Sitter* universe). The blue line with the blue band correspond to a model with $\tilde{\zeta} = 1.922 \pm 0.089$. This is the best estimated value of $\tilde{\zeta}$ coming from the SCP “Union” SNe Ia data set analysis (see section 7). The band corresponds to the error at the 68.3% confidence level. We see that for $\tilde{\zeta} \leq 0$ all the models predict a decelerated universe ($d^2a/dx^2 < 0$) *forever*. For $\tilde{\zeta} \geq 3$ the models predict an accelerated universe ($d^2a/dx^2 > 0$) *forever*. And in the range $0 < \tilde{\zeta} < 3$ the models predict a transition from deceleration to acceleration.

Note that this model does not have a Big-Bang because the curvature scalar and the matter density are regular for any value of the cosmic time.

4.2.3. Case $\tilde{\zeta} > 3$. This case predicts an universe *expanding forever*, and this expansion is *always accelerated* (there is not any decelerating epoch or acceleration-deceleration transition).

In this case when $t \rightarrow \infty$ then the scale factor tends to the form of the de Sitter universe in the future (see expression (25)) and when $t \rightarrow -\infty$ then the universe tends to an *Einstein static universe* (defined by $\dot{a}, \ddot{a} = 0$) where the value of the scale factor is not zero but tends to its *minimum* value:

$$\lim_{t \rightarrow -\infty} a(t) \equiv a^* = \left(1 - \frac{3}{\tilde{\zeta}}\right)^{2/3} \quad (31)$$

and during the evolution of the universe the scale factor is an increasing monotonic function with respect to the cosmic time. Thus, for this case there was not a Big-Bang and the age of the universe *is not defined*. Figure 5 shows some plots of the expression (22) for different values of $\tilde{\zeta} > 3$.

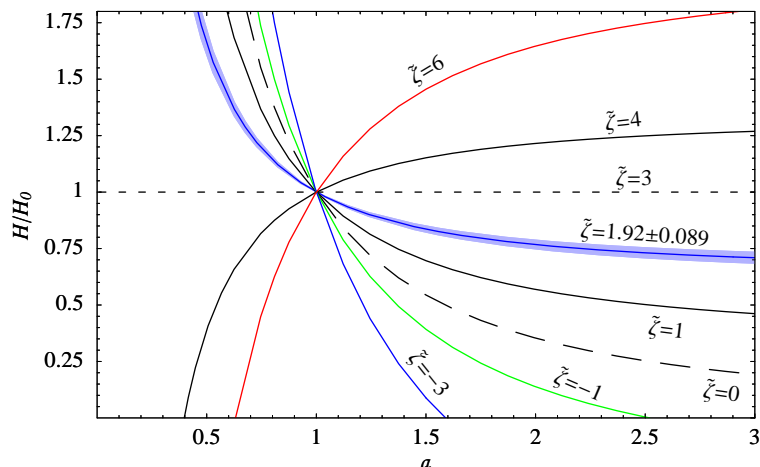


Figure 4. Plot of $H(a, \tilde{\zeta})$ for different values of $\tilde{\zeta}$ (see expression (14)). a is the scale factor and H_0 is the Hubble constant. The long dashed line corresponds to $\tilde{\zeta} = 0$ (a flat matter-dominated universe with null bulk viscosity) and the short dashed line corresponds to $\tilde{\zeta} = 3$ (the *de Sitter* universe). The blue line with the blue band correspond to a model with $\tilde{\zeta} = 1.922 \pm 0.089$. This is the best estimated value of $\tilde{\zeta}$ coming from the SCP “Union” SNe Ia data set analysis (see section 7). The band corresponds to the error at the 68.3% confidence level.

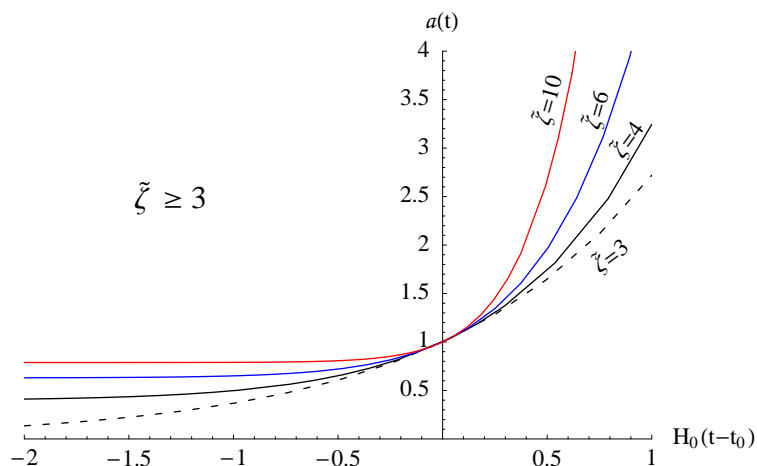


Figure 5. Plot of $a(t, \tilde{\zeta})$ for different values of $\tilde{\zeta}$ in the range $\tilde{\zeta} \geq 3$ (see expression (22)). H_0 is the Hubble constant, t is the cosmic time and t_0 is its present value. The short dashed line corresponds to $\tilde{\zeta} = 3$ (the *de Sitter* universe).

When the scale factor has the value a^* the curvature scalar is zero and it is *always increasing* from zero until to reach its maximum value $R = \frac{4}{3}H_0^2\tilde{\zeta}$ when $a \rightarrow \infty$. See section 5.2 and figure 8 for details.

The matter density ρ_m has a similar behavior to the curvature scalar in the sense of that when the scale factor has the value a^* the matter density is zero and it is *always increasing* from zero until to reach its maximum value $\rho_m = (H_0^2/24\pi G)\tilde{\zeta}^2$ when $a \rightarrow \infty$. See section 5.3 and figure 9 for details.

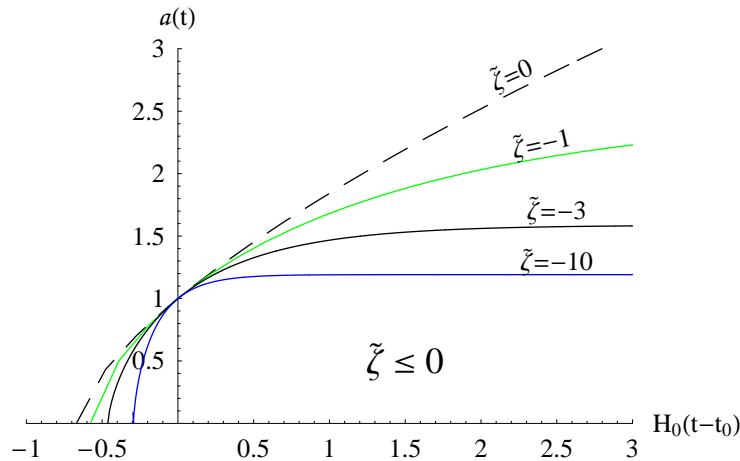


Figure 6. Plot of $a(t, \tilde{\zeta})$ for different values of $\tilde{\zeta}$ in the range $\tilde{\zeta} \leq 0$ (see expressions (16) and (22)). H_0 is the Hubble constant, t is the cosmic time and t_0 is its present value. The long dashed line corresponds to $\tilde{\zeta} = 0$ (a flat matter-dominated universe with null bulk viscosity, see expression (16)).

4.2.4. *Case $\tilde{\zeta} < 0$.* This case predicts an *eternal decelerated expanding* universe. The universe begins with a Big-Bang and expands forever until to reach its *maximum* value $a^* = (1 - 3/\tilde{\zeta})^{2/3}$ when $t \rightarrow \infty$, becoming an Einstein static universe in the future. The Big-Bang took place in the cosmic time t_B defined by expression (26). Figures 2–4 and 6 show the behavior of the scale factor for this case.

In this case the curvature scalar has a *transition* from positive to negative values at $a_0 \equiv [(3 - \tilde{\zeta})/4\tilde{\zeta}]^{2/3}$. When $a \rightarrow 0$ then $R \rightarrow \infty$ (the Big-Bang) and when $a = a^*$ the curvature scalar is zero. The minimum value of the curvature scalar is *negative* and it is reached at the value of the scale factor $\tilde{a} \equiv [(2 - 6/\tilde{\zeta})/5]^{2/3}$. See section 5.2 and figure 8 for details.

The matter density is always a decreasing function over its evolution starting from infinity (when $a \rightarrow 0$) to zero (when the scale factor is a^*). See section 5.3 and figure 9 for details.

5. Cosmological parameters.

5.1. Deceleration parameter q .

We study the behavior of the *deceleration parameter* q function that is defined as

$$q(a) \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{a} \frac{1}{H^2} \quad (32)$$

The term \ddot{a}/a can be calculated from the second Friedmann equation, that for a matter-dominated universe with bulk viscosity reads:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_m - 9\zeta H) \quad (33)$$

From the definition of $\tilde{\zeta}$ (see expression in (12)) we have that

$$\zeta = \left(\frac{H_0}{24\pi G} \right) \tilde{\zeta} \quad (34)$$

On the other hand, from the first Friedmann equation (7) we have

$$\rho_m = \frac{3}{8\pi G} H^2(a) \quad (35)$$

So, substituting the expressions (34) and (35) in (33) we obtain:

$$\frac{\ddot{a}}{a} = \frac{1}{2} \left(\tilde{\zeta} H_0 - H(a) \right) H(a) \quad (36)$$

Thus, substituting (36) in (32) yields

$$q(a) = \frac{1}{2} \left(1 - \tilde{\zeta} \frac{H_0}{H(a)} \right)$$

Using equation (14) for $H(a)$ we arrive to

$$q(a, \tilde{\zeta}) = \frac{1}{2} \left[\frac{3 - \tilde{\zeta}(1 + 2a^{3/2})}{3 - \tilde{\zeta}(1 - a^{3/2})} \right] \quad (37)$$

Figure 7 shows the behavior of expression (37). Note that this expression does not depend on H_0 . We summarize the behavior of the deceleration parameter according to different values of $\tilde{\zeta}$ as follows:

- When $\tilde{\zeta} = 0$ we have $q = \frac{1}{2}$ that corresponds to a matter-dominated universe with null bulk viscosity.
- When $\tilde{\zeta} = 3$ we have $q = -1$ that corresponds to de Sitter universe.
- Case $0 < \tilde{\zeta} < 3$: It is always a decreasing function from $q(0) = \frac{1}{2}$ to $q(\infty) = -1$ with a transition from positive to negative values in the value of the scale factor $a_t \equiv [(3 - \tilde{\zeta})/2\tilde{\zeta}]^{2/3}$ (see expression (28)).
- Case $\tilde{\zeta} > 3$: When $a \rightarrow a^*$ then $q \rightarrow -\infty$ where $a^* = (1 - 3/\tilde{\zeta})^{2/3}$ is the *minimum* value of the scale factor (see expression (31)) and when $a \rightarrow \infty$ then $q \rightarrow -1$. It is a negative and increasing function but it never becomes positive, its maximum value is -1.
- Case $\tilde{\zeta} < 0$: When $a = 0$ then $q = \frac{1}{2}$ and when $a \rightarrow a^*$ then $q \rightarrow \infty$ where $a^* = (1 - 3/\tilde{\zeta})^{2/3}$ is the *maximum* value of the scale factor. It is a positive and increasing monotonic function where its minimum value is $\frac{1}{2}$.

From the expression (37) we obtain the deceleration parameter evaluated *today* as

$$q(a = 1, \tilde{\zeta}) = \frac{1 - \tilde{\zeta}}{2} \quad (38)$$

Note that when $\tilde{\zeta} = 1$ the transition from the decelerated to accelerated epochs of the universe takes place *today*. For $\tilde{\zeta} < 1$ we have a *decelerated* universe in the present and for $\tilde{\zeta} > 1$ we have an *accelerated* one today.

Assuming the best estimated values for $\tilde{\zeta}$ from table 1, the deceleration parameter *today* is (see expression (38)): $q_{\text{today}} = -0.46 \pm 0.04$.

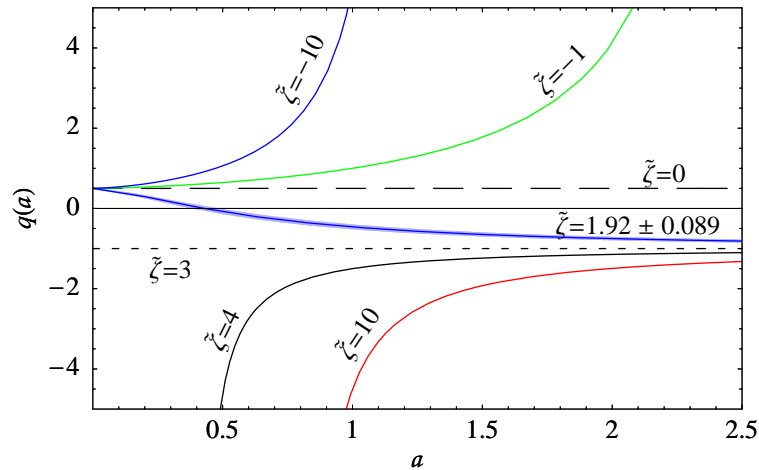


Figure 7. Plot of deceleration parameter $q(a, \tilde{\zeta})$ for different values of $\tilde{\zeta}$ (see expression (37)). The long dashed line corresponds to $\tilde{\zeta} = 0$ (a flat matter-dominated universe with null bulk viscosity). The short dashed line corresponds to $\tilde{\zeta} = 3$ (the *de Sitter* universe). The blue line with the blue band correspond to a model with $\tilde{\zeta} = 1.922 \pm 0.089$. This is the best estimated value of $\tilde{\zeta}$ coming from the SCP “Union” SNe Ia data set analysis (see section 7). The band corresponds to the error at the 68.3% confidence level.

5.2. The curvature scalar R .

We calculate the curvature scalar R in order to study the singularities of the model. For a flat universe it is defined as

$$R = 6 \left[\frac{\ddot{a}}{a} + H^2 \right] \quad (39)$$

Using the expression (36) in (39) we obtain

$$R = 3 \left[\tilde{\zeta} H_0 + 3H(a) \right] H(a) \quad (40)$$

Substituting the expression (14) for $H(a)$ into (40) yields

$$R(a, \tilde{\zeta}) = \frac{H_0^2}{3} \left[\frac{(3 - \tilde{\zeta})^2}{a^3} + \frac{5\tilde{\zeta}(3 - \tilde{\zeta})}{a^{3/2}} + 4\tilde{\zeta}^2 \right] \quad (41)$$

Figure 8 shows some plots of expression (41). From this expression we note that:

- When $\tilde{\zeta} = 0$ then $R = 3H_0^2/a^3$ that corresponds to a matter-dominated universe with null bulk viscosity.
- When $\tilde{\zeta} = 3$ then $R = 12H_0^2$ that corresponds to de Sitter universe.
- Case $0 < \tilde{\zeta} < 3$: When $a \rightarrow 0$ then $R \rightarrow \infty$ (the Big-Bang singularity). It is *always* a positive and decreasing function until to reach its minimum value $R = \frac{4}{3}H_0^2\tilde{\zeta}$ when $a \rightarrow \infty$.
- Case $\tilde{\zeta} > 3$: It is *always* a positive and increasing function from zero at $a = a^*$ (where $a^* = (1 - 3/\tilde{\zeta})^{2/3}$ is the minimum value of the scale factor) until to reach its maximum value $R = \frac{4}{3}H_0^2\tilde{\zeta}$ when $a \rightarrow \infty$.

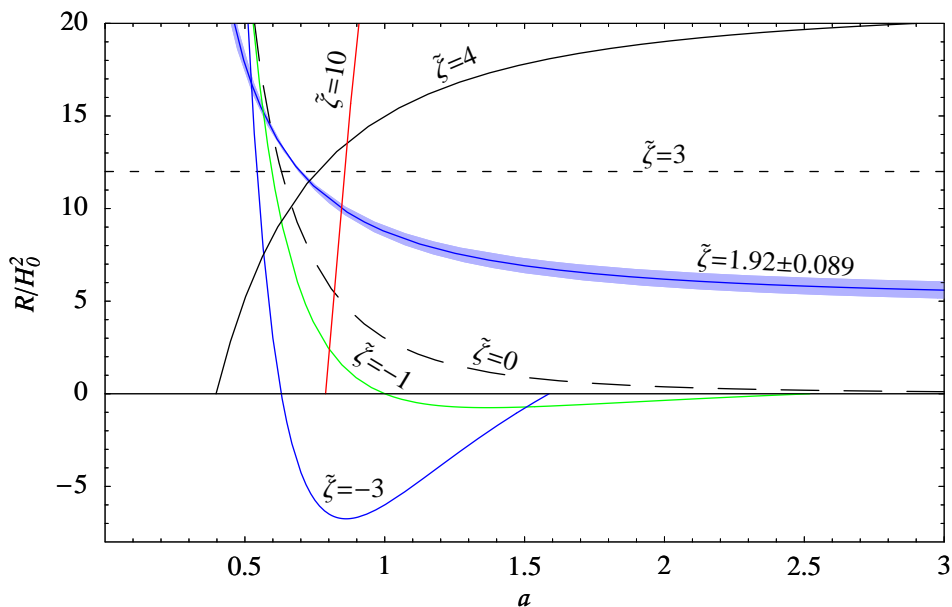


Figure 8. Plot of the curvature scalar $R(a, \tilde{\zeta})$ as a function of the scale factor for different values of $\tilde{\zeta}$ (see expression (41)). The long dashed line corresponds to $\tilde{\zeta} = 0$ (a flat matter-dominated universe with null bulk viscosity). The short dashed line corresponds to $\tilde{\zeta} = 3$ (the *de Sitter* universe). The blue line with the blue band correspond to a model with $\tilde{\zeta} = 1.922 \pm 0.089$. This is the best estimated value of $\tilde{\zeta}$ coming from the SCP “Union” SNe Ia data set analysis (see section 7). The band corresponds to the error at the 68.3% confidence level.

- Case $\tilde{\zeta} < 0$: When $a \rightarrow 0$ then $R \rightarrow \infty$ (the Big-Bang singularity). The curvature scalar is zero at the values of the scale factor a_0 and a^* , where

$$a_0 = \left(\frac{\tilde{\zeta} - 3}{4\tilde{\zeta}} \right)^{2/3} \quad (42)$$

and a^* is the *maximum* value for the scale factor defined in expression (31). At a_0 there is a *transition* from positive to negative values of R . When $0 < a < a_0$ the curvature scalar is a positive and decreasing function. For $a_0 < a < a^*$ the value of R is *negative*. The minimum value of the curvature scalar is $R = -\frac{3}{4}H_0^2\tilde{\zeta}^2$ at the value of the scale factor \tilde{a}

$$\tilde{a} = \left[\frac{2}{5} \left(\frac{\tilde{\zeta} - 3}{\tilde{\zeta}} \right) \right]^{2/3} \quad (43)$$

5.3. The matter density ρ_m .

We calculate the matter density ρ_m predicted for this model. We substitute the Hubble parameter $H(a)$ (expression (14)) into the expression (35) for the matter density, yielding

$$\rho_m(a, \tilde{\zeta}) = \frac{H_0^2}{24\pi G} \left[\tilde{\zeta} + \frac{3 - \tilde{\zeta}}{a^{3/2}} \right]^2 \quad (44)$$

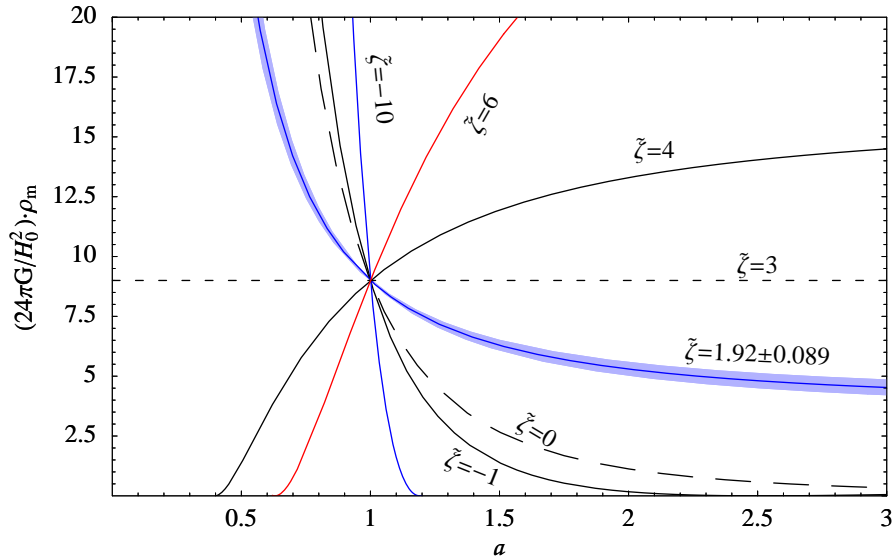


Figure 9. Plot of the matter density $\rho_m(a, \tilde{\zeta})$ as a function of the scale factor for different values of $\tilde{\zeta}$ (see expression (44)). The long dashed line corresponds to $\tilde{\zeta} = 0$ (a flat matter-dominated universe with null bulk viscosity). The short dashed line corresponds to $\tilde{\zeta} = 3$ (the *de Sitter* universe). The blue line with the blue band correspond to a model with $\tilde{\zeta} = 1.922 \pm 0.089$. This is the best estimated value of $\tilde{\zeta}$ coming from the SCP “Union” SNe Ia data set analysis (see section 7). The band corresponds to the error at the 68.3% confidence level.

Figure 9 shows some plots of expression (44). From this expression we note that:

- When $\tilde{\zeta} = 0$ then $\rho_m = (3H_0^2/8\pi G) a^{-3}$ that corresponds to a matter-dominated universe with null bulk viscosity.
- When $\tilde{\zeta} = 3$ then $\rho_m = 3H_0^2/8\pi G$ that corresponds to de Sitter universe.
- Case $0 < \tilde{\zeta} < 3$: When $a \rightarrow 0$ then $\rho_m \rightarrow \infty$ (the Big-Bang singularity). It is a decreasing function until to reach its *minimum* value $\rho_m = (H_0^2/24\pi G) \tilde{\zeta}^2$ when $a \rightarrow \infty$.
- Case $\tilde{\zeta} > 3$: When $a = a^*$ then $\rho_m(a^*) = 0$ where $a^* = (1 - 3/\tilde{\zeta})^{2/3}$ is the *minimum* value of the scale factor (see expression (31) and section 4.2.3). It is an increasing monotonic function from zero until to reach its *maximum* value of $\rho_m = (H_0^2/24\pi G) \tilde{\zeta}^2$ when $a \rightarrow \infty$. In this case, the matter density increases as in the Hoyle’s steady state cosmology [84].
- Case $\tilde{\zeta} < 0$: When $a \rightarrow 0$ then $\rho_m \rightarrow \infty$ (the Big-Bang singularity). It is a decreasing function until to reach its *minimum* value in $\rho_m(a^*) = 0$.

5.4. The age of the universe.

We define the *age* of the universe as the elapsed time between the Big-Bang time t_B until today t_0 . So, from the expression (19) for the case $\tilde{\zeta} = 0$ we have

$$\text{Age} \equiv |t_B - t_0| = \frac{2}{3H_0}, \quad \text{for } \tilde{\zeta} = 0 \quad (45)$$

And from the expression (26) for the case $\tilde{\zeta} < 3$

$$\text{Age} \equiv |t_B - t_0| = -\frac{2}{\tilde{\zeta}H_0} \ln \left(1 - \frac{\tilde{\zeta}}{3} \right), \quad \text{for } \tilde{\zeta} < 3 \quad (46)$$

Note that for the case $\tilde{\zeta} \geq 3$ the age of the universe *is not defined*. On the other hand, evaluating the expression (46) in the best estimated values of $\tilde{\zeta}$ and H_0 from the SCP ‘‘Union’’ SNe Ia data set (see table 1) yields an age of 14.957 ± 0.422 Gyr. The errors are at 68.3% confidence level. We found that this age is in perfect agreement with the constraints on the age of the universe coming from the oldest globular clusters [85]. Figure 10 shows the age of the universe for different values of $\tilde{\zeta}$ and the best estimate, the vertical lines correspond to $H_0 = [55, 75]$ (km/s)Mpc⁻¹ §, it is the permitted region according to values of H_0 consistent with the distance moduli used to derive ages for Galactic globular clusters from the *Hipparcos* parallaxes. As a reference, the predicted age of the universe for Λ CDM (Lambda Cold Dark Matter) cosmological model with a flat universe and using the best estimated values of Ω_{m0} and H_0 shown in table 1 is 13.750 ± 0.29 Gyr.

6. Thermodynamics and the local entropy.

The law of generation of the *local* entropy in the FRW space-time is found to be [70, 71, 72]

$$T \nabla_\nu s^\nu = \zeta \nabla_\nu u^\nu = 3H\zeta \quad (47)$$

where T is the temperature and $\nabla_\nu s^\nu$ is the rate at which entropy is being generated in an unit volume. Then, the second law of the thermodynamics can be written as

$$T \nabla_\nu s^\nu \geq 0 \quad (48)$$

which, from the expression (47), it implies that $3H\zeta \geq 0$.

Since the Hubble parameter H is positive in an expanding universe then ζ has to be positive in order to preserve the validity of the second law of the thermodynamics. Thus, equation (48) can be written for this model as

$$\tilde{\zeta} \geq 0 \quad (49)$$

§ Hereinafter the units of H_0 are expressed in (km/s)Mpc⁻¹

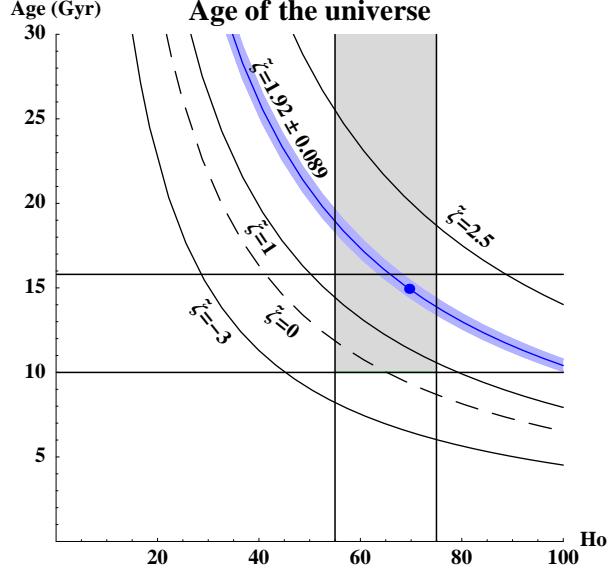


Figure 10. Plot of the age of the universe in units of Gigayears (Gyr) with respect to H_0 in units of $(\text{km/s})\text{Mpc}^{-1}$, for different values of $\tilde{\zeta}$ (see expression (45), (46)). The blue point is located at 14.957 ± 0.422 Gyr that corresponds to the best estimated value for the age of the universe coming from the SCP “Union” SNe Ia data set (where $H_0 = 69.62 \pm 0.59(\text{km/s})\text{Mpc}^{-1}$, see section 7). The blue line with the blue band correspond to a model evaluated at the best estimated value for $\tilde{\zeta} = 1.922 \pm 0.089$. The vertical lines correspond to the interval $H_0 = [55, 75]$ $(\text{km/s})\text{Mpc}^{-1}$, it is the permitted region according to values of H_0 consistent with the distance moduli used to derive ages for Galactic globular clusters from the *Hipparcos* parallaxes (see ref. [85]). The horizontal lines corresponds to the constraint for the age of the universe from the oldest globular clusters (Age= 12.9 ± 2.9 Gyr [85]). So, the shaded area is the consistent region for the age of the universe. The dashed line corresponds to $\tilde{\zeta} = 0$, a flat matter-dominated universe with null bulk viscosity.

7. Type Ia Supernovae test.

We analyze and constrain the viability of the model using the type Ia supernovae (SNe Ia) observations and the entropy tests (see expression (49) and section 6). For that, we calculate the *best estimated values* for the parameters $\tilde{\zeta}$ and H_0 and the *goodness-of-fit* of the model to the data by χ^2 -minimization and then compute the confidence intervals for $(\tilde{\zeta}, H_0)$ and the probability density functions for $\tilde{\zeta}$ (marginalizing over H_0) to constrain their possible values.

We perform the statistical analysis using the “Union” SNe Ia data set of “The Supernova Cosmology Project” (SCP) [73] composed by 307 type Ia supernovae brought together from 13 independent data sets.

We use the definition of luminosity distance d_L [3, 5], [86]–[88] in a flat cosmology, it is

$$d_L(z, \tilde{\zeta}, H_0) = c(1+z) \int_0^z \frac{dz'}{H(z', \tilde{\zeta}, H_0)} \quad (50)$$

where $H(z, \tilde{\zeta}, H_0)$ is the Hubble parameter (expression (13)) and ‘ c ’ is the speed of light. The *theoretical distance moduli* for the k -th supernova with redshift z_k is defined as

$$\mu^t(z_k, \tilde{\zeta}, H_0) \equiv m - M = 5 \log_{10} \left[\frac{d_L(z_k, \tilde{\zeta}, H_0)}{\text{Mpc}} \right] + 25 \quad (51)$$

where m and M are the apparent and absolute magnitudes of the SNe Ia respectively and the superscript ‘ t ’ stands for *theoretical*. We construct the statistical χ^2 function as

$$\chi^2(\tilde{\zeta}, H_0) \equiv \sum_{k=1}^n \frac{[\mu^t(z_k, \tilde{\zeta}, H_0) - \mu_k]^2}{\sigma_k^2} \quad (52)$$

where μ_k is the *observational* distance moduli for the k -th supernova, σ_k^2 is the variance of the measurement and n is the amount of supernovae in the data set.

Once constructed the χ^2 function (52), we numerically minimize it to compute the “*best estimates*” for the free parameters of the model (i.e., $\tilde{\zeta}, H_0$). The χ^2 function measures the goodness-of-fit of the model to data. The *probability density function* (**pdf**) is defined as

$$\text{pdf}(\tilde{\zeta}, H_0) = cte \cdot e^{-\chi^2/2} \quad (53)$$

where ‘ cte ’ is a normalization constant.

So, we minimize the χ^2 function to obtain the results shown in table 1. The confidence intervals for $\tilde{\zeta}$ and H_0 are shown in figure 11. Note that at least at the 99.73% confidence level the possible values for $\tilde{\zeta}$ are inside of the interval $1 < \tilde{\zeta} < 3$. The models in this interval indicate the existence of a Big-Bang in the past of the universe, they have an accelerated expansion today and predict an age of the universe $\sim 13 - 18$ Gyr at the 99.73 % confidence level (see figure 11 and subsections 4.2.1, 5.4).

In order to compare the results shown in table 1 (in particular the measurement of the goodness-of-fit to data $\chi_{\text{d.o.f.}}^2$ and the best estimated value for H_0) with other models we compute also the best estimated values for the Λ CDM model using the same SNe Ia data set. For the Λ CDM model, we assume a flat cosmology with H_0 and Ω_{m0} as free parameters (implying that the cosmological constant density parameter is $\Omega_{\Lambda0} = 1 - \Omega_{m0}$). The results are shown in Table 1.

We find that the best estimated values for H_0 and the goodness-of-fit to data $\chi_{\text{d.o.f.}}^2$ from Λ CDM model are very similar to those obtained from the bulk viscous model using the same SCP “Union” SNe Ia data set and in agreement with that reported by 5 year WMAP ($H_0 = 70.5 \pm 1.3$ (km/s)Mpc $^{-1}$, see [83]).

8. Marginalization over H_0

Now, we construct a **pdf** for the bulk viscous coefficient $\tilde{\zeta}$ marginalizing over the Hubble constant H_0 in order to have to $\tilde{\zeta}$ as the only free parameter of the model. We use three different priors to marginalize H_0 :

- (i) *Constant* prior over H_0 (see Appendix A).

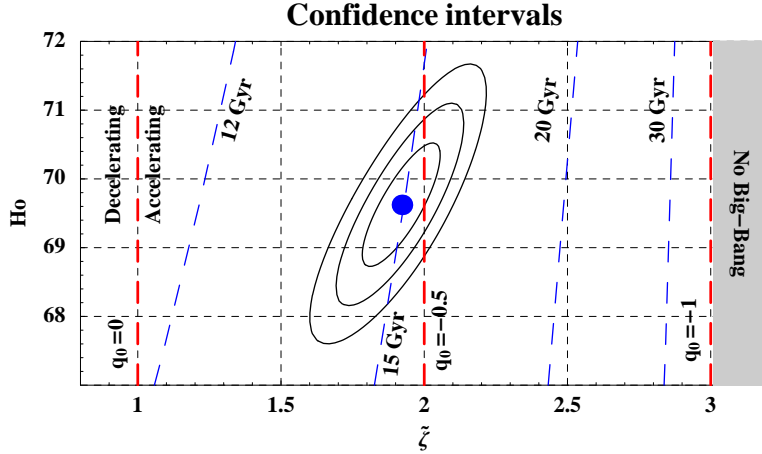


Figure 11. Confidence intervals for $\tilde{\zeta}$ and H_0 of a bulk viscous matter-dominated universe model with constant bulk viscosity. The best estimated values and confidence intervals were computed using the SCP “Union” SNe Ia data set where it is found $\tilde{\zeta} = 1.922 \pm 0.089$ and $H_0 = 69.62 \pm 0.59$ (km/s)Mpc $^{-1}$ as best estimates (see table 1). The shown confidence intervals correspond to 68.3%, 95.4% and 99.73% of probability for the estimated value. H_0 is in units of (km/s)Mpc $^{-1}$ and $\tilde{\zeta}$ is dimensionless. Note that at least at the 99.73% confidence level the possible values for $\tilde{\zeta}$ are inside of the interval $1 < \tilde{\zeta} < 3$ indicating an accelerating universe *today* with high confidence level (see subsection 4.2.1). The blue dashed lines show different ages of the universe according to the values of $\tilde{\zeta}$ and H_0 (see expression (46)). The red vertical dashed lines indicate values of the deceleration parameter evaluated *today* (see expression (38)) being the line $q_0 = 0$ the limit between a decelerated–accelerated universe *today*. The estimated age of the universe is 14.957 ± 0.422 Gyr, where the error is at 68.3% confidence level. We found that this age is in perfect agreement with the constraints on the age of the universe coming from the oldest globular clusters [85] (see figure 10). The models that are in the grey region ($\tilde{\zeta} \geq 3$) do not have a Big-Bang in the past of the universe as the label indicates.

- (ii) *Gaussian* prior over H_0 centered at $H_0 = 70.5 \pm 1.3$ (km/s)Mpc $^{-1}$ and $H_0 = 72 \pm 8$ (km/s)Mpc $^{-1}$ as reported by 5 year WMAP data [83] and HST Cepheid variable star observations [89] respectively (see Appendix B).
- (iii) *Dirac delta* prior over H_0 centered at $H_0 = 70.5$ (km/s)Mpc $^{-1}$ and $H_0 = 72$ (km/s)Mpc $^{-1}$ reported as mentioned above (see Appendix C).

8.1. Marginalization assuming a constant prior over H_0 .

We marginalize over H_0 by assuming a *constant* prior as described in Appendix A. We use the χ_{cp}^2 function (A.18) instead of χ^2 function (52) to perform the supernova analysis. The results are:

- $\tilde{\zeta} = 1.922 \pm 0.089$, with a $\chi_{\text{min}}^2 = 331.30$ ($\chi_{\text{d.o.f.}}^2 = 1.08$).

The error is at 68.3% confidence level. We plot the **pdf** of $\tilde{\zeta}$ to show the value that maximizes such a function, i.e., the best estimated value of $\tilde{\zeta}$. This result is shown in

Viscous model $\zeta = \text{constant}$					
Model	H_0	$\tilde{\zeta}$	Ω_{m0}	χ^2_{\min}	$\chi^2_{\text{d.o.f.}}$
Constant bulk viscosity	69.62 ± 0.59	1.922 ± 0.089	1	314.57	1.031
Λ CDM	70.01 ± 0.59	—	0.278 ± 0.027	311.84	1.022

Table 1. Summary of the best estimates of $\tilde{\zeta}$ and H_0 for the bulk viscous matter-dominated universe model with a constant bulk viscosity coefficient. Also, it is shown the best estimates for the Λ CDM model in order to compare the values of H_0 and χ^2_{\min} of both models. It is found that the best estimates for H_0 and their corresponding χ^2_{\min} computed for both models and using the same SNe Ia data set are almost the same value and in agreement with the reported value by 5 year WMAP ($H_0 = 70.5 \pm 1.3$ (km/s)Mpc $^{-1}$, see [83]). For the Λ CDM model, we assumed a flat cosmology with H_0 and Ω_{m0} as free parameters, where Ω_m is the matter density parameter. Note that in the constant bulk viscous model the value of Ω_m is not estimated but it is assumed to have the value of one as part of the model. The best estimates were computed by a Bayesian statistical analysis using the SCP “Union” 2008 compilation data set composed by 307 SNe Ia from 13 independent data sets [73]. H_0 is in units of (km/s)Mpc $^{-1}$ and $\tilde{\zeta}$, Ω_m are dimensionless. The subscript “d.o.f.” stands for *degrees of freedom* and the errors are at 68.3% confidence level. Figure 11 shows the confidence intervals.

Viscous model $\zeta = \text{constant}$			
Best estimates for $\tilde{\zeta}$			
<i>Marg.</i>	$\tilde{\zeta}$	χ^2_{\min}	$\chi^2_{\text{d.o.f.}}$
(i)	1.922 ± 0.089	331.30	1.08
(ii)	1.941 ± 0.084	315.26	1.03
(iii)	1.924 ± 0.089	314.91	1.02
(iv)	2.026 ± 0.051	316.75	1.03
(v)	2.195 ± 0.049	330.41	1.07

Table 2. Summary of the best estimates of $\tilde{\zeta}$ for the bulk viscous matter-dominated universe model with a constant bulk viscosity coefficient. The best estimates were computed by a Bayesian statistical analysis using the SCP “Union” 2008 compilation data set composed by 307 SNe Ia from 13 independent data sets [73]. The errors are at 68.3% confidence level and $\tilde{\zeta}$ is dimensionless. Figure 12 shows the probability density functions for $\tilde{\zeta}$. We have marginalized over the Hubble constant H_0 using three different priors described in section 8. In the first column of the table the label “*Marg.*” indicates the marginalization method over H_0 used to obtain the results. The numbers correspond to:

- (i) *Constant* prior over H_0 (see Appendix A).
- (ii) *Gaussian* prior over H_0 centered at $H_0 = 70.5 \pm 1.3$ (km/s)Mpc $^{-1}$, as reported by 5 year WMAP data [83] (see Appendix B).
- (iii) *Gaussian* prior over H_0 centered at $H_0 = 72 \pm 8$ (km/s)Mpc $^{-1}$, as reported by HST Cepheid variable star observations [89] (see Appendix B).
- (iv) *Dirac delta* prior over H_0 located at $H_0 = 70.5$ (km/s)Mpc $^{-1}$, as reported by 5 year WMAP data (see Appendix C).
- (v) *Dirac delta* prior over H_0 located at $H_0 = 72$ (km/s)Mpc $^{-1}$, as reported by HST Cepheid variable star observations (see Appendix C).

table 2 and figure 12.

8.2. Marginalization assuming a Gaussian prior over H_0 .

We marginalize over H_0 by assuming a *Gaussian* prior as described in Appendix B. We use two different central values for the Gaussian prior for H_0 , they are $H_0 = 70.5 \pm 1.3$ (km/s)Mpc $^{-1}$ and $H_0 = 72 \pm 8$ (km/s)Mpc $^{-1}$ as suggested by 5 year WMAP data [83] and HST Cepheid variable star observations [89] respectively. See section Appendix B for details. The results are shown in table 2 and figure 12.

8.3. Marginalization assuming a Dirac delta prior over H_0 .

Finally, we marginalize over H_0 by assuming a *Dirac delta* prior as described in Appendix C. In practice, it means to assume a specific value for H_0 . In the same way as above we use two different values for H_0 , they are $H_0 = 70.5$ (km/s)Mpc $^{-1}$ and $H_0 = 72$ (km/s)Mpc $^{-1}$. The results are shown in table 2 and figure 12.

9. Conclusions.

We performed a detailed study of a bulk viscous matter-dominated universe model with constant bulk viscous coefficient ζ . The only component of this model is a pressureless fluid with an *assumed* constant bulk viscosity driving the present accelerated expansion of the universe. We analyzed the different possible scenarios for the behavior and evolution of the universe predicted by the model according to the value of the dimensionless bulk viscous coefficient $\tilde{\zeta}$. In general, the model predicts an *expanding* universe for any value of $\tilde{\zeta}$ and we find that for values of $\tilde{\zeta}$ in the range $0 < \tilde{\zeta} < 3$ the model predicts an universe with a Big-Bang in the past and then starting with a *decelerated* expansion epoch followed by a *transition* to an accelerated epoch in late times.

The case $\tilde{\zeta} = 3$ corresponds to the de Sitter universe. When $\tilde{\zeta} > 3$ the universe is always in an accelerated expansion and the value of the scale factor tends to infinity in the future (in a similar way than the de Sitter universe) and where the curvature scalar goes to a positive constant value. In this case there is not a Big-Bang nor a defined age nor origin of the universe because the scale factor has a minimum value greater than zero in the past of the universe (when $t \rightarrow -\infty$), where also, the curvature scalar and the matter density are zero, i.e., we have the Einstein static universe in the past.

When $\tilde{\zeta} < 0$ the universe is always in a decelerated expansion and the scale factor tends to a finite value in the future (it tends to the Einstein static universe in the future) where also, the curvature scalar and the matter density go to zero. For $\tilde{\zeta}$ in this range there is always a Big-Bang in the past.

For any value of $\tilde{\zeta}$ there *does not* exist any recollapse epoch or transition between expansion to contraction epochs. When $\tilde{\zeta} = 1$ the transition between a decelerated to accelerated epoch takes place *today* (i.e., $q_0 = 0$ for $\tilde{\zeta} = 1$). For $0 < \tilde{\zeta} < 1$ the transition

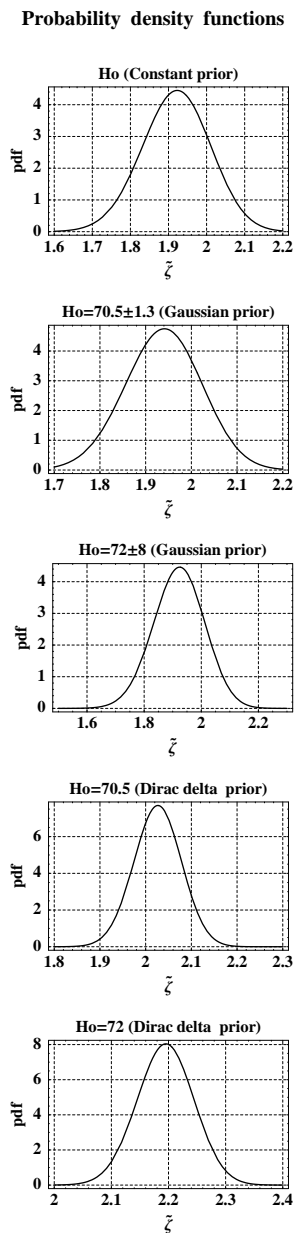


Figure 12. Probability distribution functions for the coefficient $\tilde{\zeta}$ in a bulk viscous matter-dominated universe model with a constant bulk viscosity. We have marginalized over the Hubble constant H_0 using three different priors described in section 8. The best estimates and pdf’s were computed by a Bayesian statistical analysis using the SCP “Union” 2008 compilation data set composed by 307 SNe Ia from 13 independent data sets [73]. Table 2 summarizes the best estimated values of $\tilde{\zeta}$ using the different marginalizations. The first figure (on top) corresponds to the marginalization using a *constant* prior over H_0 (see Appendix A). The second and third figures (from top to bottom) correspond to the marginalization using a *Gaussian* prior over H_0 centered at $H_0 = 70.5 \pm 1.3$ (km/s)Mpc $^{-1}$ and $H_0 = 72 \pm 8$ (km/s)Mpc $^{-1}$ as reported by 5 year WMAP data [83] and HST Cepheid variable star observations [89] respectively. The fourth and fifth figures correspond to the marginalization using a *Dirac delta* prior over H_0 located at $H_0 = 70.5$ (km/s)Mpc $^{-1}$ and $H_0 = 72$ (km/s)Mpc $^{-1}$, as mentioned above.

between the deceleration–acceleration epoch takes place in the future and for $1 < \tilde{\zeta} < 3$ the transition took place in the past.

On the other hand, we estimate the values of $\tilde{\zeta}$ and H_0 using the SCP Union SNe Ia data set where it is found that the estimated values of $\tilde{\zeta}$ are positives, in agreement with the second law of thermodynamics. We note that the best estimated value for H_0 is in agreement with that reported by 5 year WMAP and estimated from Λ CDM model (using the same SNe Ia data set) and the minimum value of the χ^2 function ($\chi_{\text{d.o.f.}}^2$) obtained is of the same order than that obtained for Λ CDM model indicating that this is a competitive model to fit the supernova observations.

We estimate and compare the age of the universe predicted by this model with the constraints on the age coming from the oldest globular cluster observations. We find that the best estimated values for the age of the universe is 14.95 ± 0.42 Gyr that is in agreement with the constraint in the age coming from the oldest globular clusters.

The present model has been constrained using SNe Ia data. It means that we have constrained the model using information (observations) of the past of the universe until a redshift $z \lesssim 1.5$. So, SNe Ia observations are not enough to trace the whole evolution of the universe from the Big-Bang until the present time. Therefore, it would be important to constrain the model using also cosmological observations that can provide information of the early universe like the observations of the CMB anisotropies from the WMAP experiment and the large scale structure (LSS) from SDSS experiment. A preliminary work in that sense has been already done in [56] showing (but not in a conclusive way) that when the model is constrained using the shift parameter R of CMB and the parameter A measuring the BAO peak from SDSS, in addition to the Gold 2006 SNe Ia data set, the estimated value of $\tilde{\zeta}$ is negative at 99.7% confidence level, violating the second law of thermodynamics and the estimated value of H_0 is lower ($H_0 \sim 53(\text{km/s})\text{Mpc}^{-1}$) than that reported by 5 year WMAP and HST Cepheid variable star observations and giving a bad goodness-of-fit to data. Considering these facts we may conjecture that this model does not work well for early times of the universe and that in order to be a viable model the bulk viscosity should be triggered just until late times.

Another problem of the model is to explain the origin of the bulk viscosity from known or new physics. Some proposals in that sense have been explored, for instance in [49] and [50], where it is proposed a mechanism to generate the bulk viscosity by the decay of dark matter particles into relativistic products.

Finally, we explore also the sensitivity of the results to different ways of marginalization over H_0 when we estimate just the coefficient $\tilde{\zeta}$. We find that it is almost negligible the difference in the estimations of $\tilde{\zeta}$ when H_0 is marginalized assuming *constant* and *Gaussian* priors centered in the values reported by 5 year WMAP and HST Cepheid variable star observations. However, when H_0 is marginalized by assuming a specific value (a Dirac delta prior located at the values reported as mentioned above) the estimated values of $\tilde{\zeta}$ have a small increase and the values of $\chi_{\text{d.o.f.}}^2$ increase as well (i.e., we have a worse fit to data).

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Appendix A. Marginalization assuming a constant prior over H_0 .

To construct a **pdf** that depends only on the parameter $\tilde{\zeta}$ we use the process of *marginalization* over H_0 in order to eliminate the dependence of the **pdf** with respect to the parameter H_0 .

The following procedure can be applied for any other model. It is, in a model with several parameters where it is necessary to reduce the number of free parameters or, for some reason, to eliminate the dependence of the model from some particular free parameters (for instance: H_0) in order to compute the best estimated values of the other free parameters that we keep (in our case: $\tilde{\zeta}$) the solution is to *marginalize* over the parameters that we want to eliminate.

In the present work we assume three different prior distribution functions for H_0 : constant, Gaussian and Dirac delta. In this appendix we describe the marginalization using a constant prior for H_0 .

We start by multiplying and dividing by H_0 the equation (50)

$$d_L(z, \tilde{\zeta}, H_0) = \frac{c(1+z)}{H_0} \int_0^z \frac{H_0 dz'}{H(z', \tilde{\zeta}, H_0)} \quad (\text{A.1})$$

Next, we define a new dimensionless luminosity distance as $D_L(z, \tilde{\zeta}) \equiv H_0 \cdot d_L(z, \tilde{\zeta}, H_0)/c$. Then

$$D_L(z, \tilde{\zeta}) = c(1+z) \int_0^z \frac{dz'}{E(z', \tilde{\zeta})} \quad (\text{A.2})$$

where $D_L(z, \tilde{\zeta})$ does not depend on H_0 anymore and $E(z, \tilde{\zeta}) \equiv H(z, \tilde{\zeta}, H_0)/H_0$.

The theoretical distance moduli (51) becomes

$$\mu^t(z, \tilde{\zeta}, H_0) = 5 \log_{10} \left(\frac{D_L(z, \tilde{\zeta}) \cdot c}{H_0 \cdot \text{Mpc}} \right) + 25 \equiv 5 \log_{10} \left(\frac{D_L(z, \tilde{\zeta})}{\tilde{H}_0} \right) + 25 \quad (\text{A.3})$$

where we have defined a dimensionless ‘‘Hubble parameter’’ $\tilde{H}_0 \equiv H_0 \cdot \text{Mpc}/c$. It is useful to define a new theoretical distance moduli that does not depend on H_0 anymore as

$$\tilde{\mu}^t(z, \tilde{\zeta}) \equiv 5 \log_{10}[D_L(z, \tilde{\zeta})] + 25 \quad (\text{A.4})$$

Then, the expression (A.3) for the distance moduli becomes

$$\mu^t(z, \tilde{\zeta}, H_0) = \tilde{\mu}^t(z, \tilde{\zeta}) - 5 \log_{10}[\tilde{H}_0] \quad (\text{A.5})$$

Now we construct the χ^2 function (52) with these new definitions as

$$\chi^2(\tilde{\zeta}, H_0) = \sum_{i=1}^n \left(\frac{\tilde{\mu}^t(z_i, \tilde{\zeta}) - \mu_i^{\text{obs}} - 5 \log_{10} \tilde{H}_0}{\sigma_i} \right)^2 \quad (\text{A.6})$$

where μ_i^{obs} is the observed distance moduli and σ_i its variance.

We rewrite the expression (A.6) as

$$\begin{aligned} \chi^2(\tilde{\zeta}, H_0) &= \sum_{i=1}^n \left(\frac{\tilde{\mu}_i^t - \mu_i^{\text{obs}}}{\sigma_i} \right)^2 - 2 \left(5 \log_{10} \tilde{H}_0 \right) \sum_{i=1}^n \left(\frac{\tilde{\mu}_i^t - \mu_i^{\text{obs}}}{\sigma_i^2} \right) \\ &\quad + \left(5 \log_{10} \tilde{H}_0 \right)^2 \sum_{i=1}^n \left(\frac{1}{\sigma_i^2} \right) \end{aligned} \quad (\text{A.7})$$

If we define ||

$$A \equiv \sum_{i=1}^n \left(\frac{\tilde{\mu}_i^t - \mu_i^{\text{obs}}}{\sigma_i} \right)^2, \quad B \equiv \sum_{i=1}^n \frac{\tilde{\mu}_i^t - \mu_i^{\text{obs}}}{\sigma_i^2}, \quad C \equiv \sum_{i=1}^n \frac{1}{\sigma_i^2} \quad (\text{A.8})$$

then we can express (A.7) as

$$\chi^2(\tilde{\zeta}, H_0) = A - 2Bx + Cx^2 \quad (\text{A.9})$$

where

$$x \equiv 5 \log_{10}(\tilde{H}_0). \quad (\text{A.10})$$

Note that all the dependence of the χ^2 function with respect to H_0 is now in the x variable. Remind that the **pdf** is defined as

$$\mathbf{pdf}(\tilde{\zeta}, \tilde{H}_0) = cte \cdot e^{-\chi^2/2} \quad (\text{A.11})$$

where “*cte*” is a normalization constant. So, we marginalize the **pdf** (A.11) over \tilde{H}_0 computing the following integration

$$\mathbf{pdf}(\tilde{\zeta}) = \int_{-\infty}^{\infty} \mathbf{pdf}(\tilde{\zeta}, \tilde{H}_0) \cdot \mathbf{pdf}(\tilde{H}_0) d\tilde{H}_0 \quad (\text{A.12})$$

where $\mathbf{pdf}(\tilde{H}_0)$ is the *prior* probability density function for \tilde{H}_0 . In general, the integration (A.12) has to be done over the range of all the possible values of the parameter to be marginalized. In this case the range of possible values for H_0 , or equivalently \tilde{H}_0 , is $(-\infty, \infty)$.

If we take the case where the prior for \tilde{H}_0 is a *constant* then:

$$\mathbf{pdf}(\tilde{\zeta}) = cte \int_{-\infty}^{\infty} \mathbf{pdf}(\tilde{\zeta}, \tilde{H}_0) d\tilde{H}_0 \quad (\text{A.13})$$

|| Note that these expressions do not depend on H_0 anymore.

A *constant* prior means that we do not prefer any particular value for \tilde{H}_0 , i.e., any value for \tilde{H}_0 has the same probability of being. To solve the integral (A.13) we do a change of variable from \tilde{H}_0 to x (see expression (A.10)). At the same time, we substitute the expression (A.9) into (A.11) and then (A.11) into (A.13), yielding

$$\mathbf{pdf}(\tilde{\zeta}) = cte \left(\frac{\ln 10}{5} \right) \exp \left[\frac{1}{2} \left(\frac{\tilde{B}^2}{C} - A \right) \right] \int_{-\infty}^{\infty} \exp \left[-\frac{C}{2} \left(x - \frac{\tilde{B}}{C} \right)^2 \right] dx \quad (\text{A.14})$$

where $\tilde{B} \equiv B + (\ln 10)/5$.

We can see that the integral (A.14) has the form of a Gaussian distribution with value:

$$1 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[-\frac{(x - \bar{x})^2}{2\sigma^2} \right] dx \quad (\text{A.15})$$

where \bar{x} is the mean and σ^2 is the variance of the distribution.

Therefore the expression (A.14) becomes

$$\mathbf{pdf}(\tilde{\zeta}) = cte \left(\frac{\ln 10}{5} \right) \sqrt{\frac{2\pi}{C}} \exp \left[-\frac{1}{2} \left(A - \frac{\tilde{B}^2}{C} \right) \right] \quad (\text{A.16})$$

so that the $\mathbf{pdf}(\tilde{\zeta})$ does not depend on H_0 anymore. Note that it was not necessary any *numerical* integration of the expression (A.13). We can express this $\mathbf{pdf}(\tilde{\zeta})$ in terms of a new χ_{cp}^2 function like

$$\mathbf{pdf}(\tilde{\zeta}) = a \cdot e^{-\chi_{\text{cp}}^2/2} \quad (\text{A.17})$$

where $a \equiv cte \cdot \sqrt{2\pi} \ln 10 / (5\sqrt{C})$ and

$$\chi_{\text{cp}}^2(\tilde{\zeta}) \equiv A(\tilde{\zeta}) - \frac{\left[B(\tilde{\zeta}) + \ln(10)/5 \right]^2}{C} \quad (\text{A.18})$$

This new χ_{cp}^2 function does not depend on H_0 anymore. The label ‘‘cp’’ stands for *constant prior* for H_0 .

Appendix B. Marginalization assuming a Gaussian prior over H_0 .

We perform the marginalization assuming a *Gaussian* probability distribution function for H_0 centered at H_0^* and with standard deviation σ^* . So, the $\mathbf{pdf}(H_0)$ prior with the form of a Gaussian distribution for H_0 is

$$\mathbf{pdf}(H_0) = \exp \left[-\frac{1}{2} \left(\frac{H_0 - H_0^*}{\sigma^*} \right)^2 \right] \quad (\text{B.1})$$

With this, the expression (A.12) becomes

$$\mathbf{pdf}(\tilde{\zeta}) = cte \cdot \int_{-\infty}^{\infty} e^{-\chi^2/2} \exp \left[-\frac{1}{2} \left(\frac{H_0 - H_0^*}{\sigma^*} \right)^2 \right] dH_0 \quad (\text{B.2})$$

where the χ^2 function is given by (A.9) and “*cte*” is a normalization constant.

In the present work we use two different central values for the Gaussian prior coming from two different observations. One of them, is that coming from the 5 year WMAP observations where $H_0^* = 70.5 \text{ (km/s)Mpc}^{-1}$ with an standard deviation of $\sigma^* = 1.3 \text{ (km/s)Mpc}^{-1}$. The other central value is $H_0^* = 72 \text{ (km/s)Mpc}^{-1}$ with an standard deviation of $\sigma^* = 8 \text{ (km/s)Mpc}^{-1}$ as reported by the HST Cepheid variable star observations.

In practice we perform the numerical integration of the expression (B.2) in the interval $H_0 = [55, 85] \text{ (km/s)Mpc}^{-1}$ considering that this is a suitable and representative interval that include almost the 100% of the probability density for H_0 .

Appendix C. Marginalization assuming a Dirac delta prior over H_0 .

It means to assume a specific value of H_0 . It is like to think that the value of the Hubble constant is H_0^* that has been measured with an infinite accuracy (with standard deviation equal to zero), so that its **pdf**(H_0) has the form of a *Dirac delta* function.

Clearly, this assumption is just an idealization to simplify the work and it does not correspond to the reality. However, for some cases it is a good approximation. This prior has the advantage of that once assumed it is very simple to perform the integration of the expression (A.12) using the Dirac delta properties. Thus, the prior with the form of a Dirac delta for H_0 is

$$\mathbf{pdf}(H_0) = \delta(H_0 - H_0^*) \tag{C.1}$$

With this, the expression (A.12) becomes

$$\mathbf{pdf}(\tilde{\zeta}) = cte \cdot e^{-\chi^2(\tilde{\zeta}, H_0^*)/2} \tag{C.2}$$

where “*cte*” is a normalization constant.

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