TESTING FOR LOCAL ACTIVITY AND EDGE OF CHAOS

TAO YANG and LEON O. CHUA
Electronics Research Laboratory and
Department of Electrical Engineering and Computer Sciences,
University of California at Berkeley,
Berkeley, California, CA 94720, USA

Received February 24, 2001; Revised April 18, 2001

In this paper we study the local activity, local passivity and edge of chaos of continuous-time reaction–diffusion cellular nonlinear networks (CNN) with one-port first-order, one-port second-order, two-port second-order, two-port third-order and three-port third-order cells. We prove that the local passive regions determined by cell impedance $Z_Q(s)$ and cell admittance $Y_Q(s)$ for first- and second-order cells are equivalent to each other. We also present an efficient procedure to determine the edge-of-chaos parameter region by combining the local active regions derived from $Y_Q(s)$ and the pole locations of $Z_Q(s)$. In order to characterize the fundamental limitations of local passivity on the emergence of complexity we study the local active property from a parameter space spanned by both the cell parameters and the external excitations called the cell's port currents (in view of its interpretations from classical circuit theory). Analytical results of locally passive, restricted locally passive, edge-of-chaos and locally active parameter regions for CNN cells modeled by cubic nonlinearities are presented. We illustrate our results by analyzing CNN cells modeled by Chua’s circuits with a cubic nonlinearity. We find that the morphology of the edge-of-chaos and the local active parameter regions have a close connection to the pattern formation behaviors of CNNs. Simulation results are presented to verify our theoretical results.

1. Introduction

The concept of local activity for reaction–diffusion cellular nonlinear networks (CNN) was first presented in [Chua, 1998]. The principles and applications of local activity and edge of chaos for different classes of continuous-time CNNs had been presented in [Min et al., 2000; Dogaru & Chua, 1998c, 1998b, 1998a; Chua, 1999]. In this paper, we present new analytical results of locally passive and locally active parameter regions for different classes of CNNs. Since the basic concepts and terminologies of CNN had been presented in the book [Chua, 1998], we will only present some basic concepts of CNN to make this paper self-contained.

A general reaction–diffusion CNN made of “m-port” and nth-order cells ($m \leq n$) is defined by

$$
\frac{dV_1(t)}{dt} = f_1(V_1(t), V_2(t), \ldots, V_n(t)) + I_1(t)
$$

$$
\frac{dV_2(t)}{dt} = f_2(V_1(t), V_2(t), \ldots, V_n(t)) + I_2(t)
$$

$$
\vdots
$$

$$
\frac{dV_m(t)}{dt} = f_m(V_1(t), V_2(t), \ldots, V_n(t)) + I_m(t)
$$

$$
\frac{dV_{m+1}(t)}{dt} = f_{m+1}(V_1(t), V_2(t), \ldots, V_n(t))
$$

$$
\vdots
$$

$$
\frac{dV_n(t)}{dt} = f_n(V_1(t), V_2(t), \ldots, V_n(t)).
$$

(1)
For any constant excitation $\mathbf{I}(t) = [\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_m], \mathbf{T}_i \in \mathbb{R}$, the associated Local State Equation at a cell equilibrium point $Q = (\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_m, \mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_m)$ obtained by solving Eq. (1) with $dV_i(t)/dt = 0$, subject to zero initial state $v_i(0) = v_2(0) = \cdots = v_n(0) = 0$, is given by

\begin{align*}
\dot{v}_1 &= a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n + i_1 \\
\dot{v}_2 &= a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n + i_2 \\
&\vdots \\
\dot{v}_m &= a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n + i_m \\
\dot{v}_{m+1} &= a_{m+1,1}v_1 + a_{m+1,2}v_2 + \cdots + a_{m+1,n}v_n \\
&\vdots \\
\dot{v}_n &= a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n 
\end{align*}

(2)

where

\begin{align*}
a_{ij} \triangleq \left. \frac{\partial f_i(V_1, V_2, \ldots, V_n)}{\partial V_j} \right|_Q
\end{align*}

are the local cell coefficients at the cell equilibrium point $Q$. Using vector notations, we can recast Eq. (2) into

\begin{align*}
\dot{\mathbf{v}}_a &= \mathbf{A}_{aa}\mathbf{v}_a + \mathbf{A}_{ab}\mathbf{v}_b + \mathbf{i}_a \\
\dot{\mathbf{v}}_b &= \mathbf{A}_{ba}\mathbf{v}_a + \mathbf{A}_{bb}\mathbf{v}_b 
\end{align*}

(4)

where

\begin{align*}
\mathbf{v}_a &= [v_1, v_2, \ldots, v_m]^T, \\
\mathbf{i}_a &= [i_1, i_2, \ldots, i_m]^T, \\
\mathbf{v}_b &= [v_{m+1}, v_{m+2}, \ldots, v_n]^T,
\end{align*}

and where $\mathbf{A}_{aa}$, $\mathbf{A}_{ab}$, $\mathbf{A}_{ba}$, and $\mathbf{A}_{bb}$ are appropriately partitioned submatrices.

1.1. Local activity criteria for one-port reaction-diffusion CNN cells

A one-port reaction–diffusion CNN cell is locally active at a cell equilibrium point $Q \triangleq (\mathbf{V}_1, \mathbf{T}_1)$ if, and only if, its cell impedance $Z_Q(s)$ at $Q$ satisfies at least one of the following four conditions:

A1. $Z_Q(s)$ has a pole in $\text{Re}[s] > 0$.

A2. $\text{Re}[Z_Q(\omega)] = 1/2[Z_Q^*(\omega) + Z_Q(\omega)] < 0$ for some $\omega = \omega_0$, where $\omega_0$ is any real number.

A3. $Z_Q(s)$ has a simple pole $s = i\omega_p$ on the imaginary axis, where its associated residue

\begin{align*}
k_{-1} \triangleq \begin{cases} 
\lim_{s \to i\omega_p} (s - i\omega_p)Z_Q(s), & \text{if } \omega_p < \infty, \\
\lim_{\omega_p \to \infty} \frac{Z_Q(i\omega_p)}{i\omega_p}, & \text{if } \omega_p = \infty.
\end{cases}
\end{align*}

(5)

A4. $Z_Q(s)$ has a multiple pole on the imaginary axis.

1.2. Local passivity criteria for one-port reaction-diffusion CNN cells

A one-port Reaction–Diffusion CNN cell is locally passive at a cell equilibrium point $Q \triangleq (\mathbf{V}_1, \mathbf{T}_1)$ if, and only if, its cell impedance $Z_Q(s)$ at $Q$ satisfies the following conditions:

P1. $Z_Q(s)$ has no poles in $\text{Re}[s] > 0$.

P2. $\text{Re}[Z_Q(\omega)] = 1/2[Z_Q^*(\omega) + Z_Q(\omega)] \geq 0$ for all real $\omega$ where $s = i\omega$ is not a pole of $Z_Q(s)$.

P3. If $s = s_p$ is a pole of $Z(s)$ on the imaginary axis, i.e. $s_p = i\omega_p$, then $s_p$ must be a simple pole and its residue must be a positive real number.

1.3. Local activity criteria for two-port reaction-diffusion CNN cells

A two-port Reaction–Diffusion CNN cell is locally active at a cell equilibrium point $Q \triangleq (\mathbf{V}_1, \mathbf{V}_2, \mathbf{T}_1, \mathbf{T}_2)$ if, and only if, its cell impedance matrix $Z_Q(s)$ at $Q$ satisfies at least one of the following four conditions:

A1. $Z_Q(s)$ has a pole in $\text{Re}[s] > 0$.

A2. $Z_Q^H(i\omega) \triangleq Z_Q^*(i\omega) + Z_Q(i\omega)$ is not a positive semi-definite matrix at some $\omega = \omega_0$, where $\omega_0$ is any real number; i.e., there exist some $\omega = \omega_0$ and some constant (real or complex) vector $\mathbf{i} = [i_1, i_2]$ such that the real number

\begin{align*}
\mathbf{i}^TZ_Q^H(i\omega)\mathbf{i} < 0
\end{align*}


1We use the superscripts “H” and “T” to denote the “Hermitian” and “transpose” operators, respectively.
A3. $Z_Q(s)$ has a simple pole $s = i\omega_p$ on the imaginary axis where its associated residue matrix

$$K_{-1} \triangleq \begin{cases} 
\lim_{s \to i\omega_p} (s - i\omega_p)Z_Q(s), & \text{if } \omega_p < \infty \\
\lim_{\omega_p \to \infty} Z(i\omega_p), & \text{if } \omega_p = \infty 
\end{cases}$$

(6)

is either not a Hermitian matrix, or else not a positive semi-definite matrix.

A4. $Z_Q(s)$ has a multiple pole on the imaginary axis.

1.4. Local passivity criteria for two-port reaction-diffusion CNN cells

A two-port Reaction–Diffusion CNN cell is locally passive at a cell equilibrium point $Q = (\overline{v}_1, \overline{v}_2, \overline{I}_1, \overline{I}_2)$ if, and only if, its cell impedance matrix $Z_Q(s)$ at $Q$ satisfies the following conditions:

P1. $Z_Q(s)$ has no pole in $\text{Re}[s] > 0$.

P2. $Z_Q^H(i\omega) \triangleq Z_Q^H(i\omega) + Z_Q(i\omega)$ is positive semi-definite for all real $\omega$ where $s = i\omega$ is not a pole of $Z_Q(s)$.

P3. If $s = s_p$ is a pole of $Z_Q(s)$ on the imaginary axis, i.e. $s = i\omega_p$, then it must be a simple pole and its associated residue matrix

$$K_{-1} \triangleq \begin{cases} 
\lim_{s \to i\omega_p} (s - i\omega_p)Z_Q(s), & \text{if } \omega_p < \infty \\
\lim_{\omega_p \to \infty} Z(i\omega_p), & \text{if } \omega_p = \infty 
\end{cases}$$

(7)

is a positive semi-definite Hermitian matrix.

1.5. Remarks

Since the local state equation at $Q$ in Eq. (2) depends on the cell equilibrium point $Q = (\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_m, \overline{I}_1, \overline{I}_2, \ldots, \overline{I}_m)$ where the constants $\overline{I}_1 \in \mathbb{R}, \overline{I}_2 \in \mathbb{R}, \ldots, \overline{I}_m \in \mathbb{R}$ are independent parameters, we need to distinguish between the following two terminologies:

1. Local passive cells must be locally passive at every cell equilibrium points with arbitrary $\overline{I}_1 \in \mathbb{R}$, $\overline{I}_2 \in \mathbb{R}, \ldots, \overline{I}_m \in \mathbb{R}$.

2. Restricted local passive cells need only be locally passive at all cell equilibrium points with some proper subset of $\overline{I}_1 \in \mathbb{R}$, $\overline{I}_2 \in \mathbb{R}, \ldots, \overline{I}_m \in \mathbb{R}$.

Remark. It follows from classical circuit theory that to test local passivity or local activity, we can substitute the cell impedance $Z_Q(s)$ by the cell admittance $Y_Q(s) \triangleq 1/Z_Q(s)$ in the above test criteria.

In this paper we use extensively the basic results on Hermitian matrices summarized in the following two theorems [Franklin, 1968].

Theorem 1. A Hermitian matrix is positive definite if, and only if, all its eigenvalues are positive.

Theorem 2. Let $H = (h_{ij})_{n \times n}$ be an $n \times n$ Hermitian matrix. Then $H$ is positive definite if, and only if, all of its determinants are positive; namely,

$$h_{11} > 0, \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} > 0, \ldots, \begin{vmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{vmatrix} > 0. $$

(8)

2. One-Port CNN Cells

In this section we study CNNs made of interconnections of one-port cells. In particular, we will study one-port first-order cells and one-port second-order cells. We will show that the local passive results derived from $Z_Q(s)$ and $Y_Q(s)$ are equivalent to each other for one-port cells studied in this section.

2.1. One-port first-order CNN cells

Let us study first the local passive regions of one-port first-order cells by using $Z_Q(s)$ and $Y_Q(s)$.

Results from $Z_Q(s)$

For one-port first-order cells, the linearized cell state equation at an equilibrium point $Q$ is given by

$$\dot{v}_1 = a_{11}v_1 + i_1$$

(9)

from which we derive the cell impedance

$$Z_Q(s) = \frac{1}{s - a_{11}}. $$

(10)

P1($Z_Q(s)$). Conditions for “$\text{Re}[s_p] \leq 0$”.
The parameter regions satisfying P1 are given by
\[a_{11} \leq 0.\]  
(11)

P2(\(Z_Q(s)\)). Conditions for “\(\text{Re}[\lambda(s)] \geq 0\), for all \(s = \lambda + j\mu\)”.

The parameter regions are given as follows.
\[Z(\lambda) = \frac{1}{\lambda - a_{11}} = \frac{-\lambda - a_{11}}{\lambda^2 + a_{11}^2}.\]  
(12)

Therefore from \(\text{Re}[Z(\lambda)] \geq 0\) we have
\[a_{11} \leq 0.\]  
(13)

When \(a_{11} = 0\), \(s = 0\) is a pole of \(Z(s)\), therefore the parameter region satisfying P2 is given by
\[a_{11} < 0\]  
(14)

which is a subset of P1. This means that the edge-of-chaos region can only possibly be located at \(a_{11} = 0\). However, our next condition shows that the parameter point \(a_{11} = 0\) satisfies P3. Therefore, the edge-of-chaos region is empty for one-port first-order cells.

P3(\(Z_Q(s)\)). Conditions for “\(\lambda_p = \lambda + j\mu\) is a simple pole and has positive residue.”

Only when \(a_{11} = 0\) does \(Z(s)\) have a simple pole on the imaginary axis. In this case, the associated residue is \(K_{-1} = 1 > 0\). It follows that P3 is satisfied when

\[a_{11} = 0.\]

Therefore, the local passive parameter regions are given by \([\text{P1 AND P2]} \text{ OR P3}\); namely,
\[a_{11} \leq 0.\]  
(15)

If follows that for one-port first-order cells, the locally active regions are located at
\[a_{11} > 0.\]  
(16)

There is no edge of chaos region.

Results from \(Y_Q(s)\)

The cell admittance is given by
\[Y_Q(s) = Z_Q^{-1}(s) = s - a_{11}.\]  
(17)

P1(\(Y_Q(s)\)). Conditions for “\(\text{Re}[\lambda_p] \leq 0\)”.

Since there is only one pole; namely, at infinity, the residue is given by
\[\lim_{\omega \rightarrow \infty} \frac{Y_Q(i\omega)}{i\omega} = 1.\]  
(18)

This means that this criterion is satisfied for all parameters.

P2(\(Y_Q(s)\)). Conditions for “\(\text{Re}[\lambda(i\omega)] \geq 0\), for all \(\omega \in \mathbb{R}\) and \(s = i\omega\) is not a pole of \(Z(s)\)”.

The parameter regions are given as follows.
\[Y(i\omega) = i\omega - a_{11}.\]  
(19)

Therefore from \(\text{Re}[Y(i\omega)] \geq 0\) we have
\[a_{11} \leq 0.\]  
(20)

P3(\(Y_Q(s)\)). Conditions for “\(\lambda_p = \lambda + j\mu\) is a simple pole and has positive residue.”

In this case, the residue is given by
\[K_{-1} = \lim_{\omega \rightarrow \infty} \frac{i\omega - a_{11}}{i\omega} = 1 > 0.\]  
(21)

Therefore, this condition is satisfied at the pole at infinity. In this case, it contributes nothing to the passive parameter region.

Therefore, the local passive parameter regions are given by \([\text{P1 AND P2]} \text{ OR P3}\); namely,
\[a_{11} \leq 0.\]  
(22)

From above we can see that the conclusions from \(Z_Q(s)\) and \(Y_Q(s)\) for one-port first-order cells are identical, as expected.

2.2. One-port second-order CNN cells

Many well-known reaction–diffusion CNNs have only one nonzero diffusion coefficient and two state variables. The linearized cell state equation at an equilibrium point \(Q\) is given by
\[\dot{v}_1 = a_{11} v_1 + a_{12} v_2 + i_1,\]  
\[\dot{v}_2 = a_{21} v_1 + a_{22} v_2\]  
(23)

which is a one-port second-order cell.

The CNN cell impedance \(Z_Q(s)\) associated with Eq. (23) is given by
\[Z_Q(s) = \frac{(s - a_{22})}{s^2 - Ts + \Delta}.\]  
(24)

where
\[T \triangleq a_{11} + a_{22}, \quad \Delta \triangleq a_{11} a_{22} - a_{12} a_{21}\]  
(25)

are the trace and determinant, respectively, of the associated Jacobian matrix
\[J_Q = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\]  
(26)
evaluated at the cell equilibrium point \(Q\).
2.2.1. Passivity of one-port second-order CNN Cells

Let us study next the local passivity of one-port second-order cells based on $Z_Q(s)$ and $Y_Q(s)$.

Conclusions from $Z_Q(s)$

Let us find the parameter regions where the properties P1 to P3 are satisfied with respect to $Z_Q(s)$.

P1($Z_Q(s)$). Conditions for “Re[$s_p$] ≤ 0.”

The parameter regions are given by

$$T \leq 0 \quad \text{AND} \quad \Delta \geq 0.$$  \hspace{1cm} (27)

P2($Z_Q(s)$). Conditions for “Re[$Z_Q(i\omega)$] ≥ 0 for all $\omega \in \mathbb{R}$ and $s = i\omega$ is not a pole of $Z(s)$.”

The parameter regions are given as follows:

$$Z_Q(i\omega) = \frac{\omega - a_{22}}{-\omega^2 - iT\omega + \Delta}.$$  \hspace{1cm} (28)

It follows from Re[$Z_Q(i\omega)$] ≥ 0 that

$$-a_{22}\Delta - a_{11}\omega^2 \geq 0,$$  \hspace{1cm} (29)

from which we obtain, for very large or very small $\omega$,

$$a_{11} \leq 0 \quad \text{AND} \quad a_{22}\Delta \leq 0$$  \hspace{1cm} (30)

which is equivalent to

$$T \leq a_{22} \quad \text{AND} \quad a_{22}\Delta \leq 0.$$  \hspace{1cm} (31)

P3($Z_Q(s)$). Conditions for “$Z_Q(s)$ has a simple pole at $s_p = i\omega$ with real and positive residue.”

The parameter regions are given by

$$a_{11} = 0 \quad \text{AND} \quad a_{22} = 0 \quad \text{AND} \quad a_{12}a_{21} < 0.$$  \hspace{1cm} (32)

Conclusions from $Y_Q(s)$

Let us find the parameter regions where the properties P1 to P3 are satisfied with respect to $Y_Q(s)$ which is given by

$$Y_Q(s) = \frac{s^2 - Ts + \Delta}{s - a_{22}}.$$  \hspace{1cm} (33)

P1($Y_Q(s)$). Conditions for “Re[$s_p$] ≤ 0.”

The parameter regions are given by

$$a_{22} \leq 0.$$  \hspace{1cm} (34)

P2($Y_Q(s)$). Conditions for “Re[$Y(i\omega)$] ≥ 0 for all $\omega \in \mathbb{R}$ and $s = i\omega$ is not a pole of $Y_Q(s)$.”

The parameter regions must satisfy

$$\text{Re}[Y_Q(i\omega)] = \text{Re}\left[\frac{-\omega^2 - iT\omega + \Delta}{i\omega - a_{22}}\right]$$

$$= \frac{a_{22}\omega^2 - a_{22}\Delta - T\omega^2}{\omega^2 + a_{22}^2} \geq 0, \quad \forall \omega \in \mathbb{R}$$  \hspace{1cm} (35)

from which we have

$$a_{22}\omega^2 - a_{22}\Delta - T\omega^2 \geq 0, \quad \forall \omega \in \mathbb{R}.$$  \hspace{1cm} (36)

It follows that

$$a_{22}\Delta \leq 0 \quad \text{AND} \quad T \leq a_{22}.$$  \hspace{1cm} (37)

P3($Y_Q(s)$). Conditions for “$Y_Q(s)$ has a simple pole at $s_p = i\omega$ with real and positive residue.”

In this case $a_{22} = 0$ must be satisfied and the parameter regions are given by

$$K_{-1} = \Delta > 0$$  \hspace{1cm} (38)

which leads to

$$a_{22} = 0 \quad \text{AND} \quad \Delta > 0.$$  \hspace{1cm} (39)

Remarks

Let us now prove that the local passive regions derived from $Z_Q(s)$ and $Y_Q(s)$ are identical. Denote the conditions derived with respect to $Y_Q(s)$ from P1, P2 and P3, as P1($Y_Q(s)$), P2($Y_Q(s)$) and P3($Y_Q(s)$), and denote the conditions derived with respect to $Z_Q(s)$ from P1, P2 and P3, as P1($Z_Q(s)$), P2($Z_Q(s)$) and P3($Z_Q(s)$), respectively.

**Theorem 3** [P1($Y_Q(s)$) AND P2($Y_Q(s)$)] ⇔ [P1($Z_Q(s)$) AND P2($Z_Q(s)$)].

**Proof**

1. [P1($Y_Q(s)$) AND P2($Y_Q(s)$)] ⇔ [P1($Z_Q(s)$) AND P2($Z_Q(s)$)] is given by

$$T \leq 0 \quad \text{AND} \quad \Delta \geq 0 \quad \text{AND} \quad T \leq a_{22} \quad \text{AND} \quad a_{22}\Delta \leq 0.$$  \hspace{1cm} (40)
From $\Delta \geq 0$ and $a_{22}\Delta \leq 0$ we have $a_{22} \leq 0$, and Eq. (40) becomes

$$T \leq 0 \quad \text{AND} \quad a_{22} \leq 0 \quad \text{AND} \quad T \leq a_{22} \quad \text{AND} \quad a_{22}\Delta \leq 0.$$  \hfill (41)

Since $T \leq a_{22}$ and $a_{22} \leq 0$ we know $T \leq 0$; therefore Eq. (41) becomes

$$a_{22} \leq 0 \quad \text{AND} \quad T \leq a_{22} \quad \text{AND} \quad a_{22}\Delta \leq 0$$ \hfill (42)

which is exactly the conditions given by $\{P1(Y_{Q}(s)) \quad \text{AND} \quad P2(Y_{Q}(s))\}$.

2. $\{P1(Y_{Q}(s)) \quad \text{AND} \quad P2(Y_{Q}(s))\} \Rightarrow \{P1(Z_{Q}(s)) \quad \text{AND} \quad P2(Z_{Q}(s))\}$: $\{P1(Y_{Q}(s)) \quad \text{AND} \quad P2(Y_{Q}(s))\}$ is given by

$$a_{22} \leq 0 \quad \text{AND} \quad T \leq a_{22} \quad \text{AND} \quad a_{22}\Delta \leq 0.$$ \hfill (43)

Since $a_{22} \leq 0$ and $T \leq a_{22}$, we know $T \leq 0$, and Eq. (43) is equivalent to

$$T \leq 0 \quad \text{AND} \quad a_{22} \leq 0 \quad \text{AND} \quad T \leq a_{22} \quad \text{AND} \quad a_{22}\Delta \leq 0.$$ \hfill (44)

Since $a_{22} \leq 0$ and $a_{22}\Delta \leq 0$, we have $\Delta \geq 0$. It follows that Eq. (44) is equivalent to

$$T \leq 0 \quad \text{AND} \quad \Delta \geq 0 \quad \text{AND} \quad T \leq a_{22} \quad \text{AND} \quad a_{22}\Delta \leq 0.$$ \hfill (45)

which is exactly the conditions given by $\{P1(Z_{Q}(s)) \quad \text{AND} \quad P2(Z_{Q}(s))\}$.

Summarizing the above results, we have

$$\{P1(Y_{Q}(s)) \quad \text{AND} \quad P2(Y_{Q}(s))\} \Leftrightarrow \{P1(Z_{Q}(s)) \quad \text{AND} \quad P2(Z_{Q}(s))\}. \quad \square$$

Remark. This theorem shows that the local passive parameter regions derived from $Y_{Q}(s)$ and $Z_{Q}(s)$ are identical. Furthermore, since the local active regions and local passive regions are complementary sets it follows that the local active regions derived from $Y_{Q}(s)$ and $Z_{Q}(s)$ are also identical. Can we go further and argue that the edge-of-chaos regions derived from $Y_{Q}(s)$ and $Z_{Q}(s)$ are also identical? The answer is NO, as demonstrated in the counter example given in the next section.

2.2.2. Edge-of-chaos parameter regions for one-port second-order CNN cells

The edge-of-chaos region [Chua, 1998] is defined by $\{P1(Z_{Q}(s)) \quad \text{AND} \quad \text{NOT} \quad P2(Z_{Q}(s))\}$. Here we show for this example that

$\{P1(Z_{Q}(s)) \quad \text{AND} \quad \text{NOT} \quad P2(Z_{Q}(s))\}$ and

$\{P1(Y_{Q}(s)) \quad \text{AND} \quad \text{NOT} \quad P2(Y_{Q}(s))\}$ are not equivalent.

$\{P1(Z_{Q}(s)) \quad \text{AND} \quad \text{NOT} \quad P2(Z_{Q}(s))\}$ is given by

$$[T < 0 \quad \text{AND} \quad \Delta > 0] \quad \text{AND} \quad [T > a_{22} \quad \text{OR} \quad a_{22}\Delta > 0]$$ \hfill (46)

and $\{P1(Y_{Q}(s)) \quad \text{AND} \quad \text{NOT} \quad P2(Y_{Q}(s))\}$ is given by

$$[a_{22} < 0] \quad \text{AND} \quad [T > a_{22} \quad \text{OR} \quad a_{22}\Delta > 0]$$ \hfill (47)

1. From Eq. (46) we have

$$[a_{11} + a_{22} < 0 \quad \text{AND} \quad \Delta > 0 \quad \text{AND} \quad a_{11} + a_{22} > a_{22}] \quad \text{OR} \quad [a_{11} + a_{22} < 0 \quad \text{AND} \quad \Delta > 0 \quad \text{AND} \quad a_{22}\Delta > 0]$$ \hfill (48)

Equation (48) implies

$$[a_{11} + a_{22} < 0 \quad \text{AND} \quad \Delta > 0 \quad \text{AND} \quad a_{11} > 0] \quad \text{OR} \quad [a_{11} + a_{22} < 0 \quad \text{AND} \quad \Delta > 0 \quad \text{AND} \quad a_{22} > 0]$$ \hfill (49)
and
\[
[a_{11} + a_{22} < 0 \, \text{AND} \, \Delta > 0 \, \text{AND} \, a_{11} > 0] \, \text{OR} \, [a_{11} + a_{22} < 0 \, \text{AND} \, \Delta > 0 \, \text{AND} \, a_{22} \Delta > 0]
\]
\[
[a_{22} \leq 0 \, \text{AND} \, a_{11} + a_{22} < 0 \, \text{AND} \, \Delta > 0 \, \text{AND} \, a_{11} > 0] \, \text{OR} \, [a_{22} > 0 \, \text{AND} \, a_{11} + a_{22} < 0 \, \text{AND} \, \Delta > 0]
\]
(50)

2. From Eq. (47) we have
\[
[a_{22} < 0 \, \text{AND} \, a_{11} + a_{22} > a_{22} \, \text{OR} \, a_{22} \Delta > 0]
\]
Equation (52) implies
\[
[a_{22} < 0 \, \text{AND} \, a_{11} > 0] \, \text{OR} \, [a_{22} < 0 \, \text{AND} \, a_{22} \Delta > 0]
\]
(53)
Hence,
\[
[a_{22} < 0 \, \text{AND} \, a_{11} > 0] \, \text{OR} \, [a_{22} < 0 \, \text{AND} \, \Delta < 0]
\]
(54)

We conclude that the two set of conditions in Eqs. (51) and (54) are not equivalent. Therefore, the edge-of-chaos regions cannot be found from the conditions derived from \(Y_Q(s)\). Based on the original definition in [Chua, 1998], to find edge-of-chaos region it is necessary to find the poles of \(Z_Q(s)\); namely,

\[
\text{Edge-of-chaos region} = (\text{local active region}) \, \text{AND} \, (\text{region where poles of } Z_Q(s) \text{ are on the left-half } s\text{-plane}).
\]

### 2.2.3. Active parameter regions for one-port second-order CNN cells

Let us find next the parameter regions where the properties A1(\(Z_Q(s)\)) to A4(\(Z_Q(s)\)) are satisfied.

A1(\(Z_Q(s)\)). Conditions for “\(\text{Re}[s_p] > 0\)” are those for “NOT (P1)” and are given by
\[
T > 0 \, \text{OR} \, \Delta < 0.
\]
(55)

A2(\(Z_Q(s)\)). Conditions for “NOT (P2)” are given by
\[
a_{11} > 0 \, \text{OR} \, a_{22} \Delta > 0.
\]
(56)

A3(\(Z_Q(s)\)). Conditions for “\(Z(s)\) has a simple pole at \(s_p = \pm i\omega\) with complex or negative residue” are given by
\[
T = 0 \, \text{AND} \, \Delta > 0 \, \text{AND} \, a_{22} \neq 0.
\]
(57)

A4(\(Z_Q(s)\)). Conditions for “\(Z(s)\) has a multiple pole” are given by
\[
T = 0 \, \text{AND} \, \Delta = 0 \, \text{AND} \, a_{22} \neq 0.
\]
(58)

### 3. Two-Port and Second-Order CNN Cells

The linearized cell state equation at an equilibrium point \(Q\) is given by
\[
\dot{v}_1 = a_{11}v_1 + a_{12}v_2 + i_1,
\]
\[
\dot{v}_2 = a_{21}v_1 + a_{22}v_2 + i_2.
\]
(59)
The cell impedance at an equilibrium point \(Q\) is given by
\[
Z_Q(s) = \frac{s - a_{11}}{s - a_{22}} - \frac{a_{12}}{a_{21}}
\]
\[
\times \left[ \begin{array}{cc}
-1 & s - a_{22} \\
a_{21} & s - a_{11}
\end{array} \right]
\]
\[
= \frac{1}{s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21})}
\]
\[
\times \left[ \begin{array}{cc}
s - a_{22} & a_{12} \\
a_{21} & s - a_{11}
\end{array} \right]
\]
\[
= \frac{1}{s^2 - Ts + \Delta} \left[ \begin{array}{cc}
s - a_{22} & a_{12} \\
a_{21} & s - a_{11}
\end{array} \right].
\]
(60)
In this section we study the local activity and local passivity of two-port and second-order cells by using $Z_Q(s)$ and $Y_Q(s)$. We also prove that the conclusions derived from $Z_Q(s)$ and $Y_Q(s)$ are equivalent.

### 3.1. Passivity of two-port and second-order cells

#### 3.1.1. Conclusions from $Z_Q(s)$

Let us find the parameter regions where properties P1 to P3 are satisfied in terms of $Z_Q(s)$. P1($Z_Q(s)$). Conditions for $\text{Re}[s_p] \leq 0$ are given by

$$T \leq 0 \quad \text{AND} \quad \Delta \geq 0.$$  \hspace{1cm} (61)

P2($Z_Q(s)$). Conditions for "$Z_Q^\dagger(i\omega) + Z_Q(i\omega)$ is positive semi-definite for all $\omega \in \mathbb{R}$ and $s = i\omega$ is not a pole of $Z(s)$" are found as follow:

$$Z^\dagger(i\omega) + Z(i\omega) = \begin{bmatrix}
2\text{Re}\left(\frac{-a_{22} - 2\omega - iT\omega + \Delta}{-\omega^2 - iT\omega + \Delta} + \frac{a_{21}}{-\omega^2 - iT\omega + \Delta} + \frac{a_{12}}{-\omega^2 - iT\omega + \Delta} + \frac{a_{21}}{-\omega^2 + iT\omega + \Delta} + \frac{a_{12}}{-\omega^2 + iT\omega + \Delta}
\end{bmatrix}$$

$$= \begin{bmatrix}
2(-a_{22}(-\omega^2 + \Delta) - T\omega^2)
\end{bmatrix}$$

Since the associated Hermitian matrix

$$\begin{bmatrix}
2(-a_{22}\Delta - a_{11}\omega^2)
\end{bmatrix}$$

must be positive semi-definite, it follows from Theorem 1 that the eigenvalues must be real and non-negative. The eigenvalues of the above matrix are given by the solutions $\lambda_{1,2}$ of the following equation:

$$[\lambda + 2(a_{22}\Delta + a_{11}\omega^2)][\lambda + 2(a_{11}\Delta + a_{22}\omega^2)] - (-\omega^2 + \Delta)^2(a_{12} + a_{21})^2 - T^2\omega^2(a_{12} - a_{21})^2 = 0.$$  \hspace{1cm} (64)

The following three equations are used to organize and simplify the terms from this equation:

$$\lambda^2 + 2[(a_{11} + a_{22})\Delta + (a_{11}\omega^2 + a_{22}\omega^2)]\lambda + 4(a_{22}\Delta + a_{11}\omega^2)(a_{11}\Delta + a_{22}\omega^2)
= (-\omega^2 - \Delta)^2(a_{12} + a_{21})^2 + T^2\omega^2(a_{12} - a_{21})^2 = 0.$$  \hspace{1cm} (65)

Defining

$$U \triangleq 2T(\Delta + \omega^2),$$

$$V \triangleq [4a_{11}a_{22} - (a_{12} + a_{21})^2]\omega^4 + [4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2]\omega^2 + [4a_{11}a_{22} - (a_{12} + a_{21})^2]\Delta^2,$$  \hspace{1cm} (65)
we can recase Eq. (67) as follow:

$$\lambda^2 + U\lambda + V = 0. \quad (69)$$

It follows that \( \lambda_1 \geq 0 \) and \( \lambda_2 \geq 0 \) for all \( \omega \in \mathbb{R} \) if, and only if,

$$U \leq 0 \quad \text{AND} \quad V \geq 0, \quad \text{for all} \quad \omega \in \mathbb{R}. \quad (70)$$

Let us analyze the composite conditions in Eq. (70) as follow:

**Case 1.** \( U \leq 0. \)

From \( U \leq 0 \) for all \( \omega \in \mathbb{R} \) we have

$$T(\Delta + \omega^2) \leq 0 \quad \text{for all} \quad \omega \in \mathbb{R} \quad (71)$$

from which we obtain the simple condition

$$T \leq 0 \quad \text{AND} \quad \Delta \geq 0. \quad (72)$$

**Case 2.** \( V \geq 0. \)

From \( V \geq 0 \) for all \( \omega \in \mathbb{R} \) we have

$$4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0 \quad (73)$$

If \( \omega_{\text{max}}^2 = K \) we have

$$4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0 \quad \text{AND} \quad \frac{4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2}{2[4a_{11}a_{22} - (a_{12} + a_{21})^2]} + [4a_{11}a_{22} - (a_{12} + a_{21})^2]\Delta^2 \geq 0 \quad \text{AND} \quad 4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 < 0. \quad (78)$$

The above results are rather messy. To simplify the results, let us apply the first condition of Theorem 2 and obtain

$$-a_{22}\Delta - a_{11}\omega^2 \geq 0, \quad \text{for all} \quad \omega \in \mathbb{R} \quad (79)$$

It follows that

$$a_{11} \leq 0 \quad \text{AND} \quad a_{22} \Delta \leq 0. \quad (80)$$

Combining the results in Eqs. (78) and (72) we have

$$a_{11} \leq 0 \quad \text{AND} \quad a_{22} \leq 0 \quad \text{AND} \quad \Delta \geq 0. \quad (81)$$

Applying the second condition of Theorem 2, we obtain

$$4(a_{22}\Delta + a_{11}\omega^2)(a_{11}\Delta + a_{22}\omega^2) - (\Delta - \omega^2)^2(a_{12} + a_{21})^2 - (a_{12} - a_{21})^2T^2\omega^2 \geq 0 \quad (82)$$

where the minimum value \( V(\omega_{\text{min}}) \) of \( V(\omega) \) must be non-negative and \( \omega_{\text{min}} \) is given by

$$\omega_{\text{min}}^2 = \begin{cases} 0, & \text{if} \quad K \leq 0, \\ K, & \text{otherwise} \end{cases} \quad (74)$$

where

$$K = -\frac{4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta}{2[4a_{11}a_{22} - (a_{12} + a_{21})^2]} \quad (75)$$

If \( \omega_{\text{min}}^2 = 0 \) we have

$$4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0 \quad \text{AND} \quad [4a_{11}a_{22} - (a_{12} + a_{21})^2]\Delta^2 \geq 0 \quad \text{AND} \quad 4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 \geq 0. \quad (76)$$

which is the same as

$$4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0 \quad \text{AND} \quad 4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 \geq 0. \quad (77)$$

which leads to

$$[4a_{11}a_{22} - (a_{12} + a_{21})^2]\omega^4 + [4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2]\omega^2 + [4a_{11}a_{22} - (a_{12} + a_{21})^2]\Delta^2 \geq 0. \quad (83)$$

Comparing with Eq. (68) we observe that this is exactly the condition \( V \geq 0. \) Therefore this condition offers no new information.

P3(\( Z_Q(s) \)). Conditions for \( Z_Q(s) \) has a simple pole at \( s_p = i\omega \) and its associated residue matrix \( K_{-1} \) is a positive semi-definite Hermitian matrix.” In this case

$$T = 0, \quad \Delta > 0, \quad s_{1,2} = \pm i\sqrt{\Delta}. \quad (84)$$
The residue matrix is given by

\[
\mathbf{K}_{-1} = \lim_{s \to \pm i \sqrt{\Delta}} \frac{1}{s + i \sqrt{\Delta}} \frac{1}{s^2 + \Delta} \begin{bmatrix}
    s - a_{22} & a_{12} \\
    a_{21} & s - a_{11}
\end{bmatrix}
\]

which leads to

\[
\lambda^2 + 2(a_{11} + a_{22})\lambda + 4a_{11}a_{22} - (a_{12} + a_{21})^2 = 0. \tag{91}
\]

Observe that all roots of Eq. (91) are non-negative if, and only if,

\[
2(a_{11} + a_{22}) \leq 0 \quad \text{AND} \quad 4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0. \tag{92}
\]

The above conditions are equivalent to

\[
T \leq 0 \quad \text{AND} \quad 2\Delta + 2a_{11}a_{22} - a_{12}^2 - a_{21}^2 \geq 0. \tag{93}
\]

Applying the first condition of Theorem 2 we obtain

\[
a_{11} \leq 0. \tag{94}
\]

The second condition of Theorem 2 leads to

\[
4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0. \tag{95}
\]

P3(\(\mathbf{Y}_Q(s)\)). Conditions for “\(\mathbf{Y}_Q(s)\) has a simple pole at \(s_p = i\omega\) and its associated residue matrix \(\mathbf{K}_{-1}\) is a positive semi-definite Hermitian matrix”.

In this case,

\[
\mathbf{K}_{-1} = \lim_{\omega \to \infty} \frac{1}{i\omega} \begin{bmatrix}
    i\omega - a_{11} & -a_{12} \\
    -a_{21} & i\omega - a_{22}
\end{bmatrix}
\]

which is a positive semi-definite Hermitian matrix for all parameters. Therefore this criterion provides no additional information.

3.1.2. Conclusions from \(\mathbf{Y}_Q(s)\)

Let us derive next the parameter regions where properties P1 to P3 are satisfied in terms of

\[
\mathbf{Y}_Q(s) = \begin{bmatrix}
    s - a_{11} & -a_{12} \\
    -a_{21} & s - a_{22}
\end{bmatrix}. \tag{87}
\]

P1(\(\mathbf{Y}_Q(s)\)). Conditions for “Re[\(s_p\)] \leq 0.”

Observe that \(\mathbf{Y}_Q(s)\) does not have poles in the finite \(s\)-plane. At the infinite pole we have the residue matrix

\[
\mathbf{K}_{-1} = \lim_{\omega \to \infty} \frac{\mathbf{Y}(i\omega)}{i\omega} = \begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}. \tag{88}
\]

which is positive definite.

P2(\(\mathbf{Y}_Q(s)\)). Conditions for “\(\mathbf{Y}_Q^T(i\omega) + \mathbf{Y}_Q(i\omega)\) is positive semi-definite for all \(\omega \in \mathbb{R}\) and \(s = i\omega\) is not a pole of \(\mathbf{Y}_Q(s)\)” are evaluated as follows:

\[
\mathbf{Y}_Q^T(i\omega) + \mathbf{Y}_Q(i\omega)
\]

Applying Theorem 1, the condition for the above symmetric matrix to be positive semi-definite is that all eigenvalues must be non-negative. The eigenvalues are given by the solutions of the equation:

\[
\begin{vmatrix}
    \lambda + 2a_{11} & a_{12} + a_{21} \\
    a_{12} + a_{21} & \lambda + 2a_{22}
\end{vmatrix} = 0. \tag{90}
\]

3.1.3. Comparing the results from \(\mathbf{Z}_Q(s)\) and \(\mathbf{Y}_Q(s)\)

Let us now prove that the local passive regions derived from \(\mathbf{Z}_Q(s)\) and \(\mathbf{Y}_Q(s)\) are identical.
1. The local passive regions derived from $\mathbf{Z}_Q(s)$ are given by

$$a_{11} \leq 0 \quad \text{AND} \quad a_{22} \leq 0 \quad \text{AND} \quad 4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0 \quad \text{AND} \quad \Delta \geq 0 \quad \text{AND} \quad 4(a_{11}^2 + a_{22}^2) \Delta$$

$$+2(a_{12} + a_{21})^2 \Delta - T^2(a_{12} - a_{21})^2 \geq 0 \quad \text{OR} \quad \left\{ - \frac{4(a_{11}^2 + a_{22}^2) + 2(a_{12} + a_{21})^2 \Delta - T^2(a_{12} - a_{21})^2}{4[4a_{11}a_{22} - (a_{12} + a_{21})^2]} \right\} + [4a_{11}a_{22} - (a_{12} + a_{21})^2] \Delta^2 \geq 0 \quad \text{AND} \quad 4(a_{11}^2 + a_{22}^2) \Delta + 2(a_{12} + a_{21})^2 \Delta - T^2(a_{12} - a_{21})^2 < 0 \right\} \right\}.$$  

(97)

2. The local passive regions derived from $\mathbf{Y}_Q(s)$ are given by

$$a_{11} \leq 0 \quad \text{AND} \quad a_{11} + a_{22} \leq 0 \quad \text{AND} \quad 4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0.$$  

(98)

Observe that $a_{22} > 0$ is impossible, for otherwise it would follow from $a_{11} \leq 0$ that $4a_{11}a_{22} \leq 0$, contradicting the condition $4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0$. Therefore Eq. (98) can be rewritten as

<table>
<thead>
<tr>
<th>Local Passivity Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{11} \leq 0 \quad \text{AND} \quad a_{22} \leq 0 \quad \text{AND} \quad 4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0.$</td>
</tr>
</tbody>
</table>

(99)

**Theorem 3.** The conditions in Eqs. (97) and (99) are equivalent.

**Proof.** Let us first simplify the conditions in Eq. (97) as follows.

$$4\Delta = 2\Delta + 2\Delta$$

$$= 2(a_{11}a_{22} - a_{12}a_{21}) + 2(a_{11}a_{22} - a_{12}a_{21})$$

$$\geq 2(a_{11}a_{22} - a_{12}a_{21}) + 2a_{11}a_{22} - (a_{12}^2 + a_{21}^2)$$

$$= 4a_{11}a_{22} - (a_{12}^2 + a_{21}^2)$$

$$= 4a_{11}a_{22} - (a_{12} + a_{21})^2.$$  

(100)

It follows from the third inequality in Eq. (97) and Eq. (100) that $4\Delta \geq 0$, which leads to the fourth inequality in Eq. (97). Therefore the fourth inequality in Eq. (97) is redundant and Eq. (97) can be rewritten as

$$a_{11} \leq 0 \quad \text{AND} \quad a_{22} \leq 0 \quad \text{AND} \quad 4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0 \quad \text{AND} \quad 4(a_{11}^2 + a_{22}^2) \Delta + 2(a_{12} + a_{21})^2 \Delta$$

$$-T^2(a_{12} - a_{21})^2 \geq 0 \quad \text{OR} \quad \left\{ - \frac{4(a_{11}^2 + a_{22}^2) \Delta + 2(a_{12} + a_{21})^2 \Delta - T^2(a_{12} - a_{21})^2}{4[4a_{11}a_{22} - (a_{12} + a_{21})^2]} \right\} + [4a_{11}a_{22} - (a_{12} + a_{21})^2] \Delta^2 \geq 0 \quad \text{AND} \quad 4(a_{11}^2 + a_{22}^2) \Delta + 2(a_{12} + a_{21})^2 \Delta - T^2(a_{12} - a_{21})^2 < 0 \right\}.$$  

(101)

If

$$4(a_{11}^2 + a_{22}^2) \Delta + 2(a_{12} + a_{21})^2 \Delta - T^2(a_{12} - a_{21})^2 < 0$$
and

\[ 4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0 \]

then we have

\[
\frac{-4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2}{4[4a_{11}a_{22} - (a_{12} + a_{21})^2]} + [4a_{11}a_{22} - (a_{12} + a_{21})^2]\Delta^2 \geq 0.
\]

It follows from the above conclusion that the condition

\[
\left\{ \begin{array}{c}
4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 \geq 0 \quad \text{OR} \\
\left[ \frac{-4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2}{4[4a_{11}a_{22} - (a_{12} + a_{21})^2]} + [4a_{11}a_{22} - (a_{12} + a_{21})^2]\Delta^2 \right. \\
\left. \geq 0 \quad \text{AND} \quad 4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 < 0 \right\}
\]

(102)

can be reduced to

\[ 4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 < 0. \] (103)

Finally, the condition

\[
\left\{ \begin{array}{c}
4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 \geq 0 \quad \text{OR} \\
\left[ \frac{-4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2}{4[4a_{11}a_{22} - (a_{12} + a_{21})^2]} + [4a_{11}a_{22} - (a_{12} + a_{21})^2]\Delta^2 \right. \\
\left. \geq 0 \quad \text{AND} \quad 4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 < 0 \right\}
\]

(104)

becomes

\[
4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 \geq 0 \quad \text{OR} \quad 4(a_{11}^2 + a_{22}^2)\Delta + 2(a_{12} + a_{21})^2\Delta - T^2(a_{12} - a_{21})^2 < 0
\]

(105)

which is always true regardless of the choice of parameters. Therefore, as a final conclusion, the conditions in Eq. (97) are equivalent to

\[
a_{11} \leq 0 \quad \text{AND} \quad a_{22} \leq 0 \quad \text{AND} \quad 4a_{11}a_{22} - (a_{12} + a_{21})^2 \geq 0
\]

(106)

which is exactly Eq. (99). This completes our proof that Eq. (97) \( \Rightarrow \) Eq. (99). Since all deductions presented so far are reversible, if follows that Eq. (99) \( \Rightarrow \) (97).

Remark. It follows from Eq. (99) that a two-port and second-order cell is locally active at equilibrium point \( Q \) if, and only if, any one of the three inequalities in Eq. (99) is violated, which is precisely Corollary 4.4.3 in p. 296 of [Chua, 1998].

### 3.2. Active regions for two-port and second-order CNN Cells

For illustrative purposes, let us derive next the following equivalent conditions for local activity.

A1(\( Z_Q(s) \)). Conditions for “\( \text{Re}[s_p] > 0 \)” are those for \( \text{NOT} \ (P1(Z_Q(s))) \) and are given by

\[ T > 0 \quad \text{OR} \quad \Delta < 0. \] (107)

A2(\( Z_Q(s) \)). Conditions for “\( \text{NOT} \ (P2(Z_Q(s))) \)” satisfy

\[ A2(Z_Q(s)) \subseteq A1(Z_Q(s)) \] (108)

Therefore, for the purpose of studying the local activity parameter regions, \( A2(Z_Q(s)) \) contributes no new information.
A3(\(Z_Q(s)\)). Conditions for “\(Z_Q(s)\) has a simple pole \(s_p = \pm i\omega\) whose residue matrix \(K_{-1}\) is either not Hermitian, or is not positive semi-definite” are given by

\[
T = 0 \quad \text{AND} \quad \Delta > 0 \quad \text{AND} \\
[a_{11} \neq 0 \quad \text{OR} \quad a_{22} \neq 0 \quad \text{OR} \\
a_{12} \neq 0 \quad \text{OR} \quad a_{21} \neq 0]. \quad (109)
\]

A4(\(Z_Q(s)\)). Conditions for “\(Z_Q(s)\) has a multiple pole \(s_p = \pm i\omega\)” are given by

\[
T = 0 \quad \text{AND} \quad \Delta = 0 \quad \text{AND} \\
[a_{11} \neq 0 \quad \text{OR} \quad a_{22} \neq 0]. \quad (110)
\]

4. One-Port and First-Order CNN Cells: Examples

As an illustrative example, consider the state equation of a one-port first-order cell given by

\[
\dot{V}_1 = -V_1^3 + \alpha V_1 + I^* \quad (111)
\]

where \(I^* \in (-\infty, \infty)\) is an external constant called the “cell bias”. An equilibrium point \(Q\) is obtained by solving the equation

\[
V_1^{*3} - \alpha V_1^* - I^* = 0 \quad (112)
\]

as a function of \(I^*\) and \(\alpha\). Depending on the value \(\mathcal{D} = \frac{I^*}{4} - \frac{\alpha^3}{27}\) (113)

Eq. (112) can have one, two, or three distinct solutions; namely,

1. \(\mathcal{D} > 0\): only one equilibrium point

\[
V_1^* = \left(\frac{I^*}{2} + \sqrt{\mathcal{D}}\right)^{1/3} + \left(\frac{I^*}{2} - \sqrt{\mathcal{D}}\right)^{1/3}. \quad (114)
\]

2. \(\mathcal{D} = 0\): two equilibrium points

\[
V_{1a}^* = 2 \left(\frac{I^*}{2}\right)^{1/3} \quad V_{1b}^* = \left(\frac{I^*}{2}\right)^{1/3}. \quad (115)
\]

3. \(\mathcal{D} < 0\): three equilibrium points

\[
V_{1a}^* = 2\sqrt[3]{\frac{|\alpha|}{3}} \cos \frac{\psi}{3}, \quad V_{1b}^* = -2\sqrt[3]{\frac{|\alpha|}{3}} \cos \left(\frac{\psi}{3} - \frac{\pi}{3}\right), \quad V_{1c}^* = -2\sqrt[3]{\frac{|\alpha|}{3}} \cos \left(\frac{\psi}{3} + \frac{\pi}{3}\right). \quad (116)
\]

where

\[
\psi = \cos^{-1}\left(\frac{I^*}{2}\right)^{1/3}. \quad (117)
\]

The linearized cell state equation about each equilibrium point is given by

\[
\dot{v}_1 = (-3V_1^{*2} + \alpha)v_1 + i_1. \quad (118)
\]

where \(v_1\) and \(i_1\) denote small perturbations about \(V_1^*\) and \(I^*\), respectively.

The local passive parameter region at each equilibrium point \(Q = (V_1^*, I^*)^T\) is given by

\[
\alpha \leq 3V_1^{*2} \quad (119)
\]

which depends on the positions \(V_1^*\) of the equilibrium point. Since there are three different cases, we need to consider them separately as follows.

1. \(\mathcal{D} > 0\): The local passive regions are given by

\[
\frac{I^*}{4} - \frac{\alpha^3}{27} > 0, \quad -3 \left[\left(\frac{I^*}{2} + \sqrt{\mathcal{D}}\right)^{1/3} + \left(\frac{I^*}{2} - \sqrt{\mathcal{D}}\right)^{1/3}\right]^2 \quad (120)
\]

Since the second inequality is always satisfied if the first inequality is satisfied, it follows that

\[
\frac{I^*}{4} - \frac{\alpha^3}{27} > 0 \quad (121)
\]

defines the local passive parameter region.

2. \(\mathcal{D} = 0\): Since there are two equilibrium points in this degenerate case, the local passive regions are defined by the set intersection of the following parameter regions associated with the two equilibrium points

\[
\frac{I^*}{4} - \frac{\alpha^3}{27} = 0, \quad -12 \left(\frac{I^*}{2}\right)^{2/3} + \alpha \leq 0 \quad \text{AND} \quad -3 \left(\frac{I^*}{2}\right)^{2/3} + \alpha \leq 0. \quad (122)
\]
Solving for $\alpha$, we obtain

$$\alpha = 3 \left( \frac{I^*}{2} \right)^{2/3},$$

$$\alpha \leq 3 \left( \frac{I^*}{2} \right)^{2/3}. \quad (123)$$

It follows that the local passive region for $\mathcal{O} = 0$ consists of an isolated curve

$$\alpha = 3 \left( \frac{I^*}{2} \right)^{2/3}. \quad (124)$$

3. $\mathcal{O} < 0$. Since there are there equilibrium points in this case, the local passive regions are the intersection of the corresponding local passive parameter regions; namely,

$$\frac{I^*^2}{4} - \frac{\alpha^3}{27} < 0,$$

$$-3 \left( 2\sqrt{\alpha/3} \cos(\psi/3) \right)^2 + \alpha \leq 0 \quad \text{AND}$$

$$-3 \left( -2\sqrt{\alpha/3} \cos(\psi/3 - \pi/3) \right)^2 + \alpha \leq 0 \quad \text{AND}$$

$$-3 \left( -2\sqrt{\alpha/3} \cos(\psi/3 + \pi/3) \right)^2 + \alpha \leq 0. \quad (125)$$

We will now show that the intersection of the local passive regions at these three equilibrium points is empty. Let us assume

$$\frac{I^*}{2} = (1 - \delta) \sqrt{\frac{\alpha}{3}}, \quad (126)$$

where $0 < \delta \ll 1$, then $0 < \psi \ll \pi/2$. The intersection of the local passive regions at these three equilibrium points is given by

$$\alpha \leq 3 \left( -2\sqrt{\alpha/3} \cos(\psi/3 + \pi/3) \right)^2$$

$$= 3 \left( -2\sqrt{\alpha/3} (1/2 - \delta_1) \right)^2$$

$$= |\alpha| (1 - 2\delta_1)^2. \quad (127)$$

where $0 < \delta_1 \ll 1$. Since $\alpha > 0$ when $\mathcal{O} < 0$, it follows that the above condition gives an empty set.

In Fig. 1 the locally passive, restricted locally passive and active parameter regions in the $I^*-$\(\alpha\) plane are identified in blue, cyan and green colors, respectively. Observe that for $\alpha \leq 0$ (blue region) the cell is locally passive since the local passivity criteria are satisfied for $-\infty < I^* < \infty$. In contrast, for $0 < \alpha < 3(I^*/2)^{2/3}$ (cyan region) the local passivity criteria hold only for certain ranges of $I^*$. The boundary curve separating the cyan and green regions (defined by $\mathcal{O} = 0$) also belong to the restricted local passive region. Notice that there is no edge-of-chaos region in this case.

We now present our simulation results for an $N \times N$ CNN denoted by the following equations:

$$\dot{V}_1(i, j) = -V_1^3(i, j) + \alpha V_1(i, j) + D_1[V_1(i - 1, j)$$

$$+ V_1(i, j - 1) + V_1(i, j + 1)$$

$$+ V_1(i + 1, j) - 4V_1(i, j)], \quad 1 \leq i \leq N, 1 \leq j \leq N. \quad (128)$$

The simulation results are shown in Fig. 2. Figure 2(a) shows a random initial condition. Figures 2(b) and 2(c) show the evolving process of the CNN with $D_1 = 1$ and $\alpha = -2$. As shown in Fig. 1, in this case the CNN is locally passive. From Figs. 2(b) and 2(c) we can see that the CNN approaches a homogeneous output. Figures 2(d) and 2(e) show the evolving process of the CNN with $D_1 = 1$ and $\alpha = 2$. As shown in Fig. 1, in this case the CNN is locally active. From Figs. 2(d) and 2(e) we can see that the CNN outputs a static pattern.

In fact, these two kinds of patterns are the typical ones exhibited by this CNN. Observe that this CNN has a relatively low possibility for generating different kinds of patterns. We conjecture that this fact is due to the relatively simple configurations of different kinds of parameter regions in the $I^*-$\(\alpha\) plane.

5. One-Port and Second-Order CNN Cells: Examples

In this section let us study the CNN consisting of one-port second-order cells whose equations are given by

$$\dot{V}_1 = -V_1^3 + (0.5\alpha - b)V_1 - V_2^3$$

$$+ (1.5\alpha + b)V_2 + 2I^*,$$

$$\dot{V}_2 = V_1 - V_2. \quad (129)$$
Fig. 1. Locally passive, restricted locally passive and locally active parameter regions for one-port first-order cells. There is no edge-of-chaos region.

Fig. 2. The simulation results of a $64 \times 64$ CNN made of one-port first-order cells. The parameter $D_1 = 1$ is fixed. (a) Initial condition. (b) $\alpha = -2$, $t = 0.1$, locally passive. (c) $\alpha = -2$, $t = 1$, locally passive. (d) $\alpha = 2$, $t = 1$, locally active. (e) $\alpha = 2$, $t = 5$, locally active.
Fig. 2. (Continued)
The equilibrium points are given by
\[ V_1^* = V_2^*, \]
\[ V_1^{*3} - \alpha V_1^* - I^* = 0. \]  
(130)

Notice that \( V_1^* \) has the same form as that of one-port first-order cells. The Jacobian matrix at an equilibrium point \( Q \) is given by
\[ J_Q = \begin{pmatrix} -3V_1^{*2} + 0.5\alpha - b & -3V_1^{*2} + 1.5\alpha + b \\ 1 & -1 \end{pmatrix} \]  
(131)

with
\[ T = -3V_1^{*2} + 0.5\alpha - b - 1, \]
\[ \Delta = 6V_1^{*2} - 2\alpha. \]  
(132)

1. From \( P1(Z_Q(s)) \) we have
\[-3V_1^{*2} + 0.5\alpha - b - 1 \leq 0 \text{ AND } 6V_1^{*2} - 2\alpha \geq 0\]  
(133)

from which we have
\[ \alpha \leq 6V_1^{*2} + 2b + 2 \text{ AND } \alpha \leq 3V_1^{*2}. \]  
(134)

Consequently,
\[ \alpha \leq \begin{cases} 6V_1^{*2} + 2b + 2, & \text{if } V_1^{*2} \leq -\frac{2}{3}(b + 1), \\ 3V_1^{*2}, & \text{otherwise}. \end{cases} \]  
(135)

2. From \( P2(Z_Q(s)) \) we have
\[-3V_1^{*2} + 0.5\alpha - b \leq 0 \text{ AND } 6V_1^{*2} - 2\alpha \geq 0\]  
(136)

from which we have
\[ \alpha \leq 6V_1^{*2} + 2b \text{ AND } \alpha \leq 3V_1^{*2}. \]  
(137)

Consequently,
\[ \alpha \leq \begin{cases} 6V_1^{*2} + 2b, & \text{if } V_1^{*2} \leq -\frac{2}{3}b, \\ 3V_1^{*2}, & \text{otherwise}. \end{cases} \]  
(138)

3. From \( P3(Z_Q(s)) \) we have
\[-3V_1^{*2} + 0.5\alpha - b = 0 \text{ AND } -1 = 0 \text{ AND } -3V_1^{*2} + 1.5\alpha + b < 0, \]  
(139)

from which we have an empty set.

The parameter region defined by \( P2(Z_Q(s)) \) can be a subset of that defined by \( P1(Z_Q(s)) \) if we choose a proper parameter \( b \). The local passive region is given by
\[ \alpha \leq \begin{cases} 6V_1^{*2} + 2b, & \text{if } V_1^{*2} \leq -\frac{2}{3}b, \\ 3V_1^{*2}, & \text{otherwise}. \end{cases} \]  
(140)

The edge-of-chaos region is given by
\[ 6V_1^{*2} + 2b < \alpha < 6V_1^{*2} + 2b + 2 \text{ AND } V_1^{*2} \leq -\frac{2}{3}(b + 1). \]  
(141)

Comparing with the one-port first-order case where the local passive region is given by \( \alpha \leq (3/2)V_1^{*2} \), we can see that the local passive region for the one-port second-order case has shrunk and given birth to an edge-of-chaos region. Observe that these conditions are only valid at each individual equilibrium point. The behavior of the entire cell is defined by the combination of conditions considered at all equilibrium points as follows.

Remarks

From Sec. 2.2.1 we know that we can also find the passive region from \( Y_Q(s) \). In fact, \( P2(Y_Q(s)) = P2(Z_Q(s)) \) and \( P1(Y_Q(s)) \) is much simpler than \( P1(Z_Q(s)) \). From \( P1(Y_Q(s)) \) we have
\[ a_{22} = -1 \leq 0 \]  
(142)

which is always satisfied. This means that in this case we do not need to consider \( P1(Z_Q(s)) \) as we have already done in Eq. (135). On the other hand, it shows that the results shown in Eq. (135) are redundant. This fact can be seen from Eqs. (135), (138) and (140) where the local passive regions defined in Eq. (140) are exactly the same as those given by \( P2(Z_Q(s)) \); namely, Eq. (138).

5.1. Passive parameter regions

1. When \( \mathcal{D} > 0 \), the local passive regions are given by
\[
\frac{I^2}{4} - \frac{\alpha^3}{27} > 0, \\
\alpha \leq \begin{cases} 
6 \left[ \left( \frac{I^*}{2} + \sqrt{3} \right)^{1/3} + \left( \frac{I^*}{2} - \sqrt{3} \right)^{1/3} \right]^2 + 2b, & \text{if } \left[ \left( \frac{I^*}{2} + \sqrt{3} \right)^{1/3} + \left( \frac{I^*}{2} - \sqrt{3} \right)^{1/3} \right]^2 \leq -\frac{2}{3} b, \\
3 \left[ \left( \frac{I^*}{2} + \sqrt{3} \right)^{1/3} + \left( \frac{I^*}{2} - \sqrt{3} \right)^{1/3} \right]^2, & \text{otherwise}
\end{cases}
\]

which is the intersection of the following two parameter regions:

\[
\frac{I^2}{4} - \frac{\alpha^3}{27} > 0 \AND \alpha \leq 6 \left[ \left( \frac{I^*}{2} + \sqrt{3} \right)^{1/3} + \left( \frac{I^*}{2} - \sqrt{3} \right)^{1/3} \right]^2 + 2b \AND \\
\left[ \left( \frac{I^*}{2} + \sqrt{3} \right)^{1/3} + \left( \frac{I^*}{2} - \sqrt{3} \right)^{1/3} \right]^2 \leq -\frac{2}{3} b,
\]

and

\[
\frac{I^2}{4} - \frac{\alpha^3}{27} > 0 \AND \left[ \left( \frac{I^*}{2} + \sqrt{3} \right)^{1/3} + \left( \frac{I^*}{2} - \sqrt{3} \right)^{1/3} \right]^2 > -\frac{2}{3} b.
\]

2. When \( D = 0 \), since there are two equilibrium points, the local passive region is the intersection of the local passive parameter regions for both equilibrium points and are given by

\[
\frac{I^2}{4} - \frac{\alpha^3}{27} = 0 \AND \alpha \leq \begin{cases} 
24 \left( \frac{I^*}{2} \right)^{2/3} + 2b, & \text{if } 4 \left( \frac{I^*}{2} \right)^{2/3} \leq -\frac{2}{3} b, \\
12 \left( \frac{I^*}{2} \right)^{2/3}, & \text{otherwise}
\end{cases} \AND \\
\alpha \leq \begin{cases} 
6 \left( \frac{I^*}{2} \right)^{2/3} + 2b, & \text{if } \left( \frac{I^*}{2} \right)^{2/3} \leq -\frac{2}{3} b, \\
3 \left( \frac{I^*}{2} \right)^{2/3}, & \text{otherwise}
\end{cases}
\]

from which we have

\[
\alpha \geq 0 \AND \alpha \leq \begin{cases} 
8\alpha + 2b, & \text{if } \alpha \leq -0.5b, \\
4\alpha, & \text{otherwise}
\end{cases} \AND \alpha \leq \begin{cases} 
2\alpha + 2b, & \text{if } \alpha \leq -0.5b, \\
\alpha, & \text{otherwise}
\end{cases}
\]

If \( b \geq 0 \) then the local passive region is a point \( \alpha = 0, I^* = 0 \). If \( b < 0 \) then the local passive region is given by two points at

\[
(\alpha = -0.5b, I^* = 2(-0.5b/3)^{3/2}), (\alpha = -0.5b, I^* = -2(-0.5b/3)^{3/2}).
\]

3. When \( D < 0 \), since there are three equilibrium points, the local passive regions are given by the intersection of the local passive parameter regions at these equilibrium points as

\[
\frac{I^2}{4} - \frac{\alpha^3}{27} < 0 \AND \alpha \leq \begin{cases} 
6(2\sqrt{|\alpha|/3\cos(\psi/3)})^2 + 2b, & \text{if } (2\sqrt{|\alpha|/3\cos(\psi/3)})^2 \leq \frac{2}{3} b, \\
3(2\sqrt{|\alpha|/3\cos(\psi/3)})^2, & \text{otherwise}
\end{cases}
\]
\[ \alpha \leq \begin{cases} 
6(-2\sqrt{|\alpha|/3}\cos(\psi/3 - \pi/3))^2 + 2b, & \text{if } (-2\sqrt{|\alpha|/3}\cos(\psi/3 - \pi/3))^2 \leq -\frac{2}{3}b, \\
3(-2\sqrt{|\alpha|/3}\cos(\psi/3 - \pi/3))^2, & \text{otherwise} 
\end{cases} \quad \text{AND} \]

\[ \alpha \leq \begin{cases} 
6(-2\sqrt{|\alpha|/3}\cos(\psi/3 + \pi/3))^2 + 2b, & \text{if } (-2\sqrt{|\alpha|/3}\cos(\psi/3 + \pi/3))^2 \leq -\frac{2}{3}b, \\
3(-2\sqrt{|\alpha|/3}\cos(\psi/3 + \pi/3))^2, & \text{otherwise} 
\end{cases} \]  

This set of conditions give an empty set.

### 5.2. Edge-of-chaos parameter regions

From the analysis of the local passive parameter region presented Sec. 5.1 we know that the edge-of-chaos region can only be found when \( \mathcal{D} > 0 \). The edge-of-chaos region is then given by the following conditions:

\[
b < 0 \quad \text{AND} \quad \frac{I^*}{4} - \frac{\alpha^3}{27} > 0 \quad \text{AND} \quad 6 \left[ \left( \frac{I^*}{2} + \sqrt{\mathcal{D}} \right)^{1/3} + \left( \frac{I^*}{2} - \sqrt{\mathcal{D}} \right)^{1/3} \right]^2 + 2b \\
< \alpha < 6 \left[ \left( \frac{I^*}{2} + \sqrt{\mathcal{D}} \right)^{1/3} + \left( \frac{I^*}{2} - \sqrt{\mathcal{D}} \right)^{1/3} \right]^2 + 2b + 2 \quad \text{AND} \quad 6 \left[ \left( \frac{I^*}{2} + \sqrt{\mathcal{D}} \right)^{1/3} + \left( \frac{I^*}{2} - \sqrt{\mathcal{D}} \right)^{1/3} \right]^2 \\
\leq -\frac{2}{3}(b + 1). \quad (150)
\]

### 5.3. Illustrative examples

In Fig. 3 the locally passive, edge-of-chaos and locally active parameter regions are shown in the \( I^*-\alpha \) plane. The locally passive regions and restricted locally passive regions are shown in blue and cyan, respectively. The edge-of-chaos regions and the locally active unstable regions are shown in red and green, respectively. When \( b \geq 0 \), the edge-of-chaos parameter region is empty and the \( I^*-\alpha \) plane is the same as that shown in Fig. 1. When \( 0 < b \leq -1 \), the upper boundary of the edge-of-chaos region is given by the curve \( \mathcal{D} = 0 \). In Fig. 3 we show the edge-of-chaos regions with \( b = -1 \), \(-10 \) and \(-20 \), respectively. As we can see from Fig. 3, when \( b \) decreases, the edge-of-chaos region changes from the shape of a needle to the shape of a “two-tooth” fork. Also, as \( b \) decreases, the length of both fork teeth and the distance between them increase. As we will see later in this section, this shape bifurcation of the edge-of-chaos regions corresponds to a bifurcation of pattern formation in the corresponding CNNs.

We will now present our simulation results of an \( N \times N \) CNN described by the following equations:

\[
\begin{align*}
\dot{V}_1(i, j) &= -V_1^3(i, j) + (0.5\alpha - b)V_1(i, j) - V_2^3(i, j) + (1.5\alpha + b)V_2(i, j) + D_1[V_1(i - 1, j) \\
&\quad + V_1(i, j - 1) + V_1(i, j + 1) + V_1(i + 1, j) - 4V_1(i, j)], \\
\dot{V}_2(i, j) &= V_1(i, j) - V_2(i, j), 1 \leq i \leq N, 1 \leq j \leq N.
\end{align*}
\]

The simulation results with parameters \( \alpha = -3 \), \( b = -1 \) and \( D_1 = 1 \) are shown in Fig. 4. As shown in Figs. 3(a)–3(c), this CNN is locally passive with these parameters. Figures 4(a) and 4(b) show the initial conditions of \( V_1(i, j) \) and \( V_2(i, j) \), respectively. Figures 4(c) and 4(d) show \( V_1(i, j) \) and \( V_2(i, j) \) at \( t = 1 \), respectively. Figures 4(e) and 4(f) show \( V_1(i, j) \) and \( V_2(i, j) \) at \( t = 4 \), respectively. Observe that this CNN converges to a homogeneous pattern.
The simulation results with parameters \( \alpha = 1 \), \( b = -1 \) and \( D_1 = 1 \) are shown in Fig. 5. As shown in Figs. 3(a)–3(c), this CNN is locally active with these parameters. The initial conditions are the same as those in Figs. 4(a) and 4(b). Figures 5(a) and 5(b) show \( V_1(i, j) \) and \( V_2(i, j) \) at \( t = 1 \), respectively. Figures 5(c) and 5(d) show \( V_1(i, j) \) and \( V_2(i, j) \) at \( t = 10 \), respectively. Figures 5(e) and 5(f) show \( V_1(i, j) \) and \( V_2(i, j) \) at \( t = 500 \), respectively. Figures 5(g) and 5(h) show \( V_1(i, j) \) and \( V_2(i, j) \) at \( t = 1000 \), respectively. Observe that the final states converge to a static pattern where \( V_1(i, j) = V_2(i, j) \). The mechanism leading to this kind of pattern can be understood intuitively as follows. The cells push the states away from a homogeneous pattern because \( I^* = 0 \) is locally active in this case. However, as the coupling current \( I^* \) generated by the evolving pattern becomes large enough such that \( |I^*| \) enters the restricted locally passive regions, this evolving pattern will eventually be frozen in the locally passive regions. In other words, this CNN will settle down at an equilibrium point in the restricted locally passive regions.

The simulation results with parameters \( \alpha = -1 \), \( b = -1 \) and \( D_1 = 1 \) are shown in Fig. 6. As shown in Figs. 3(a)–3(c), this CNN is operating in the edge-of-chaos parameter region. The initial conditions for \( V_1(i, j) \) and \( V_2(i, j) \) are the same as those shown in Figs. 4(a) and 4(b). Figures 6(a) and 6(b) show \( V_1(i, j) \) and \( V_2(i, j) \) at \( t = 1 \), respectively. Figures 6(c) and 6(d) show \( V_1(i, j) \) and \( V_2(i, j) \) at \( t = 4 \), respectively. Figures 6(e) and 6(f) show \( V_1(i, j) \) and \( V_2(i, j) \) at \( t = 10 \), respectively. Figures 6(g)–6(i) show \( V_1(i, j) \) at \( t = 20 \), 40 and 80, respectively. Observe that this CNN converges to relatively “smooth” patterns. This is because the edge-of-chaos parameter region as shown in Figs. 3(a)–3(c) is very near the zero cell bias; namely, \( I^* \ll 1 \). In this case, the pattern is

---

Fig. 3. Parameter regions for one-port second-order cells. The locally passive regions and restricted locally passive regions are shown in blue and cyan, respectively. The edge-of-chaos regions and locally active unstable regions are shown in red and green, respectively. (a) \( b = -1 \). (b) \( b = -1 \), enlargement of (a). (c) \( b = -1 \), enlargement of (b). (d) \( b = -10 \). (e) \( b = -10 \), enlargement of (d). (f) \( b = -10 \), enlargement of (e). (g) \( b = -20 \). (h) \( b = -20 \), enlargement of (g). (i) \( b = -20 \), enlargement of (h).
Fig. 3  (Continued)
Fig. 4. Simulation results of CNN made of one-port second-order cells. The parameters are given by $\alpha = -3$, $b = -1$ and $D_1 = 1$. The cells are locally passive. (a) Initial condition for $V_1(i, j)$. (b) Initial condition for $V_2(i, j)$. (c) $V_1(i, j)$ at $t = 1$. (d) $V_2(i, j)$ at $t = 1$. (e) $V_1(i, j)$ at $t = 4$. (f) $V_2(i, j)$ at $t = 4$. 
Fig. 4. (Continued)

Fig. 5. Simulation results of CNN made of one-port second-order cells. The parameters are given by $\alpha = 1$, $b = -1$ and $D_i = 1$. The cells are locally active. (a) $V_1(i, j)$ at $t = 1$. (b) $V_2(i, j)$ at $t = 1$. (c) $V_1(i, j)$ at $t = 10$. (d) $V_2(i, j)$ at $t = 10$. (e) $V_1(i, j)$ at $t = 500$. (f) $V_2(i, j)$ at $t = 500$. (g) $V_1(i, j)$ at $t = 1000$. (h) $V_2(i, j)$ at $t = 1000$. 
Fig. 5. (Continued)