Theory and Methodology

Portfolio selection based on upper and lower exponential possibility distributions

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Abstract

In this paper, two kinds of possibility distributions, namely, upper and lower possibility distributions are identified to reflect experts’ knowledge in portfolio selection problems. Portfolio selection models based on these two kinds of distributions are formulated by quadratic programming problems. It can be said that a portfolio return based on the lower possibility distribution has smaller possibility spread than the one on the upper possibility distribution. In addition, a possibility risk can be defined as an interval given by the spreads of the portfolio returns from the upper and the lower possibility distributions to reflect the uncertainty in real investment problems. A numerical example of a portfolio selection problem is given to illustrate our proposed approaches. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Possibility theory has been proposed by Zadeh [1] and advanced by Dubois and Prade [2] where fuzzy variables are associated with possibility distributions in the similar way that random variables are associated with probability distributions. Possibility distributions are represented as normal convex fuzzy sets, such as L–R fuzzy numbers, quadratic and exponential functions [3–6]. The theory of exponential possibility distributions has been proposed and applied to possibilistic data analysis [7–9]. As an application of possibility theory to portfolio analysis, possibility portfolio selection models were initially proposed in papers [10,11] where portfolio models are based on exponential possibility distributions rather than the mean-variance form in Markowitz’s model. Although there are some similarities between Markowitz’s model and possibility portfolio selection models, these two kinds of models analyze the security data in very different ways. Markowitz’s model regards the portfolio selection as probability phenomena so that it minimizes the variance of portfolio return subject to a given average return. On the contrary, possibility models...
based on possibility distributions reflect portfolio experts’ knowledge, which is characterized by the given possibility grades to security data. The basic assumption for using Markowitz’s model is that the situation of stock markets in future can be correctly reflected by security data in the past, that is, the mean and covariance of securities in future is similar to the past one. It is hard to ensure this kind of assumption for the real ever-changing stock markets. On the other hand, possibility portfolio models integrate the past security data and experts’ judgement to catch variations of stock markets more feasibly. Because experts’ knowledge is very valuable for predicting the future state of stock markets, it is reasonable that possibility portfolio models are useful in the real investment world.

This paper deals with portfolio selection problems based on two kinds of possibility distributions, namely, the upper and lower possibility distributions that are similar to rough sets to some extent to reflect two extreme opinions of experts. Using these two kinds of distributions, the corresponding portfolio selection models are formalized by quadratic optimization problems minimizing spreads of possibility portfolio returns subject to the given center returns. It can be concluded that the portfolio return based on the lower possibility distribution has smaller possibility spread than the one based on the upper possibility distribution. A numerical example of a portfolio selection problem is given to illustrate our proposed approaches.

2. Markowitz’s portfolio selection model

Let us give a brief description of Markowitz’s model. Assume that there are \( n \) securities denoted by \( S_j \) \((j = 1, \ldots, n)\), the return of the security \( S_j \) is denoted as \( x_j \) and the proportion of total investment funds devoted to this security is denoted as \( r_j \).

Thus,

\[
\sum_{j=1}^{n} r_j = 1. \tag{1}
\]

Since the return of the security \( x_j \) \((j = 1, \ldots, n)\) vary from time to time, those are assumed to be random variables which can be represented by the pair of the average vector and covariance matrix. For instance, it is assumed that the observation data on returns of securities over \( m \) periods are given. At the discrete time \( t \), the vector of \( n \) returns is denoted as \( x_t = [x_{1t}, \ldots, x_{nt}]^T \). Thus, the total data over \( m \) periods are denoted as the following matrix.

\[
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
\]

The average vector of returns over \( m \) periods is denoted as \( x^0 = [x_1^0, \ldots, x_n^0]^T \) and is written as

\[
x^0 = \frac{1}{m} \sum_{i=1}^{m} x_i. \tag{3}
\]

Also the covariance matrix \( Q = [q_{ij}] \) can be written as

\[
q_{ij} = \frac{1}{m} \sum_{k=1}^{m} (x_{ki} - x_i^0)(x_{kj} - x_j^0),
\]

\[
(i = 1, \ldots, n, j = 1, \ldots, n). \tag{4}
\]

Therefore random variables \( x_j \) \((j = 1, \ldots, n)\) can be represented by the average vector \( x^0 \) and the covariance matrix \( Q \), denoted as \( (x^0, Q) \).

Now, the return associated with a portfolio \( r = [r_1, \ldots, r_n]^T \) is given by

\[
z = r^T x. \tag{5}
\]

The average and variance of \( z \) are given as

\[
E(z) = E(r^T x) = r^T E x = r^T x^0, \tag{6}
\]

\[
V(z) = V(r^T x) = r^T Q r. \tag{7}
\]

Since the variance of a portfolio return is regarded as the risk of investment, the best investment is one with the minimum variance \( (7) \) subject to a given average return \( c \). This leads to the following quadratic programming problem.
\[
\begin{align*}
\min_{r} & \quad r^{1} Q r \\
\text{s.t.} & \quad r^{1} x^{0} = c, \\
& \quad \sum_{i=1}^{n} r_{i} = 1, \\
& \quad r_{i} \geq 0.
\end{align*}
\]

(8)

3. Possibility distributions in portfolio selection problems

Let us begin with the given data \((x_{i}, h_{i}) (i = 1, \ldots, m)\) where \(x_{i} = [x_{i1}, \ldots, x_{in}]^{1}\) is a vector of returns of \(n\) securities \(S_{j} (j = 1, \ldots, n)\) at the \(i\)th period and \(h_{i}\) is an associated possibility grade given by expert knowledge to reflect a similarity degree between the future state of stock markets and the state of the \(i\)th sample. These grades \(h_{i} (i = 1, \ldots, m)\) are assumed to be expressed by a possibility distribution \(A\) defined as

\[
\Pi_{A}(x) = \exp\{- (x - a)^{1} D_{A}^{-1} (x - a)\} = (a, D_{A})_{e},
\]

where \(a = [a_{1}, a_{2}, \ldots, a_{n}]^{1}\) is a center vector and \(D_{A}\) is a symmetric positive definite matrix, denoted as \(D_{A} > 0\).

Given the data, the problem is to determine an exponential possibility distribution (9), i.e., a center vector \(a\) and a symmetric positive definite matrix \(D_{A}\). According to two different viewpoints, two kinds of possibility distributions of \(A\), namely, the upper and the lower possibility distributions are introduced in this paper. The upper and the lower possibility distributions denoted as \(\Pi_{u}\) and \(\Pi_{l}\), respectively, should satisfy the inequality \(\Pi_{u}(x) \geq \Pi_{l}(x)\) with considering some similarities between our proposed methods and rough sets.

From the formulation (9), it is obvious that the vector \(x\) with the highest possibility grade should be closest to the center vector \(a\) among all \(x_{i} (i = 1, \ldots, m)\). Thus, the center vector \(a\) can be approximately estimated as

\[
a = x_{r},
\]

where \(x_{r}\) denotes the vector with grade \(h_{r} = \max_{k=1,\ldots,m} h_{k}\). The associated possibility grade of \(x_{r}\) is revised to be 1. Taking the transformation \(y = x - a\), the possibility distribution with a zero center vector is obtained as,

\[
\Pi_{A}(y) = \exp\{- y^{1} D_{A}^{-1} y\} = (0, D_{A})_{e}.
\]

(11)

In what follows, the matrices \(D_{u}\) in the upper and the lower possibility distributions are denoted as \(D_{u} \) and \(D_{l}\), respectively.

3.1. Identification of upper and lower possibility distributions by integrated model

The upper and the lower distributions are used to reflect two kinds of distributions from the upper and the lower directions. In order to determine the matrix \(D_{u}\) in the upper distribution, the following assumptions are given:

(1) \(\Pi_{u}(y_{i}) \geq h_{i}, \quad i = 1, \ldots, m\) (the constraint conditions),

(2) minimize \(\Pi_{u}(y_{1}) \times \cdots \times \Pi_{u}(y_{m})\) (the objective function).

The problem can be described as determining \(D_{u}\) that minimizes the objective function (2) subject to the constraint conditions (1). It means to obtain the smallest distribution among \(\Pi_{u}(y_{i}) \geq h_{i} (i = 1, \ldots, m)\) in the sense of minimizing \(\Pi_{u}(y_{1}) \times \cdots \times \Pi_{u}(y_{m})\). Fig. 1 illustrates the basic idea of the identification method.

![Fig. 1. Upper possibility distribution.](image-url)
More detailed, the assumption (1) can be written as follows:

\[ \Pi_u(y_i) \geq h_i \iff y_i^T D_u^{-1} y_i \leq -\ln h_i, \quad i = 1, \ldots, m. \tag{12} \]

The objective function can be rewritten as follows:

\[ \max \sum_{i=1}^{m} y_i^T D_u^{-1} y_i. \tag{13} \]

Thus, the problem for obtaining \( D_u \) becomes the following optimization problem.

\[ \max_{D_u} \sum_{i=1}^{m} y_i^T D_u^{-1} y_i \tag{14} \]

\[ \text{s.t. } y_i^T D_u^{-1} y_i \leq -\ln h_i, \quad i = 1, \ldots, m, \quad D_u > 0. \]

Similarly, in order to determine the matrix \( D_l \) in the lower distribution, the following assumptions are given:

1. \( \Pi_l(y_i) \leq h_i, \quad i = 1, \ldots, m \) (the constraint conditions),
2. \( \Pi_l(y_1) \times \cdots \times \Pi_l(y_m) \) (the objective function).

Likewise, the assumptions (1) and (2) can be converted into the following optimization problem to obtain the distribution matrix \( D_l \).

\[ \min_{D_l} \sum_{i=1}^{m} y_i^T D_l^{-1} y_i \tag{15} \]

\[ \text{s.t. } y_i^T D_l^{-1} y_i \geq -\ln h_i, \quad i = 1, \ldots, m, \quad D_l > 0. \]

It means to obtain the largest distribution among \( \Pi_l(y_i) \leq h_i, i = 1, \ldots, m \) in the sense of maximizing \( \Pi_l(y_1) \times \cdots \times \Pi_l(y_m) \). Fig. 2 shows the basic idea of the lower distribution.

From the optimization problems (14) and (15), we know that the upper possibility distribution is obtained as the smallest possibility distribution subject to \( \Pi_l(y_i) \geq h_i \). On the contrary, the lower possibility distribution is obtained as the largest possibility distribution subject to \( \Pi_l(y_i) \leq h_i \). If we solve these two optimization problems separately, it cannot be ensured that \( \Pi_u(y) \geq \Pi_l(y) \) holds for an arbitrary \( y \). Let us consider the following model, which integrates Eqs. (14) and (15) to find out \( D_l \) and \( D_u \), simultaneously with the condition that \( D_u - D_l \) is a semi-positive definite matrix.

\[ \min_{D_u, D_l} \sum_{i=1}^{m} y_i^T D_l^{-1} y_i - \sum_{i=1}^{m} y_i^T D_u^{-1} y_i \tag{16} \]

\[ \text{s.t. } y_i^T D_l^{-1} y_i \leq -\ln h_i, \quad y_i^T D_l^{-1} y_i \geq -\ln h_i, \quad i = 1, \ldots, m, \quad D_u - D_l \geq 0, \quad D_l > 0. \]

In this case, \( \Pi_u(y) \) and \( \Pi_l(y) \) are similar to rough set concept shown in Fig. 3 because \( D_u - D_l \geq 0 \) ensures \( \Pi_u(y) \geq \Pi_l(y) \). It is obvious that Eq. (16)
is a nonlinear optimization problem which is difficult to be solved.

In order to solve the problem (16) easily let us firstly consider a simple linear programming problem without the conditions \( D_u - D_l \geq 0 \) and \( D_l > 0 \) in this problem. If the obtained matrix \( D_u \) and \( D_l \) cannot satisfy the conditions \( D_u - D_l \geq 0 \) and \( D_l > 0 \), we introduce the following auxiliary conditions to the constraint conditions of the problem (16) to obtain positive definite matrices \( D_l \) and \( D_u \) such that \( D_u - D_l \geq 0 \) holds by the linear programming.

\[
y_i^T D_u^{-1} y_i \geq \varepsilon \quad \text{for} \ i \in E,
\]

(17)

\[
y_i^T D_u^{-1} y_j = 0 \quad \text{for all} \ i \neq j \quad \text{and} \ i, j \in E
\]

(18)

\[
y_i^T (D_l^{-1} - D_u^{-1}) y_i \geq 0 \quad \text{for} \ i \in E,
\]

(19)

\[
y_i^T D_l^{-1} y_j = 0 \quad \text{for all} \ i \neq j \quad \text{and} \ i, j \in E,
\]

(20)

where \( E \) is the index set of \( n \) selected independent vectors \( \{y_1, \ldots, y_n\} \) among \( y_i \ (i = 1, \ldots, m) \) with considering the condition \( m \gg n \) and \( \varepsilon \) is a small positive number. It should be noted that the center vector \( y_r = 0 \) is not included in \( \{y_1, \ldots, y_n\} \). The equalities (18) and (20) are called the orthogonal conditions. It is proved afterwards that the constraint conditions (17)–(20) can ensure that \( D_l > 0, D_u > 0 \) and \( D_u - D_l \geq 0 \) hold.

Thus, the following LP problem is formed.

\[
\min_{D_u, D_l} \sum_{i=1}^{m} y_i^T D_u^{-1} y_i - \sum_{i=1}^{m} y_i^T D_l^{-1} y_i
\]

s.t. \( y_i^T D_u^{-1} y_i \leq -\ln h_i \quad \text{for} \ i = 1, \ldots, m, \)

\( y_i^T D_l^{-1} y_i \geq -\ln h_i \quad \text{for} \ i = 1, \ldots, m, \)

\( y_i^T D_u^{-1} y_i \geq \varepsilon \quad \text{for} \ i \in E, \)

\( y_i^T (D_l^{-1} - D_u^{-1}) y_i \geq 0 \quad \text{for} \ i \in E, \)

\( y_i^T D_l^{-1} y_j = 0 \quad \text{for all} \ i \neq j \quad \text{and} \ i, j \in E, \)

\( y_i^T D_u^{-1} y_j = 0 \quad \text{for all} \ i \neq j \quad \text{and} \ i, j \in E. \)

(21)

Theorem 1. The matrices \( D_u \) and \( D_l \) obtained from Eqs. (17)–(20) satisfy the condition \( D_u - D_l \geq 0, D_l > 0 \) and \( D_u > 0 \).

**Proof.** Because \( \{y_1, \ldots, y_n\} \) are independent vectors in the \( n \)-dimensional space, an arbitrary vector \( z \) can be represented as

\[
z = \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n,
\]

where \( \lambda_i \) is a real number.

Thus, using Eqs. (17) and (18), we have for \( z \neq 0 \)

\[
z^T D_u^{-1} z = (\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n)^T D_u^{-1} y_i > 0.
\]

(23)

It means that \( D_u > 0 \). It follows from Eq. (19) that

\[
y_i^T D_l^{-1} y_i \geq y_i^T D_u^{-1} y_i > 0.
\]

(24)

Thus, using Eqs. (20) and (24), we have

\[
z^T D_l^{-1} z = (\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n)^T D_l^{-1} \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n)
\]

\[
= \sum_{i=1}^{n} \lambda_i^2 y_i^T D_l^{-1} y_i > 0,
\]

(25)

which means that \( D_l > 0 \). Similarly,

\[
z^T (D_l^{-1} - D_u^{-1}) z = (\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n)^T (D_l^{-1} - D_u^{-1}) (\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n)
\]

\[
= \sum_{i=1}^{n} \lambda_i^2 y_i^T (D_l^{-1} - D_u^{-1}) y_i \geq 0,
\]

(26)

which means that \( D_u - D_l \geq 0 \). □

**Theorem 2.** An optimal solution of Eq. (21) always exists.

**Proof.** Let us consider a \( n \times n \) matrix \( K \) by which a set of linearly independent vectors \( \{y_1, \ldots, y_n\} \) is transferred into a set of orthonormal vectors \( \{z_1, \ldots, z_n\} \) where \( z_i^T = [0, \ldots, 0, 1, 0, \ldots, 0]^T \). Thus,

\[
K[y_1, \ldots, y_n] = I,
\]

(27)

where \( I \) is the identical matrix. If we take positive definite matrices

\[
D_u^{-1} = q_1 K^T K,
\]

(28)
and

\[ D_1^{-1} = q_2 K' K, \]

where \( q_1 \leq q_2 \), it is obvious that \( D_u^{-1} \) and \( D_l^{-1} \) satisfy Eqs. (18) and (20) and \( D_1^{-1} - D_u^{-1} = (q_2 - q_1) K' K \). The constraint condition of Eq. (21) becomes

\[ q_1 (K_{y_i})^t (K_{y_i}) \leq - \ln h_i \quad \text{for } i = 1, \ldots, m, \]

\[ q_2 (K_{y_i})^t (K_{y_i}) \geq - \ln h_i \quad \text{for } i = 1, \ldots, m, \]

\[ q_1 (K_{y_i})^t (K_{y_i}) \geq \varepsilon, \quad \text{for } i \in E, \]

\[ (q_2 - q_1) (K_{y_i})^t (K_{y_j}) \geq 0, \quad \text{for } i \in E, \]

\[ q_1 (K_{y_i})^t (K_{y_j}) = 0, \]

\[ \text{for all } i \neq j \text{ and } i, j \in E, \]

\[ q_2 (K_{y_i})^t (K_{y_j}) = 0 \]

\[ \text{for all } i \neq j \text{ and } i, j \in E. \]

If we take

\[ q_1 = \min_{i = 1, \ldots, m-1, i \neq i'} (- \ln h_i / (K_{y_i})^t (K_{y_i})), \]

\[ q_2 = \max_{i = 1, \ldots, m-1, i \neq i'} (- \ln h_i / (K_{y_i})^t (K_{y_j})), \]

and a very small positive value for \( \varepsilon \), the obtained \( D_u \) and \( D_l \) satisfy all of constraint conditions. It means that there is an admissible set in the constraint conditions of Eq. (21). It should be noted that the center vector \( y_{i'} = 0 \) is omitted in the first two inequalities of Eq. (30), because \( (K_{y_{i'}})^t (K_{y_{i'}}) = - \ln 1 = 0 \). Thus, we consider \( i = 1, \ldots, m-1 \) without \( y_{i'} = 0 \) in determining \( q_1 \) and \( q_2 \) in Eqs. (31) and (32).

Here, orthogonal conditions are added to constraint conditions that can confine the matrices \( D_u \) and \( D_l \) to positive definite matrices and \( D_u - D_l \) to a semi-positive definite matrix. However, since there are many orthogonal conditions among independent vectors, it is very hard to select appropriate ones.

To cope with this difficulty, we use principle component analysis (PCA) to rotate the given data \((y_i, h_i)\) to obtain a positive definite matrix easily. Fig. 4 illustrates the rotation of orthogonal axes. The data can be transformed by linear transformation \( T \). Columns of \( T \) are eigenvectors of the matrix \( \Sigma = [\sigma_{ij}] \), where \( \sigma_{ij} \) is defined as

\[ \frac{\partial}{\partial y_{i'}} \left( - \ln h_{i'} / (K_{y_i})^t (K_{y_j}) \right). \]

![Fig. 4. Illustration of the linear transform by principle component analysis.](image)
\[ \sigma_{ij} = \left\{ \sum_{k=1}^{m} (x_{ki} - a_i)(x_{kj} - a_j)h_k \right\} \over \sum_{k=1}^{m} h_k. \] (33)

Without loss of generality, we assume that rank(\Sigma) = n. It should be noted that \( T^* T = I \). Using the linear transformation, the data \( y \) can be transformed into \{ \( z = T^* y \) \}. Then we have

\[ \Pi_A(z) = \exp(-z^T D_A^{-1} T z). \] (34)

According to the feature of PCA, \( T^* D_A^{-1} T \) is assumed to be a diagonal matrix as follows:

\[ T^* D_A^{-1} T = C_A = \begin{pmatrix} c_1 & 0 \\ \vdots & \vdots \\ 0 & c_n \end{pmatrix}. \] (35)

Denote \( C_A \) as \( C_u \) and \( C_l \) for the upper and the lower possibility distribution cases, respectively and denote \( c_{uj} \) and \( c_{vj} \) (\( j = 1, \ldots, n \)) as the diagonal elements in \( C_u \) and \( C_l \), respectively. The integrated model can be rewritten as follows.

\[
\begin{align*}
\min_{C_u, C_l} & \sum_{i=1}^{m} z_i^T C_u z_i - \sum_{i=1}^{m} z_i^T C_l z_i \\
\text{s.t.} & \quad z_i^T C_u z_i \geq -\ln h_i, \quad i = 1, \ldots, m, \quad z_i^T C_l z_i \leq -\ln h_i, \quad i = 1, \ldots, m, \\
& \quad c_{uj} \geq c_{vj}, \quad j = 1, \ldots, n,
\end{align*}
\] (36)

where the condition \( c_{uj} \geq c_{vj} \geq \varepsilon > 0 \) makes the matrix \( D_u - D_l \) semi-positive definite and matrices \( D_u \) and \( D_l \) positive. Thus, we have

\[ D_u = T C_u^{-1} T^*, \]
\[ D_l = T C_l^{-1} T^*. \] (37)

This identification procedure is called as PCA method.

In what follows, PCA method is used. It is simpler than the method based on orthogonal conditions.

**Theorem 3.** In the linear programming (LP) problem (36), matrices \( C_u \) and \( C_l \) always exist.

**Proof.** Let us take \( C_u = qI \) and \( C_l = pI \) in Eq. (36). Thus the constraint conditions of Eq. (36) can be written as

\[ \begin{align*}
pz_i^T z_i & \geq -\ln h_i, & i = 1, \ldots, m, \\
qz_i^T z_i & \leq -\ln h_i, & i = 1, \ldots, m, \\
p & \geq q.
\end{align*} \] (38)

If we take \( p = \max_{i=1, \ldots, m-1, j \neq i} (-\ln h_i/z_i z_j), q = \min_{i=1, \ldots, m-1, j \neq i} (-\ln h_i/z_i z_j) \) and \( \varepsilon \leq q \), inequalities (38) can hold. Therefore, there is an admissible set in the constraint conditions of the LP problem (36). It should be noted that the vector \( z_r = 0 \) is omitted, because \( z_r^T z_r = -\ln 1 = 0 \) in Eq. (38). Thus, we consider \( i = 1, \ldots, m-1 \) without \( z_r = 0 \) in determining the values for \( p \) and \( q \). \( \square \)

This theorem implies that using PCA method we always can obtain the matrices \( D_u \) and \( D_l \) in the upper and lower possibility distributions.

**3.2. Verification of our proposed methods**

Assume that the given data \( \{(y_i, h_i), i = 1, \ldots, m, \} \) are obtained from an exponential possibility distribution \( Y = (0, A^\nabla) \) where the center vector is zero. In other words, the following equations hold.

\[ \Pi_Y(y_i) = \exp\{-y_i^T A^\nabla y_i\} = h_i, \quad \text{for } i = 1, \ldots, m. \] (39)

Let us consider the following optimization problem for finding out the upper possibility matrix \( A_u \) and the lower possibility matrix \( A_l \) from the above given data.

\[
\begin{align*}
\min_{A_u, A_l} & \quad J(A_l, A_u) = \sum_{i=1}^{m} y_i^T A_l^{-1} y_i - \sum_{i=1}^{m} y_i^T A_u^{-1} y_i \\
\text{s.t.} & \quad y_i^T A_l^{-1} y_i \geq -\ln h_i, \\
& \quad y_i^T A_u^{-1} y_i \leq -\ln h_i,
\end{align*}
\] (40)

where \( i = 1, \ldots, m \).

**Theorem 4.** The optimal solutions of \( A_u \) and \( A_l \) in the problem (40) are \( A^\nabla \).

**Proof.** The optimization problem (40) can be separated into the following two optimization problems:
max \( A_u \quad J_1(A_u) = \sum_{i=1}^{m} y_i A_u^{-1} y_i \) \( (41) \)

s.t. \( y_i A_u^{-1} y_i \leq -\ln h_i, \quad i = 1, \ldots, m. \)

and

\[
\min_{A_l} \quad J_2(A_l) = \sum_{i=1}^{m} y_i A_i^{-1} y_i \quad (42)
\]

s.t. \( y_i A_i^{-1} y_i \geq -\ln h_i, \quad i = 1, \ldots, m. \)

Since data \((y_i, h_i) (i = 1, \ldots, m)\) are obtained from the exponential possibility distribution \((0, A^\vee)_e\), the data \((y_i, h_i) (i = 1, \ldots, m)\) satisfy the Eq. \((39)\). Therefore, \(A^\vee\) is an admissible solution of Eqs. \((41)\) and \((42)\). Assume that there is another matrix \(A'\) such as \(J_1(A') > J_1(A^\vee)\) in Eq. \((41)\). Then, for some \(i\),

\[
y_i^{1} A^{-1} y_i > y_i^{1} A'^{-1} y_i = -\ln h_i, \quad (43)
\]

which shows that \(A'\) is not admissible. Thus, \(A^\vee\) is the optimal solution of Eq. \((41)\). In the same way, we can prove that the optimal solution of Eq. \((42)\) is also \(A^\vee\). Therefore, both \(A_u\) and \(A_l\) are \(A^\vee\). \(\Box\)

This theorem means that our methods for determining an exponential possibility distribution can obtain the actual matrix \(A^\vee\) if the given data are governed by an exponential possibility distribution with a distribution matrix \(A^\vee\). Moreover, the upper and the lower possibility distributions are equal to \(A^\vee\).

4. Possibility portfolio selection models

The portfolio return can be written as

\[
z = r^t x. \quad (44)
\]

Because \(x\) is governed by a possibility distribution \((a, D^A)_e\), \(z\) is a possibility variable \(Z\). According to the extension principle, the possibility distribution of a portfolio return \(Z\) is defined by

\[
\Pi_Z(z) = \max_{x: x \geq r^t x} \exp\{-(x - a)^t D A^{-1} (x - a)\}. \quad (45)
\]

Solving the optimization problem \((45)\), the possibility distribution of a portfolio return \(Z\) can be obtained as

\[
\Pi_Z(z) = \exp\{-(z - r^t a)^2 (r^t D^A r)^{-1}\}
\]

\[
= (r^t a, r^t D^A r)_e, \quad (46)
\]

where \(r^t a\) is the center value and \(r^t D^A r\) is the spread of a portfolio return \(Z\). Given the lower and the upper possibility distributions, the corresponding portfolio selection models are given as follows:

**Portfolio selection model based on upper possibility distributions.**

\[
\min_{r} \quad r^t D^u r
\]

s.t. \( r^t a = c, \quad (47) \)

\[
\sum_{i=1}^{n} r_i = 1, \quad r_i \geq 0,
\]

**Portfolio selection model based on lower possibility distributions.**

\[
\min_{r} \quad r^t D^l r
\]

s.t. \( r^t a = c, \quad (48) \)

\[
\sum_{i=1}^{n} r_i = 1, \quad r_i \geq 0,
\]

where \(c\) is an expected center value of possibility portfolio return. It is straightforward that models \((47)\) and \((48)\) are quadratic programming problems minimizing the spread of a possibility portfolio return \(Z\).

**Theorem 5.** The spread of the possibility portfolio return based on the lower possibility distribution is not larger than the one based on the upper possibility distribution.

**Proof.** Suppose that the optimal solutions obtained from the problems \((47)\) and \((48)\) are denoted as \(r_u^*\) and \(r_l^*\), respectively, with considering the same center value. According to the feature of the upper and lower possibility distributions, i.e. \(D_u - D_l \geq 0\), the following inequality holds.

\[
r_u^t D_u r_u^* \geq r_l^t D_l r_l^* \quad (49)
\]

Because \(r_l^*\) is the optimal solution of Eq. \((48)\), we have
\[ r_i^\prime D r_i^* \geq r_i^\prime D r_i^0. \]  
As a result,  
\[ r_i^\prime D x^* \geq r_i^\prime D r_i^0, \]  
which proves the theorem. \( \square \)

The nondominated solutions with considering two objective functions, i.e., the spread and the center of a possibility portfolio in the possibility portfolio selection models (47) and (48) can form efficient frontiers.

**Definition 1.** Efficient frontiers from the upper and lower possibility portfolio selection models are called possibility efficient frontier I and possibility efficient frontier II, respectively.

**Definition 2.** Two spreads of possibility portfolio returns from the upper and lower possibility distributions with the same given center value form an interval. This interval is called a possibility risk interval.

The possibility risk interval is used to reflect the uncertainty in portfolio selection problems.

5. **Numerical example**

In order to illustrate the proposed approaches let us consider an example shown in Table 1 introduced by Markowitz [12]. In this example, since we can consider that the recent sample is more similar to the future state, it is assumed that the possibility grade \( h_t \) can be obtained as

\[ h_t = 0.2 + 0.7(t - 1)/17 \quad (t = 1, \ldots, 18). \]  

In Table 1, Nos. 1–9 are American Tobacco, AT&T, United States Steel, General Motors, Atchison&Topeka&Santa Fe, Coca-Cola, Borden, Firestone and Sharon Steel, respectively. The return of the security \( S_j \) during a year is defined as

\[ x_{t,j} = (p_{t+1,j} + d_{t,j} - p_{t,j})/p_{t,j}, \]  
where \( p_{t,j} \) is the closing price of the security \( S_j \) \( (j = 1, \ldots, 9) \) in the year \( t \) and \( d_{t,j} \) is the dividend of this security in the same year. The center vector, and upper and lower possibility distribution matrices were obtained by Eqs. (10), (36) and (37) with \( \varepsilon = 0.001 \) as shown in Table 2.

Using models (47) and (48), we obtained two possibility efficient frontiers shown in Fig. 5. It can

<table>
<thead>
<tr>
<th>( h_t )</th>
<th>Year</th>
<th>#1 Am.T</th>
<th>#2 A.T.&amp;T.</th>
<th>#3 U.S.S.</th>
<th>#4 G.M.</th>
<th>#5 A.T.&amp;S.</th>
<th>#6 C.C</th>
<th>#7 Bdn.</th>
<th>#8 Frstn.</th>
<th>#9 S.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1937(1)</td>
<td>-0.305</td>
<td>-0.173</td>
<td>-0.318</td>
<td>-0.477</td>
<td>-0.457</td>
<td>-0.065</td>
<td>-0.319</td>
<td>-0.4</td>
<td>-0.435</td>
</tr>
<tr>
<td>0.241</td>
<td>1938(2)</td>
<td>0.513</td>
<td>0.098</td>
<td>0.285</td>
<td>0.714</td>
<td>0.107</td>
<td>0.238</td>
<td>0.076</td>
<td>0.336</td>
<td>0.238</td>
</tr>
<tr>
<td>0.282</td>
<td>1939(3)</td>
<td>0.055</td>
<td>0.2</td>
<td>-0.047</td>
<td>0.165</td>
<td>-0.424</td>
<td>-0.078</td>
<td>0.381</td>
<td>-0.093</td>
<td>-0.295</td>
</tr>
<tr>
<td>0.324</td>
<td>1940(4)</td>
<td>-0.126</td>
<td>0.03</td>
<td>0.104</td>
<td>-0.043</td>
<td>-0.189</td>
<td>-0.077</td>
<td>-0.051</td>
<td>-0.09</td>
<td>-0.036</td>
</tr>
<tr>
<td>0.365</td>
<td>1941(5)</td>
<td>-0.28</td>
<td>-0.183</td>
<td>-0.171</td>
<td>-0.277</td>
<td>0.637</td>
<td>-0.187</td>
<td>0.087</td>
<td>-0.194</td>
<td>-0.24</td>
</tr>
<tr>
<td>0.406</td>
<td>1942(6)</td>
<td>-0.003</td>
<td>0.067</td>
<td>-0.039</td>
<td>0.476</td>
<td>0.865</td>
<td>0.156</td>
<td>0.262</td>
<td>1.113</td>
<td>0.126</td>
</tr>
<tr>
<td>0.447</td>
<td>1943(7)</td>
<td>0.428</td>
<td>0.3</td>
<td>0.149</td>
<td>0.225</td>
<td>0.313</td>
<td>0.351</td>
<td>0.341</td>
<td>0.58</td>
<td>0.639</td>
</tr>
<tr>
<td>0.488</td>
<td>1944(8)</td>
<td>0.192</td>
<td>0.103</td>
<td>0.26</td>
<td>0.29</td>
<td>0.637</td>
<td>0.233</td>
<td>0.227</td>
<td>0.473</td>
<td>0.282</td>
</tr>
<tr>
<td>0.529</td>
<td>1945(9)</td>
<td>0.446</td>
<td>0.216</td>
<td>0.419</td>
<td>0.216</td>
<td>0.373</td>
<td>0.349</td>
<td>0.352</td>
<td>0.229</td>
<td>0.578</td>
</tr>
<tr>
<td>0.571</td>
<td>1946(10)</td>
<td>-0.088</td>
<td>-0.046</td>
<td>-0.078</td>
<td>-0.272</td>
<td>-0.037</td>
<td>-0.209</td>
<td>0.153</td>
<td>-0.126</td>
<td>0.289</td>
</tr>
<tr>
<td>0.612</td>
<td>1947(11)</td>
<td>-0.127</td>
<td>-0.071</td>
<td>0.169</td>
<td>0.144</td>
<td>0.026</td>
<td>0.355</td>
<td>-0.099</td>
<td>0.009</td>
<td>0.184</td>
</tr>
<tr>
<td>0.653</td>
<td>1948(12)</td>
<td>-0.015</td>
<td>0.056</td>
<td>-0.035</td>
<td>0.107</td>
<td>0.153</td>
<td>-0.231</td>
<td>0.038</td>
<td>0</td>
<td>0.114</td>
</tr>
<tr>
<td>0.694</td>
<td>1949(13)</td>
<td>0.305</td>
<td>0.038</td>
<td>0.133</td>
<td>0.321</td>
<td>0.067</td>
<td>0.246</td>
<td>0.273</td>
<td>0.223</td>
<td>-0.222</td>
</tr>
<tr>
<td>0.735</td>
<td>1950(14)</td>
<td>-0.096</td>
<td>0.089</td>
<td>0.732</td>
<td>0.305</td>
<td>0.579</td>
<td>-0.248</td>
<td>0.091</td>
<td>0.65</td>
<td>0.327</td>
</tr>
<tr>
<td>0.776</td>
<td>1951(15)</td>
<td>0.016</td>
<td>0.09</td>
<td>0.021</td>
<td>0.195</td>
<td>0.04</td>
<td>-0.064</td>
<td>0.054</td>
<td>-0.131</td>
<td>0.333</td>
</tr>
<tr>
<td>0.818</td>
<td>1952(16)</td>
<td>0.128</td>
<td>0.083</td>
<td>0.131</td>
<td>0.39</td>
<td>0.434</td>
<td>0.079</td>
<td>0.109</td>
<td>0.175</td>
<td>0.062</td>
</tr>
<tr>
<td>0.859</td>
<td>1953(17)</td>
<td>-0.01</td>
<td>0.035</td>
<td>0.006</td>
<td>-0.072</td>
<td>-0.027</td>
<td>0.067</td>
<td>0.21</td>
<td>-0.084</td>
<td>-0.048</td>
</tr>
<tr>
<td>0.9</td>
<td>1954(18)</td>
<td>0.154</td>
<td>0.176</td>
<td>0.908</td>
<td>0.715</td>
<td>0.469</td>
<td>0.077</td>
<td>0.112</td>
<td>0.756</td>
<td>0.185</td>
</tr>
</tbody>
</table>
Table 2
Two possibility distributions
\[ a = [0.154, 0.176, 0.908, 0.715, 0.469, 0.077, 0.112, 0.756, 0.185] \]

\[ \mathbf{D}_a = \]
\[
\begin{array}{cccccccccc}
52.242 & 39.936 & -0.146 & -52.592 & -26.514 & -61.637 & 149.506 & 67.634 & -114.276 \\
-0.146 & -21.404 & 374.000 & -381.493 & 89.657 & 102.189 & 112.117 & -148.463 & 161.681 \\
149.506 & 124.639 & 112.117 & -278.92 & -70.492 & -193.902 & 474.289 & 146.975 & -238.171 \\
67.634 & 73.068 & -148.463 & 100.631 & -81.383 & -153.722 & 146.975 & 159.533 & -183.809 \\
-114.276 & 56.717 & 161.681 & -43.709 & -78.534 & -244.533 & -238.171 & -183.809 & 720.757 \\
\end{array}
\]

\[ \mathbf{D}_b = \]
\[
\begin{array}{cccccccccc}
-61.831 & -218.888 & 101.954 & -47.724 & 245.85 & 559.982 & -194.014 & -154.161 & -244.76 \\
149.395 & 124.615 & 112.314 & -278.868 & -70.628 & -194.014 & 474.186 & 146.923 & -238.328 \\
-114.576 & 56.543 & 161.174 & -44.235 & -79.083 & -244.76 & -238.328 & -184.575 & 720.418 \\
\end{array}
\]

![Fig. 5. Possibility efficient frontiers I and II.](image-url)
be said from Fig. 5 that the spread of the possibility portfolio return from Eq. (47) is always larger than that from Eq. (48). This fact stems from the concept of the lower and the upper possibility distributions. They can be regarded as two extreme opinions playing a reference role for an investor. The corresponding risk with \( c = 0.3 \) is an interval value, i.e., \([0.17978, 0.67318]\), which reflect the uncertainty in real investment problems.

Figs. 6 and 7 show the securities selected by the possibility portfolio selection models (47) and (48) in the case of \( c = 0.3 \), respectively. The result shows that the number of the obtained securities from Eq. (48) is more than the one from Eq. (47). It implies that the portfolio from model (48) tends to take more distributive investment than the one from Eq. (47).

**6. Conclusions**

Markowitz’s model basically assumes that the situation of stock markets in future can be characterized by the past security data. It is difficult to ensure this kind of assumption in real investment problems. On the other hand, possibility portfolio models try to predict the state of stock markets in future by integrating the past security data and experts’ judgement, which can gain an insight into a change of stock markets. This paper proposes an integrated model for obtaining two kinds of possibility distributions, i.e., the upper and the lower possibility distributions to reflect the different viewpoints of experts in portfolio selection problems. The corresponding portfolio selection problems are converted into quadratic programming problems. It can be said that portfolio returns based on lower possibility distributions have smaller possibility spreads than those based on upper possibility distributions. In addition, the investment risk can be described as an interval value to reflect the uncertainty in real investment problems. Because portfolio experts’ knowledge is characterized by the upper and lower possibility distributions, the obtained portfolio would reflect portfolio experts’ judgement.

**References**


