

ON STRICTLY CONVEX AND STRICTLY 2-CONVEX 2-NORMED SPACES II

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ABSTRACT. In this paper a new duality mapping is defined, and it is our object to show that there is a similarity among these three types of characterizations of a strictly convex 2-normed space. This enables us to obtain more new results along each of two types of characterizations. We shall also investigate a strictly 2-convex 2-normed space in terms of the above two different types.

KEY WORDS AND PHRASES: Linear 2-normed space, strict convexity, strict 2-convexity, 2-semi-inner product, bounded linear 2-functional, duality mapping.

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1. INTRODUCTION.

This article is a continuation of the paper by Lin [11] where we investigated characterizations of strictly convex and strictly 2-convex 2-normed spaces which were initiated by Diminnie, Gähler and White [5,6]. The concept of strictly convex 2-normed space is 2-dimensional analogue of that of strictly convex normed linear space, an important space in functional analysis, and a strictly 2-convex 2-normed space is its natural generalization. A strictly convex 2-normed space is strictly 2-convex (Theorem 8 [6] and Theorem 3 [11]). But the converse is not generally true (Example 2 [6]). Note, however, that strict 2-convexity together with a certain condition is equivalent to strict convexity (Theorem 3 [11]). Most elementary 2-normed spaces originated by Gähler [7] are strictly convex. For example, a 2-normed space of dimension 2, and a 2-inner product space [6]. A strictly convex normed linear space may be characterized in terms of norms by Giles [8], semi-inner products by Berkson [1], or duality mappings by Browder [2], Gudder and Strawther [9] and many others. In this paper a new duality mapping is defined, and it is our object to show that there is a similarity among these three types of characterizations of a strictly convex 2-normed space. This enables us to obtain more new results along each of two types of characterizations. We shall also investigate a strictly 2-convex 2-normed space in terms of the above two different types.

Let X denote a real linear space of dimension greater than one, the following standard definition was introduced in [7]. If $\| \cdot, \cdot \|$ is a real function on $X \times X$, then X is called a 2-normed space with a 2-norm $\| \cdot, \cdot \|$ if the following conditions are satisfied:

- (i) $\| x, y \| = 0$ if and only if x and y are linearly dependent;
- (ii) $\| x, y \| = \| y, x \|$;
- (iii) $\| ax, y \| = |a| \| x, y \|$ for any real a ; and

$$(iv) \quad \|x + y, z\| \leq \|x, z\| + \|y, z\|.$$

Let X be a 2-normed space throughout this paper. If $x, y, z \in X$ are nonzero vectors, we denote by $V(x)$, $V(x, y)$ and $V(x, y, z)$ the linear manifolds of X generated by x , x and y , x and z , respectively.

2. STRICTLY CONVEX 2-NORMED SPACES.

Recall from [5] that X is said to be strictly convex if $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| = 1$ for $z \notin V(x, y)$

implies $x = y$. In this section we shall give several characterizations of this space in terms of 2-semi-inner products and duality mappings. But first we need the following lemma which is essential to our consequent theorems, and which is a portion of Theorem 1 in [11] plus three new statements (8), (9), and (10).

LEMMA 1. The following ten statements are equivalent:

- (1) X is strictly convex;
- (2) $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ for $z \notin V(x, y)$ implies $x = y$;
- (3) $\|x + y, z\| = \|x, z\| + \|y, z\|$ for $z \notin V(x, y)$ implies $x = by$ for some $b > 0$;
- (4) $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| \neq 0$ for $x \neq y$ implies $z = d(x - y)$ for some $d \neq 0$;
- (5) $\|x + ay, z\| = 2\|x, z\|$ for $z \notin V(x, y)$ and $a = \|x, z\|/\|y, z\|$ implies $x = ay$;
- (6) $\|x + y, z\| = \|x, z\| + \|y, z\|$ for $z \notin V(x, y)$ implies $\|y, z\|x = \|x, z\|y$;
- (7) $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| \neq 0$ for $x \neq y$ implies $\|x, y\| \neq 0$ and $z = \pm\|x, z\|(x - y)/\|x, y\|$;
- (8) $\|w + x, z\| = \|w + y, z\| \neq 0$ for all $w \in X$ implies $x = y$;
- (9) $\|x - y, z\| = |\|x, z\| - \|y, z\||$ for $z \notin V(x, y)$ implies $x = sy$ for some $s > 0$;
- (10) $\|x - y, z\| = |\|x, z\| - \|y, z\||$ for $z \notin V(x, y)$ implies $\|y, z\|x = (\|x - y, z\| + \|y, z\|)y$.

PROOF. The equivalence of (1) through (7) was proved in (Theorem 1 [11]), and that (10) \Rightarrow (9) is obvious. That (9) \Rightarrow (3) is clear after we verify the implication (6) \Rightarrow (10).

(6) \Rightarrow (10): We may write the relation in (10) as $\|x, z\| = \|x - y, z\| + \|y, z\|$. So $\|y, z\|(x - y) = \|x - y, z\|y$ by (6) and the result follows.

(2) \Rightarrow (8): Let $w = x$ and $w = y$ in (8), then $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ for $z \notin V(x, y)$ implies $x = y$ by (2).

(8) \Rightarrow (2): Suppose that $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ for $z \notin V(x, y)$ and $x \neq y$, then $\|w + x, z\| = \|w + y, z\| \neq 0$ for some $w \in X$ (indeed, $w = x$ and $w = y$) and $x \neq y$, i.e., (8) does not hold.

The concept of 2-semi-inner product defined by Siddiqui and Rizvi [14] is 2-dimensional analogue of that of the usual semi-inner product in functional analysis. A 2-semi-inner product is a mapping $[\cdot, \cdot | \cdot]$ on $X \times X \times X$ into real numbers such that

- (i) $[x + x', y | z] = [x, y | z] + [x', y | z]$;
- (ii) $[ax, y | z] = a[x, y | z]$ for any real a ;
- (iii) $[x, x | z] \geq 0$; $[x, x | z] = 0$ if and only if x and z are linearly dependent; and
- (iv) $|[x, y | z]|^2 \leq [x, x | z][y, y | z]$.

Every 2-normed space can be made into a 2-semi-inner product space, and the norm is given by $\|x, y\| = [x, x | y]^{\frac{1}{2}}$ [14].

THEOREM 1. The following nine statements are equivalent:

- (1) X is strictly convex (in the sense of Lemma 1);
- (2) $[x, y | z] = \|x, z\| = \|y, z\| = 1$ for $z \notin V(x, y)$ implies $x = y$;
- (3) $[x, y | z] = \|x, z\|^2 = \|y, z\|^2$ for $z \notin V(x, y)$ implies $x = y$;
- (4) $[w, x | z] = [w, y | z]$ for $z \notin V(x, y, w)$ and all $w \in X$ implies $x = y$;
- (5) $[ax, y | z] = \|x, z\|^2$ for $z \notin V(x, y)$ implies $x = ay$ for some $a > 0$, and $a = 1$ if $\|x, z\| = \|y, z\|$;
- (6) $[x, y | z] = \|x, z\| \|y, z\|$ for $z \notin V(x, y)$ implies $x = ay$ for some $a > 0$;
- (7) $[x, y | z] = \|x, z\|^2 = \|y, z\|^2 \neq 0$ for $x \neq y$ implies $z = d(x - y)$ for some $d \neq 0$;
- (6') $[x, y | z] = \|x, z\| \|y, z\|$ for $z \notin V(x, y)$ implies $\|y, z\| x = \|x, z\| y$;
- (7') $[x, y | z] = \|x, z\|^2 = \|y, z\|^2 \neq 0$ for $x \neq y$ implies $\|x, y\| \neq 0$ and $z = \pm \|x, z\| (x - y) / \|x, y\|$.

PROOF. The following implications are routine: (2) \Leftarrow (5) \Leftarrow (6') \Rightarrow (6) \Rightarrow (3) \Rightarrow (2) and (7') \Rightarrow (7).

So let us prove that (3) \Rightarrow (4) \Rightarrow (1) \Rightarrow (6'), (2) \Rightarrow (1) \Rightarrow (7') and (7) \Rightarrow (1).

(1) \Rightarrow (6'): Let $[x, y | z] = \|x, z\| \|y, z\|$ for $z \notin V(x, y)$, then $(\|x, z\| + \|y, z\|) \|y, z\| = [x + y, y | z] \leq \|x + y, z\| \|y, z\| \leq (\|x, z\| + \|y, z\|) \|y, z\|$, or $\|x + y, z\| = \|x, z\| + \|y, z\|$. Hence $\|y, z\| x = \|x, z\| y$ by (6) in Lemma 1.

(3) \Rightarrow (4): Let $w = x$ in (4), then $\|x, z\|^2 = [x, y | z] \leq \|x, z\| \|y, z\|$, or $\|x, z\| \leq \|y, z\|$. If $w = y$, then $\|y, z\| \leq \|x, z\|$ similarly. Hence $\|x, z\| = \|y, z\|$ and $x = y$ by (3).

(4) \Rightarrow (1): Suppose that X is not strictly convex, i.e., $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ for $z \notin V(x, y)$ and $x \neq y$, we have to show that $[w, x | z] = [w, y | z]$ for $z \notin V(x, y, w)$ and some z 's implies $x = y$. Since $\|x, z\| = \|y, z\|$ by the proof (3) \Rightarrow (4) we have $[x, y | z] = \|x, z\| \|y, z\|$. As in the proof (1) \Rightarrow (6') we conclude that $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$.

(2) \Rightarrow (1): Let $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| = 1$ and $x \neq y$, then, with the aid of the proof (1) \Rightarrow (6'), we can show easily that $[x, y | z] = \|x, z\| = \|y, z\| = 1$ implies $x = y$.

(1) \Rightarrow (7'): Let $x \neq y$ and $[x, y | z] = \|x, z\|^2 = \|y, z\|^2$, so $[x, y | z] = \|x, z\| \|y, z\|$, then $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ by the proof (1) \Rightarrow (6'). Hence $\|x, z\| \neq 0$ and $z = \pm \|x, z\| (x - y) / \|x, y\|$ by (7) in Lemma 1.

(7) \Rightarrow (1): Suppose by contrapositive that (4) in Lemma 1 does not hold, then by the proof (1) \Rightarrow (6') it is easily seen that (7) does not hold, and the proof of the theorem is complete.

Motivated by the concepts of bounded linear functionals, and duality mappings on normed linear spaces [2, 9], bounded linear 2-functionals on 2-normed spaces were introduced by White [15], and associated duality mappings were defined in [3]. Let M and N be linear manifolds of X , a bounded linear 2-functional is a mapping f on $M \times N$ into real numbers such that

- (i) $f(x + x', y + y') = f(x, y) + f(x, y') + f(x', y) + f(x', y')$;
- (ii) $f(ax, by) = abf(x, y)$ for any real numbers a and b ; and
- (iii) $|f(x, y)| \leq k\|x, z\|$ for some $k \geq 0$ and all $(x, y) \in M \times N$.

In this case the norm of f is defined by

$$\|f\| = \inf\{k: |f(x, y)| \leq k\|x, y\|, (x, y) \in M \times N\}.$$

It can be shown that $|f(x, y)| \leq \|f\| \|x, y\|$ and $f(x, y) = 0$ if $x \in V(y)$ [15]. We need also a result which is similar to the Hahn-Banach theorem of functional analysis: If $x, z \in X$ and $x \notin V(z)$, then there exists a bounded linear 2-functional f on $X \times V(z)$ such that $f(x, z) = \|x, z\|$ and $\|f\| = 1$ [6, 13, 15].

The following duality mappings defined in [3] are 2-dimensional analogues of usual duality mappings on a normed linear space [2, 9]:

$$I(x, z) = \{f \in X_z^*: f(x, z) = \|f\| \|x, z\|\} \text{ and}$$

$$J(x, z) = \{f \in X_z^*: f(x, z) = \|f\| \|x, z\|, \|f\| = \|x, z\|\}$$

with duality mappings $I, J: X \times V(z) \rightarrow 2^{X_z^*}$, where X_z^* is the space of all bounded linear 2-functionals on $X \times V(z)$.

Evidently the following assertions are true: (a) $J(x, z) \subseteq I(x, z)$; (b) $I(x, z) = X_z^*$ if and only if $x \in V(z)$; (c) $I(x, z) = I(cx, dz) = cdI(x, z)$ for $c, d > 0$; (d) $0 \neq f \in I(x, z)$ for $x \notin V(z)$ implies $f \in J(cx, z)$ for some $c > 0$; and (e) If $x \notin V(z)$, then there exists an $f \in J(x, z)$ with $f \neq 0$ (by the Hahn-Banach theorem stated in above).

Let us define another type of duality mapping as follows:

DEFINITION. Let $I'(x, z)$ be the same as $I(x, z)$ which has the following additional properties:

- (i) $\|x, z\| \geq \|y, z\|$ if and only if $\|f\| \geq \|g\|$ for $z \notin V(x, y)$, $f \in I(x, z)$ and $g \in I(y, z)$; and
- (ii) $\|x, z\| \geq \|x, w\|$ if and only if $\|f\| \geq \|h\|$ for $x \notin V(z, w)$, $f \in I(x, z)$ and $h \in I(x, w)$.

It follows easily from (i) that $f \in I'(x, z) \cap I'(y, z)$ for $z \notin V(x, y)$ if and only if $f(x, z) = \|f\| \|x, z\|$, $f(y, z) = \|f\| \|y, z\|$ and $\|x, z\| = \|y, z\|$. A similar result from (ii) is obtainable.

LEMMA 2. If $0 \neq f \in I'(x, z)$, $0 \neq g \in I'(y, z)$ for $x \neq y$ and $z \notin V(x, y)$, then

- (1) $(f - g)(x - y, z) \geq 0$;
- (2) $(f - g)(x - y, z) = 0$ if and only if $f(y, z) = \|f\| \|y, z\|$, $g(x, z) = \|g\| \|x, z\|$ and $\|x, z\| = \|y, z\|$;
- (3) $(f - g)(x - y, z) = 0$ if and only if $f, g \in I'(x, z) \cap I'(y, z)$.

PROOF. (1) and (2) are straightforward computations and can be found in ([10] p. 379). Indeed, $(f - g)(x - y, z) = (\|f\| - \|g\|)(\|x, z\| - \|y, z\|) + [\|f\| \|y, z\| - f(y, z)] + [\|g\| \|x, z\| - g(x, z)] \geq 0$. (3) is consequences of (2) and a previous remark.

In a similar manner we can prove the following analogous result.

LEMMA 3. If $0 \neq f \in I'(x, z)$, $0 \neq g \in I'(x, w)$ for $z \neq w$ and $x \notin V(z, w)$, then

- (1) $(f - g)(x, z - w) \geq 0$;
- (2) $(f - g)(x, z - w) = 0$ if and only if $f(x, w) = \|f\| \|x, w\|$, $g(x, z) = \|g\| \|x, z\|$ and $\|x, z\| = \|x, w\|$;
- (3) $(f - g)(x, z - w) = 0$ if and only if $f, g \in I'(x, z) \cap I'(x, w)$.

Obviously, I' in Lemma 2 and 3 may be replaced by J . Let $\#$ denote the inclusion relation \subseteq, \supseteq or

- .

THEOREM 2. If $x, y \neq 0$, then the following thirteen statements are equivalent:

- (1) X is strictly convex (in the sense of Lemma 1);

- (2) $I(x,z) \cap I(y,z) \neq \emptyset$ for $z \notin V(x,y)$ implies $x = ay$ for some $a > 0$;
- (3) $I(x,z) \# I(y,z)$ for $z \notin V(x,y)$ implies $x = ay$ for some $a > 0$;
- (4) $J(x,z) \cap J(y,z) \neq \emptyset$ for $z \notin V(x,y)$ implies $x = y$;
- (5) $J(x,z) \# J(y,z)$ for $z \notin V(x,y)$ implies $x = y$;
- (6) $I'(x,z) \cap I'(y,z) \neq \emptyset$ for $z \notin V(x,y)$ implies $x = y$;
- (7) $I'(x,z) \# I'(y,z)$ for $z \notin V(x,y)$ implies $x = y$;
- (8) If $0 \neq f \in I'(x,z)$ and $0 \neq g \in I'(y,z)$ for $x \neq y$ and $z \notin V(x,y)$, then $(f-g)(x-y,z) > 0$;
- (9) $J(x,z) \cap J(y,z) \neq \emptyset$ for $x \neq y$ implies $z = d(x-y)$ for some $d \neq 0$;
- (2') $I(x,z) \cap I(y,z) \neq \emptyset$ for $z \notin V(x,y)$ implies $\|y,z\|x = \|x,z\|y$;
- (3') $I(x,z) \# I(y,z)$ for $z \notin V(x,y)$ implies $\|y,z\|x = \|x,z\|y$;
- (8') If $0 \neq f \in J(x,z)$ and $0 \neq g \in J(y,z)$ for $x \neq y$ and $z \notin V(x,y)$, then $(f-g)(x-y,z) > 0$;
- (9') $J(x,z) \cap J(y,z) \neq \emptyset$ for $x \neq y$ implies $\|x,y\| \neq 0$ and $z = \pm \|x,z\| (x-y) / \|x,y\|$.

PROOF. The proof of (2') \Rightarrow (2) \Rightarrow (3), (2') \Rightarrow (3') \Rightarrow (3) and (9') \Rightarrow (9) are trivial. Equivalences of (1), (4), (5), (6) and (7) are clear after we verify the implications (3) \Rightarrow (1) \Rightarrow (2'). (8') is, of course, a special case of (8).

(1) \Rightarrow (2'): Let $0 \neq f \in I(x,z) \cap I(y,z) = I(x,z) \cap I(\|x,z\|y/\|y,z\|,z)$, then $\|f\| \|x + (\|x,z\|y/\|y,z\|), z\| \geq f(x + (\|x,z\|y/\|y,z\|), z) = 2\|f\| \|x,z\| \geq \|f\| \|x + (\|x,z\|y/\|y,z\|), z\|$, or $\|x + (\|x,z\|y/\|y,z\|), z\| = 2\|x,z\|$ and hence $\|y,z\|x = \|x,z\|y$ by (5) in Lemma 1.

(3) \Rightarrow (1): Without loss of generality we may assume that $0 \neq f \in I(x,z) \subseteq I(y,z)$ in (3). Suppose that $\|x + y, z\| = \|x,z\| + \|y,z\|$ and $x \neq by$ for all $b > 0$, i.e., the negation of (3) in Lemma 1, we have to show that $f \in I(x,z) \subseteq I(y,z)$ implies $x \neq by$ for all $b > 0$. This follows from the relation $\|f\| \|x + y, z\| \geq f(x + y, z) = \|f\| (\|x,z\| + \|y,z\|) \geq \|f\| \|x + y, z\|$, or $\|x + y, z\| = \|x,z\| + \|y,z\|$.

(6) \Rightarrow (8): Let $0 \neq f \in I'(x,z)$, $0 \neq g \in I'(y,z)$, $x \neq y$, $z \notin V(x,y)$ and $(f-g)(x-y,z) = 0$, then $f \in I'(x,z) \cap I'(y,z)$ by Lemma 2, and $x \neq y$. Thus (6) does not hold.

(8) \Rightarrow (6): If $f \in I'(x,z) \cap I'(y,z)$ and if $x \neq y$, then $0 = (f-f)(x-y,z) > 0$ by (8) yielding a contradiction.

(1) \Rightarrow (9'): For $x \neq y$ let $0 \neq f \in J(x,z) \cap J(y,z)$, then $\|x,z\| = \|y,z\| = \|f\| \neq 0$. It follows easily that $\frac{1}{2}\|x + y, z\| = \|x,z\| = \|y,z\| \neq 0$. Hence $\|x,y\| \neq 0$ and $z = \pm \|x,z\| (x-y) / \|x,y\|$ by (7) in Lemma 1.

(9) \Rightarrow (1): Consider the negation of (4) in Lemma 1, i.e., $\frac{1}{2}\|x + y, z\| = \|x,z\| = \|y,z\| \neq 0$, $x \neq y$ and $z \neq d(x-y)$ for all $d \neq 0$, then as in the proof (1) \Rightarrow (9') we can easily conclude that (9) does not hold.

REMARKS. (a) That $J(x,z) \cap J(y,z) \neq \emptyset$ in (9) and (9') above may be replaced, of course, by $J(x,z) \# J(y,z)$ without any other change in the statements; (b) J in (9) and (9') may be replaced by I' if $\|x,z\|$ or $\|y,z\| \neq 0$ in addition to the conditions; (c) Though (2) appeared in ([3] Theorem 1), our proof is direct and much simpler. (4) is in ([3] Corollary 3). (8) was discussed in ([10] Theorem 2.5) with a different type of duality mapping; (d) Note that a duality mapping which satisfies the statement (8) is said to be strictly monotone [10] (cf. [2, 9]). In other words, X is strictly convex if and only if I' or J is strictly monotone.

3. STRICTLY 2-CONVEX 2-NORMED SPACES.

According to [6] X is said to be strictly 2-convex if $\|x+z, y+z\|/3 = \|x, y\| = \|y, z\| = \|z, x\| = 1$ implies $z = x + y$. We now turn to the investigation of this space in terms of 2-semi-inner products and duality mappings. To this end we require first the next result which is a portion of Theorem 2 in [11].

LEMMA 4. The following four statements are equivalent:

- (1) X is strictly 2-convex;
- (2) $\|x+z, y+z\| = \|x, y\| + \|y, z\| + \|z, x\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$ for some $b, c > 0$;
- (3) $\|bx+z, cy+z\| = 3\|bx, z\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$, where $b = \|y, z\|/\|x, y\|$ and $c = \|x, z\|/\|x, y\|$.
- (4) $\|x+z, y+z\| = \|x, y\| + \|y, z\| + \|z, x\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$, where b and c are as in (3).

In order to be able to prove the next theorem we shall use one of the basic properties of a 2-norm that $\|ax + by, y\| = |a| \|x, y\|$ for any real numbers a and b [7].

THEOREM 3. The following five statements are equivalent:

- (1) X is strictly 2-convex (in the sense of Lemma 4);
- (2) $[-x, y | y+z] = (\|x, y\| + \|x, z\|) \|y, z\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$ for some $b, c > 0$;
- (3) $\frac{1}{2}[-x, y | y+z] = \|x, y\|^2 - \|y, z\|^2 - \|z, x\|^2 \neq 0$ implies $z = x + y$;
- (4) $\frac{1}{2}[-x, y | y+z] = \|x, y\| = \|y, z\| = \|z, x\| = 1$ implies $z = x + y$;
- (2') $[-x, y | y+z] = (\|x, y\| + \|x, z\|) \|y, z\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$, where $b = \|y, z\|/\|x, y\|$ and $c = \|x, z\|/\|x, y\|$.

PROOF. The following implications are trivial: (2') \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).

(1) \Rightarrow (2)': If $[-x, y | y+z] = (\|x, y\| + \|x, z\|) \|y, z\|$, then $(\|x, y\| + \|y, z\| + \|z, x\|) \|y, z\| = [y - x, y | y+z] \leq \|y - x, y+z\| \|y, z\| = \|(y+z) - (x+z), y+z\| \|y, z\| = \|x+z, y+z\| \|y, z\| \leq (\|x, y\| + \|y, z\| + \|z, x\|) \|y, z\|$, or $\|x+z, y+z\| = \|x, y\| + \|x, z\| + \|y, z\|$ and the result follows by (4) in Lemma 4.

(4) \Rightarrow (1): If $\|x+z, y+z\|/3 = \|x, y\| = \|y, z\| = \|z, x\| = 1$ and $z \neq x + y$, we have to show that $\frac{1}{2}[-x, y | y+z] = \|x, y\| = \|y, z\| = \|z, x\| = 1$ implies $z = x + y$. But this is clear from the proof in above.

THEOREM 4. In the following let $I(u, v)$, $J(u, v)$ and $I'(u, v)$ be defined as in the previous section, and let $u \notin V(v)$, then the following seven statements are equivalent:

- (1) X is strictly 2-convex (in the sense of Lemma 4);
- (2) $I(x, y) \cap I(x, z) \cap I(z, y) \neq \emptyset$ implies $z = bx + cy$ for some $b, c > 0$;
- (3) $J(x, y) \cap J(x, z) \cap J(z, y) \neq \emptyset$ implies $z = x + y$;
- (4) $I'(x, y) \cap I'(x, z) \cap I'(z, y) \neq \emptyset$ implies $z = x + y$;
- (5) If $0 \neq f \in I'(x, y)$, $0 \neq g \in I'(x, z)$ and $0 \neq h \in I'(z, y)$ for $z = x + y$, then $(f-h)(x-z, y)$ and $(f-g)(x, y-z) > 0$;
- (2') $I(x, y) \cap I(x, z) \cap I(z, y) \neq \emptyset$ implies $z = bx + cy$ for $b = \|y, z\|/\|x, y\|$ and $c = \|x, z\|/\|x, y\|$;

(5') If $0 \neq f \in J(x, y)$, $0 \neq g \in J(x, z)$ and $0 \neq h \in J(z, y)$ for $z \neq x + y$, then $(f - h)(x - z, y)$ and $(f - g)(x, y - z) > 0$.

PROOF. That (2') \Rightarrow (2) is trivial. (5') is a special case of (5), and it is clear that we need to verify that (2) \Rightarrow (1) \Rightarrow (2') and (4) \Leftrightarrow (5) only.

(1) \Rightarrow (2)': Let $0 \neq f \in I(x, y) \cap I(x, z) \cap I(z, y) = I(bx, cy) \cap I(bx, z) \cap I(z, cy)$, where $b = \|y, z\| / \|x, y\|$ and $c = \|x, z\| / \|x, y\|$, then $\|f\| \|bx + z, cy + z\| \leq \|f\| (\|bx, cy\| + \|bx, z\| + \|z, cy\|) = f(bx + z, cy + z)$, $\leq \|f\| \|bx + z, cy + z\|$, or $\|bx + z, cy + z\| = \|bx, cy\| + \|bx, z\| + \|z, cy\| = 3\|bx, z\|$, and $\|x, y\| \|y, z\| \|z, x\| \neq 0$ by assumption. So $z = bx + cy$ by (3) in Lemma 4.

(2) \Rightarrow (1): Consider the negation of (2) in Lemma 4, i.e., $\|x + z, y + z\| = \|x, y\| + \|x, z\| + \|z, y\|$, $\|x, y\| \|y, z\| \|z, x\| \neq 0$ and $z \neq bx + cy$ for all $b, c > 0$, we have to show that $0 \neq f \in I(x, y) \cap I(x, z) \cap I(z, y)$ implies $z \neq bx + cy$ for all $b, c > 0$. This follows from the relation $\|f\| \|x + z, y + z\| \geq f(x + z, y + z) = \|f\| (\|x, y\| + \|x, z\| + \|z, y\|) \geq \|f\| \|x + z, y + z\|$, or $\|x + z, y + z\| = \|x, y\| + \|x, z\| + \|z, y\|$.

(4) \Rightarrow (5): Let $0 \neq f \in I'(x, y)$, $0 \neq g \in I'(x, z)$, $0 \neq h \in I'(z, y)$, $(f - h)(x - z, y) = 0 = (f - g)(x, y - z)$ and $z \neq x + y$, i.e., the negation of (5), then $f \in I'(x, y) \cap I'(z, y) \cap I'(x, z)$ by Lemma 2 and 3, and $z \neq x + y$. Thus (4) does not hold.

(5) \Rightarrow (4): If $f \in I'(x, y) \cap I'(x, z) \cap I'(z, y)$ and suppose that $z \neq x + y$, then $0 = (f - f)(x - z, y) > 0$ by (5) yielding a contradiction, and the proof of the theorem is complete.

REMARK. (2) in Theorem 4 appeared in ([4] Theorem 1.2) except that the domain of the duality mapping I has been changed. The change is unnecessary.

REFERENCES

1. BERKSON, E., Some Types of Banach Spaces, Hermitian Operators, and Bade Functionals, Trans. Amer. Math. Soc. **116** (1965), 376-385.
2. BROWDER, F. E., Multi-valued Monotone Nonlinear Mappings and Duality Mappings in Banach Spaces, Trans. Amer. Math. Soc. **118** (1965), 338-351.
3. CHO, Y. J., HA, K. S. and KIM, W. S., Strictly Convex Linear 2-Normed Spaces, Math. Japon. **26** (1981), 475-478.
4. CHO, Y. J., PARK, B. H. and PARK, K. S., Strictly 2-Convex Linear 2-Normed Spaces, Math. Japon. **24** (1982), 609-612.
5. DIMINNIE, C., GAHLER, S. and WHITE, A., Strictly Convex Linear 2-Normed Spaces, Math. Nachr. **59** (1974), 319-324.
6. DIMINNIE, C., GAHLER, S. and WHITE, A., Remarks on Strictly Convex and Strictly 2-Convex 2-Normed Spaces, Math. Nachr. **88** (1979), 363-372.
7. GAHLER, S., Linear 2-Normierte Räume, Math. Nachr. **28** (1965), 1-43.
8. GILES, J. R., Class of Semi-Inner-Product Spaces, Trans. Amer. Math. Soc. **129** (1967), 436-446.
9. GUDDER, S. and STRAWTHER, D., Strictly Convex Normed Linear Spaces, Proc. Amer. Math. Soc. **52** (1976), 263-267.
10. HA, K. S., CHO, Y. J. and WHITE, A., Strictly Convex and Strictly 2-Convex 2-Normed Spaces, Math. Japon. **33** (1988), 375-384.
11. LIN, C. -S., On Strictly Convex and Strictly 2-Convex 2-Normed Spaces, Math. Nachr., to appear.
12. LIN, C. -S., On Strictly Convex and Normally Convex Spaces, preprint.
13. MABIZELA, S., On Bounded Linear 2-Functionals, Math. Japon. **35** (1990), 51-55.
14. SIDDIQUI, A. H. and RIZVI, S. M., 2-Semi-Inner Product Spaces I, Math. Japon. **21** (1976), 391-397.
15. WHITE, A., 2-Banach Spaces, Math. Nachr. **42** (1969), 43-60.

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