

Disentangling the Demographic Determinants of the English Take-off: 1530-1860

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Abstract

We propose a model with some of the main demographic, economic and institutional factors usually considered to matter in the transition to modern growth. We apply our theory to England over the period 1530-1860. We use the model to measure the impact of mortality, population density and technological progress on school foundations, literacy and growth through a set of experiments. We find that one third of the rise in literacy over the period 1530-1850 can be directly related to the rise in population density, while one sixth is linked to higher longevity and one half to exogenous total factor productivity growth. Moreover, the timing of the effect of population density in the model is consistent with the available evidence for England, where it is shown that schools were established at a high rate over the period 1540-1620.

JEL codes: O41, I21, R12, J11.

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1 Introduction

Economic growth, understood as the increase of the gross domestic product over a long period of time, is a contemporaneous phenomenon. As clearly explained in Maddison (2001), humanity has been caught in a very long lasting trap of economic stagnation till the 19th century. This period was accompanied by an even more catastrophic demographic situation: according to the estimations of Maddison, the size of the world population has almost remained constant in the first millennium, and life expectancy (at birth) has been systematically below fifty years till the beginning of the 20th century.

For European countries, such an evolution is traditionally associated with successive mortality crises due to wars, famines and epidemics. Recent essays due to Platt (1996) for the England case and Herlihy (1997) have stressed the structural changes induced by the bubonic plague in late medieval ages. However, since Wrigley and Schofield (1989), the view that a unitary premodern demographic regime has preceded the industrial revolution is seriously undermined, specially in the England case ¹: births, marriages and mortality have sharply fluctuated two centuries before the industrial revolution, and such variations cannot be fully attributed to mortality crises. While mortality decline has significantly contributed to population growth in England, its effect was clearly overshadowed by rising fertility until 1820, driven notably by increased nuptiality. Another major conclusion to draw from Wrigley and Schofield's careful empirical work in the England case is that mortality crises during the period studied show up a low correlation with food scarcity, and thus with standard of living of population.

Yet several issues remain unsettled on several grounds. In particular, even if we agree with the demographic appraisal above, an ultimate step is to incorporate it into a more global analysis of the transition to modern growth, and to identify the involved economic and demographic mechanisms. Incidentally, the demographic determinants have become increasingly advocated in the development literature (see Lee (1979) and McNicoll (2003)). The *unified growth theory*, a recent stream of economic growth literature, surveyed by Galor (2005), puts a special emphasis on the

¹England is not an exception though: Ireland and Holland for example have some common historical demography trends with England, notably as to the preeminent role of fertility in the acceleration of population growth compared to mortality in the middle of the 18th century. Sweden and France exhibit a completely different picture.

role of demographics in the transition to the modern economic growth regime. This paper highlights specially the role of population density, among other demographic and economic factors, in the economic development of England during the period 1530-1860. Since industrial revolutions rely on innovations, and the adoption of new technologies requires a certain density of educated people, population density and literacy are likely to be key variables in the development process. Indeed, the rise in literacy and education in the pre-industrial era may have initiated the process leading to the Industrial Revolution. Cipolla (1969) argues that improvements in literacy favored the Industrial Revolution in more than one way. It avoided shortages of skilled workers in those fields in which such workers were specifically required and, more generally, it made more people adaptable to new circumstances and receptive to change. In times of fast technological progress, literate workers assimilate new ideas more readily.

Higher education achievements might have been triggered by several economic and demographic factors. We distinguish three of them. First, technological progress increased labor productivity and wage rates in the modern sector, and thereby the return to investment in education. Facing better income perspectives in this sector, households would engage in education to benefit from the higher returns. This view is defended, among others, by Hansen and Prescott (2002), and Doepke (2004). Second, improvements in adult longevity is another interesting candidate for explaining the rise in literacy. Though, according to Wrigley and Schofield (1989), mortality decline is not the main engine of population growth in pre-industrial England, increasing longevity is potentially an important determinant of literacy: longer lives increase the rate of returns to investment in education, inducing longer schooling, according to the well known Ben-Porath mechanism. Two recent papers by Boucekine, de la Croix, and Licandro (2003) and Nicolini (2004) argue that lower mortality induced higher investment in human capital and/or in physical capital at the time of the industrial revolution, therefore paving the way to future growth. On his side, Clark (2005a) remarks that, if demographic transition and industrial revolution are the two great forces that lead to modern growth, the latter did not lead to fertility decline till over 100 years later. He explores the difficulties when trying to uncover the underlying connection between them.

A third possibility is drawn from various authors who stress that the rising density of population may have played a role in fostering the take-off. Higher density can lower

the cost of education through facilitating the creation of schools. Fujita and Thisse (2002) provide a textbook treatment of this effect. A representative empirical study is the one of Ladd (1992) according to whom a small increase in density lowers the costs of providing services, at least at very low levels of population density. There can also be externalities generated by denser population. For Kremer (1993), high population spurs technological change. For Galor and Weil (2000) and Lagerloef (2003) there is a “population-induced” technological progress. Population needs to reach a threshold for productivity to take-off.

In this paper we propose a new framework in order to disentangle the effects of the three factors on literacy and economic growth in England, 1530-1860. In our theoretical framework, we introduce the three channels outlined above: technological progress, mortality drops and population density. By looking at the linkages between literacy, school establishment and income growth, we will be able to evaluate the role of each of them. In our model, the length of schooling is chosen by individuals who maximize life-time income, which depends on future wages, longevity, and the distance to the nearest school. Then, the number and location of education facilities is determined, either chosen optimally by the state or following a free entry process (market solution). Higher population density makes it optimal to increase school density, opening the possibility to reach higher educational levels.

In our model, two sectors may coexist: a traditional sector with constant productivity, and a modern sector with exogenously rising productivity. Moreover, the remuneration of the workers in the modern sector also depends on their human capital level. Therefore the transition to this modern sector depend on both technological evolution and on education. As explained just above, a denser population induces a higher education level, which fastens the switch to the modern sector. Wrigley (1989) depicts a more detailed and concrete picture of industrial revolution. A traditional sector, called *organic economy*, merely based on agricultural goods, can eventually evolve into a definitely more productive sector, an *advanced organic economy*, thanks to animal power. However, such a regime cannot be sustainable because of fixed land supply and decreasing marginal returns. The *chance* of England, argued Wrigley, was to have abundant coal resources, which made somewhat comfortable the *change*, namely the transition to a *mineral-based economy*, which industries (producing iron, pottery or glasses) could be operated without significant pressure on land, allowing

therefore to escape from decreasing returns. Wrigley provides some empirical support to his story, specially based on collected investment data over the period studied. Indeed, the mineral-based economy opened the door to a series of innovations (notably in energy and power production) which brought productivity growth and real wages much above the figures allowed by the agricultural economy. Our modeling of the traditional and modern sectors is definitely much more stylized than Wrigley's description. It captures however a central message in the latter: the transition to modern economy features primarily an escape from decreasing returns. While in Wrigley, this transition is made possible by the much broader set of technological opportunities allowed by the mineral-based economy, it is additionally favored by human capital accumulation, itself boosted by increasing population density, in our story. As we have explained above, there are numerous good reasons to believe that rising literacy was a key factor in the dawn of the modern growth regime.

The paper is organized as follows. Section 2 gives the demographic, economic, geographic and institutional structures of our theoretical model. Section 3 describes the data and the experimental methodology we use to disentangle the effects of the three factors mentioned above on literacy and growth. Section 4 displays the findings and section 5 contains some concluding comments.

2 Theory

In order to assess the development mechanisms outlined in the introduction, we first build up a theoretical model with the relevant demographic, economic, institutional and geographic ingredients. Hereafter, we display the main assumptions adopted. The whole mathematical setting, including the rigorous proofs of the claims made along Section 2 of this paper, can be found in Boucekkine, de la Croix, and Peeters (2007).

2.1 The demographic structure

We shall consider an economy populated by overlapping cohorts. Individuals belonging to the cohort t , that is individuals born at date t , have an uncertain lifetime:

their probability of reaching age a , is given by the following survival function:

$$m_t(a) = \frac{e^{\beta_t a} - \alpha_t}{1 - \alpha_t},$$

where α_t and β_t are two numbers (for fixed t). This survival function has been introduced by Boucekkine, de la Croix, and Licandro (2002). If α_t and β_t satisfy: $\alpha_t > 1$ and $\beta_t > 0$, then the survival function is concave, i.e., the probability of death increases with age, and there is a maximum age A_t that an individual can reach. This parameter configuration allows the function $m_t(a)$ to accurately represent the empirical adult survival laws, and has the advantage of being analytically tractable. The maximum age is obtained by solving $m_t(A_t) = 0$ and is equal to

$$A_t = \frac{\log(\alpha_t)}{\beta_t}.$$

Notice that a higher longevity of individuals belonging to cohort t corresponds to larger α_t and/or lower β_t . Finally for sake of simplicity, we don't explicitly model fertility, and rather assume as (exogenously) given the size of each cohort, equal to ζ_t . The size of the generation born in t at any time $z \in [t, t + A_t]$ is therefore given by $\zeta_t m_t(z - t)$, reflecting that the measure of each generation declines deterministically through time. The demographic processes α_t , β_t and ζ_t , for varying t , will be estimated using the English data 1530-1860.

2.2 The economic structure

Now, we describe the economic structure of the model and the corresponding individuals' economic decision-making. In order to account for the role of technological advances in the transition to modern growth, we postulate that there are two distinct production sectors in the economy, a traditional vs a modern sector. The latter is subject to technological progress, inducing a rising productivity over time (at a rate say $\gamma_t > 0$ at date t), while the former has a constant productivity level. If workers are paid at their productivity level, which we postulate in our model, then the modern sector will become more attractive over time, inducing a full transition to the modern sector at a certain point in time. This way of modeling the transition captures the key mechanism put forward by Hansen and Prescott (2002).

However, such a sharp transition is by no way realistic; the process is much more gradual and much less mechanistic than the one outlined above. To generate a much more realistic picture, we account for human capital formation. Historically, human capital accumulation and the associated literacy improvements have taken place gradually over time, and this is likely to crucially matter in the actual shape of the transition to modern growth. In order to reproduce this feature, we model both the supply and demand sides of human capital. The supply side is developed in the next section, and it basically builds on the idea that school foundations depend on the attendance rates, which are in turn determined by population density.

The demand side mechanism originates in a further difference between the two production sectors. Individuals working in the traditional sector have a productivity, and thus a remuneration, which is independent of their level of human capital. In contrast, the remuneration of the workers in the modern sector does not only benefit from the (exogenously) rising productivity, it is also determined by their human capital level. This features a kind of complementarity between human capital and technological progress in the development process: for technological innovations to be exploited at their full potential, skilled workers are imperatively needed.² In particular, we take the view that technological progress and human capital interact in a multiplicative way, so that the remuneration of a given worker at a given date t is the product of her human capital and technological progress at this date t . This is consistent with a modern sector technology producing an amount Y_t at date t of the good from the multiplicative interaction of the level of technology or productivity, say $\exp\{\gamma_t t\}$, and the stock of human capital available in the economy, say H_t : $Y_t = \exp\{\gamma_t t\} H_t$.

This makes the development process much trickier provided human capital formation is costly. We precisely assume that going to school involves a transportation cost, which is proportional to the distance to the nearest school, and the payment of tuition fees. From the Schools Inquiry Commission (1868a) for example, we learn that boys can attend a city school from distances up to 20 miles, and a daily consumption of time amounting to more than one hour in the morning and in the evening. Concerning tuition fees, we know from historical surveys (see again Schools Inquiry Commission (1868a)) that schools were funded through income from an endowment

²Note that we are invoking this complementarity argument at the implementation stage of innovations, it is even more obvious if one has in mind the prior R&D stage giving rise to innovations.

and through fees paid by the students' parents in the period considered. Fees were imposed in order to supplement the endowment, and parents were willing to pay fees, provided the fees were not excessive, and the education was suitable.

With such a framework, an individual of a cohort t may not find it optimal to go to school. Indeed, going to school implies that the individual will pay a cost during the schooling time, say the first S_t periods of her life, and can only be paid back (via wages from the modern sector) after this time. Suppose the problem of the individual is the maximization of her lifetime resources. Then for a fixed cohort t , there is no guarantee that the lifetime resources allowed by schooling are superior to the lifetime revenue directly extracted from the traditional sector. An individual might not go to school for many good reasons in our framework. This could be for *demographic* reasons. In case life expectancy is markedly low for cohort t (which corresponds to a low parameter α_t and/or a high parameter β_t in our framework), the return to schooling is likely to be discouraging given the expected very short remuneration period. This could be for *technological* reasons. The (expected) pace of technological progress, measured by γ_t in our set-up, might be too slow, which also induces a low return to schooling. Finally, *institutional reasons* related to the organization and location of schools might well yield the same outcome: the absence of schools in the close neighborhood and/or prohibitively high tuition fees are of course very strong barriers to schooling.

As a consequence, the decision to go to school and the resulting schooling time depends on an exhaustive set of demographic, technological, geographic and institutional conditions. These conditions change over time, and the schooling decisions are therefore likely to vary from a cohort to another. Moreover, there is no reason to believe that all the individuals of the same cohort will take rigorously the same schooling decision: they hardly go *en masse* to school. To get the latter desired property, we postulate that the individuals of the same cohort may differ in their location, and in their innate abilities. We postulate that a given individual stays at her location for ever. In the pre-industrial era, the main reason for households to move was to reach regions with better employment opportunities or higher wages. In our theoretical model, the same technologies are available everywhere, and henceforth the main migration engine is shut down. Concerning innate abilities, we postulate that they are distributed according to a unimodal distribution. We shall use the log-normal distribution in our experimental studies.

With the latter ingredients, the demand side of human capital is complete. Within the same cohort, *ceteris paribus* only the most gifted and those located closer to schools will decide to go to school and to spend some time at school. Put in other words, there exists a threshold value for innate ability, so that individuals with an ability above (resp. below) the threshold will go to school (resp. remain uneducated). Naturally, this threshold ability value increases when the tuition fees, the distance to the nearest school or the alternative remuneration in the traditional sector goes up. The threshold ability value is also sensitive to the demographic and technological conditions: a higher life expectancy or a faster technological progress should lower the threshold. For individuals above the threshold, the schooling time can be longer or shorter depending on the same technological, demographic and institutional conditions and for the same intuitive reasons. In particular, a larger life expectancy, a faster technological progress or closer schools do induce a longer schooling time, and therefore a larger human capital level. And of course, *ceteris paribus* the more gifted individuals go to school for a longer time.

We now move to the supply side of human capital, the school foundations part of our theory.

2.3 The geographic and institutional structure

Location theory is a field of research that draws on economic geography and operations research. Its purpose is to model, formulate and solve problems of siting facilities in order to supply goods and/or services to a spatially dispersed population. The recent survey by ReVelle and Eiselt (2005) gives a bird's-eye view of the topic and its abundant literature, while the reader can refer to Daskin (1995), among others, for a deeper introduction. One of the core models of location theory is the *Simple Plant Location Problem* (in short SPLP). In words, it can be formulated as follows. Assume a geographically spread population with known demands for a certain commodity that is made available at facilities to be created. Opening a facility involves incurring a fixed cost, while distributing the commodity entails transportation costs. The problem is to determine the number, locations, and respective market areas of the facilities in order to minimize total cost defined as the sum of the transportation costs to the clients and the fixed opening costs. The SPLP captures one of the essential features of economic geography: the trade-off between transportation costs and

economies of scales. The former favor the multiplication of facilities; in contrast, the latter, expressed by the fixed costs, tend to restrict their number.

In this section, we use an extension of the SPLP to build a theory connecting school foundations to population density. We choose a very simple geographical setting: a circle of unit length. We assume that, at every point of time, the cohort of the newborn generation is uniformly spread over the circle and has the same distribution of abilities at every location. Clearly, such a representation is inconsistent with actual population patterns, as one observes strong disparities of density between urban and rural areas, between cities of different sizes and even inside a city. Nonetheless, we argue this is a minor point in our setting: rural population accounted for more than 80% of total population by the end of the period we consider. We suppose that every point of the circle can accommodate a school and that the schools are totally similar in their characteristics (same services, same quality, same reputation, etc.). It follows that a pupil will attend the closest school. Moreover, the results of the preceding section allow us to determine the demand for schooling arising from each point of the circle as a function of the distance to the nearest school. Given the hypothesis on the dispersion of the population, it is obvious that the schools will be optimally located if they are evenly spaced. Hence, for a given number of facilities, we can determine the literacy rate of the population, the total amount of fees paid by the pupils and the total transportation cost. Accordingly, the school location problem boils down to the single question: how many schools (or classrooms) will be founded at every date t to educate the newborn cohort?³ But this entails the formulation of an objective function.

To formulate such a criterion and achieve an acceptable modeling of the schooling foundation process, one must take a closer look at the relevant institutional arrangements at work in the period considered. And in particular, one needs to clarify the objectives pursued by school founders in that time. According to the Schools Inquiry Commission (1868a), the picture is far from uniform and three types of schools can be clearly distinguished: endowed schools, private schools and proprietary schools.

Endowed schools have usually some income from funds permanently appropriated to the school. Even in this category, there is a wide variation in their character and history. Some are part of large charitable foundations, others are run by the Church.

³Here we take the view that classrooms are specific to cohorts. In particular, they are assumed to be closed when the last person of cohort t graduates.

Many endowed schools have no exclusive connection. The private schools are the property of the master or mistress who teaches in them. They “owe their origin to the operation of the ordinary commercial principle of supply and demand”, according to the Schools Inquiry Commission (1868a). They provide more individual care and teaching, but the School Inquiry Commission extensively complains about the quality of these schools. Commissioners noted that “A really large and flourishing school is of course a marketable commodity, and sometimes sells well. But it is always a dangerous purchase for a stranger. (...) when the school declines the house is let for a shop or a private residence, and the master betakes himself elsewhere.” And also “ Considered commercially, few descriptions of business seem to require less capital than the keeping of a private day school of the second order. A house is taken, a cane and a map of England bought, an advertisement inserted, and the master has nothing more to do but teach. It is not likely that schools established at so slight a cost should have buildings well adapted to purposes of education.” These two quotes stress the commercial nature of private schools. The last of the three classes of schools is composed of the proprietary schools who belong to a body of shareholders. They are alike private property. This type of school is more recent, not more than 40 years old in 1860.

Since we don't have any piece of information concerning the composition of English schools sorted by each of the three types mentioned just above, we take an agnostic view and consider two different types of institutional arrangements. In the first scenario, denoted **CP** for central planning, the optimal number of classrooms is determined by a central authority every year, by maximization of aggregate profits of the education sector, reflecting that “the purpose of schools was never to save those from paying who could afford to pay”, as quoted by the Schools Inquiry Commission (1868a). The benefit drawn from building a school in a given area is roughly the difference between the tuition fees paid by the individuals in the catchment area of the school who finally decide to educate themselves (see the previous section), and the cost of building and/or implementing a classroom.⁴ The link between school foundations and population density is therefore clear: Since the profitability of a school mainly depends upon the tuition fees' revenue, the size of the population in the catchment area of the school should be a major determinant of school foundations.

A second institutional arrangement departs from the central authority view taken so

⁴This set-up cost can be seen as being net of the possible endowment.

far. In scenario **MA** for market, we assume that the density of schools results from a free entry process: schools are created as long as they earn a positive profit. This can reflect the functioning of private schools discussed above. It can be readily shown that model **MA** is equivalent to a model where a central authority maximizes aggregate attendance (for example, for religious reasons). In that case, it would create as many schools as possible, subject to a non-negative profit condition.

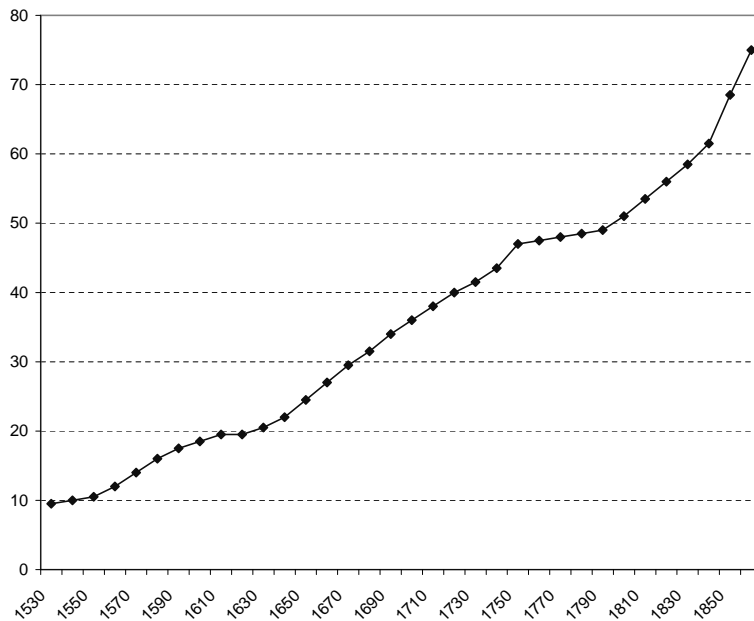
As mentioned before, the population size is a major determinant of school foundations because the main source of schools' revenues, namely tuition fees, does depend on this demographic factor. This is true for both institutional arrangements reviewed above. In all cases, no school is viable below a certain threshold of population size (or of the cohort size, ζ_t , in our theory). When the newborn population is low, the school creation or set-up costs are unlikely to be covered, hence no schools are created. Once the population reaches a threshold value, schools are created at once. The process by which illiteracy is eliminated is thus initiated by a jump. After this initial impulse, the process takes place much more smoothly in time depending on the evolution of population density and the attendance rate at schools of the successive cohorts, which in turn depends on the demographic, technological and geographic factors outlined in Section 2.2.

Hereafter, we will apply our theory to the England case 1530-1860 in order to disentangle the most salient characteristics and determinants of the English development process.

3 Data and methodology

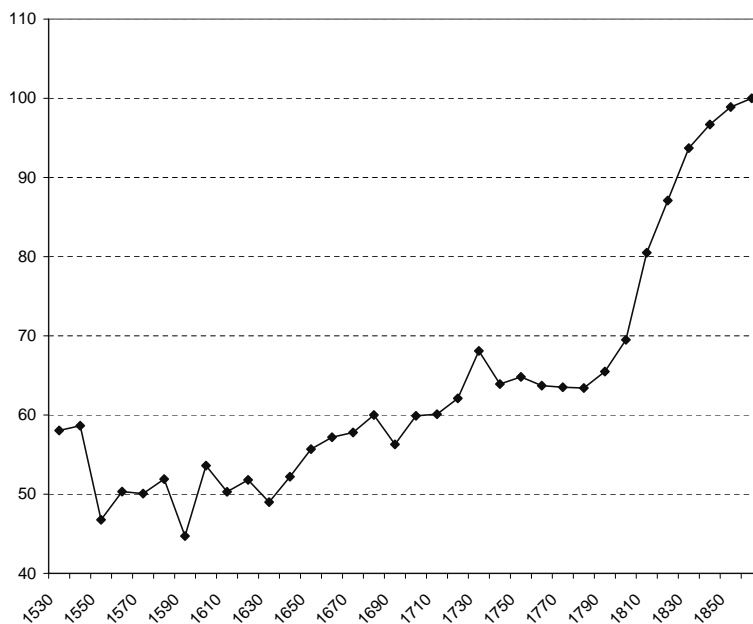
We first describe our sources with some key descriptive statistics over the period of interest. Then, we give a succinct overview of the chosen experimental setting. The section is ended with some outlines of how the data has been brought to the theoretical framework detailed in Section 2 (the so-called **calibration** step).

Figure 1: Literacy achievements (% population)



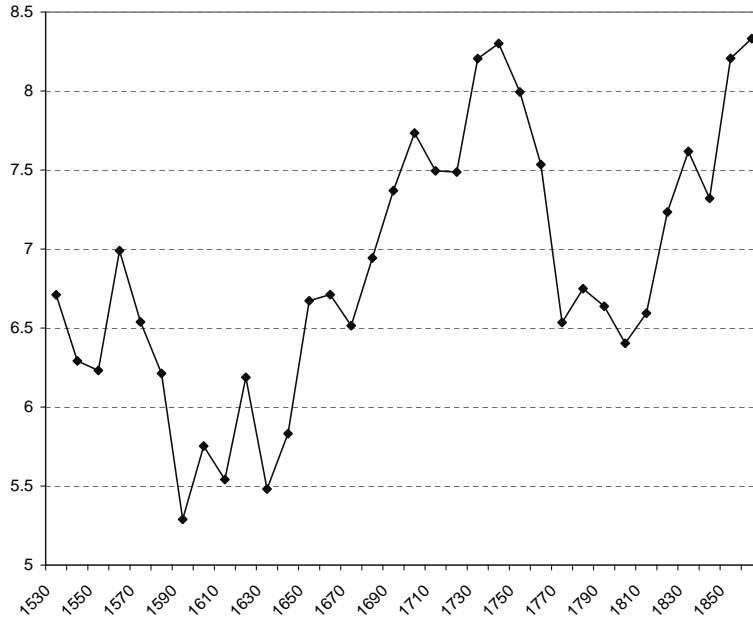
Estimation: Cressy (1980).

Figure 2: Total factor productivity



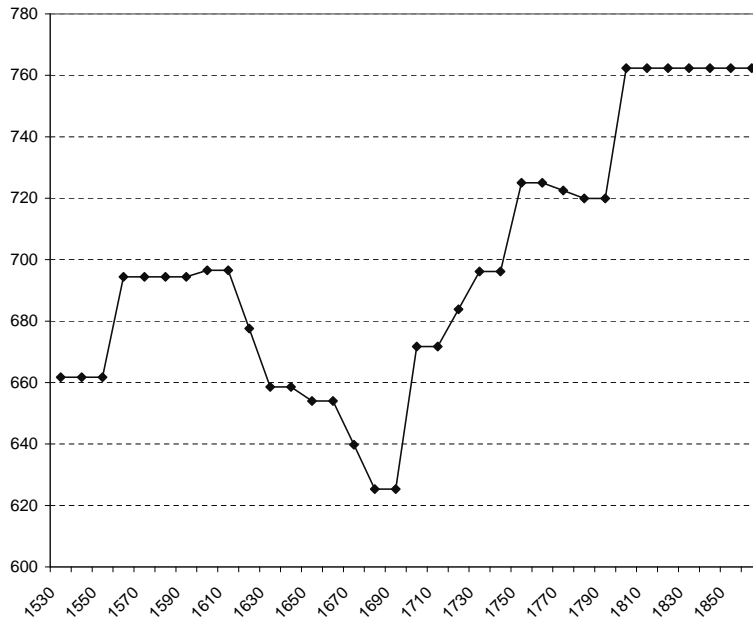
Estimation: Clark (2001).

Figure 3: Real wage of unskilled workers



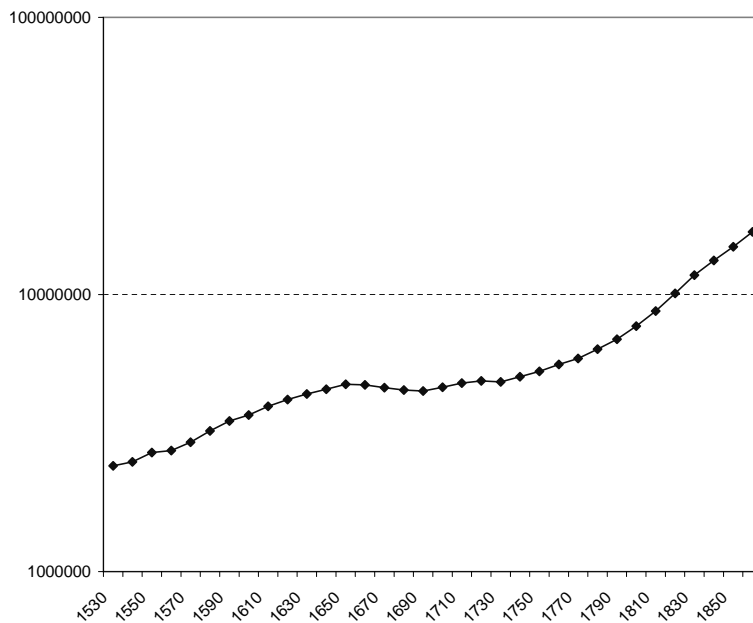
Estimation: Clark (2005b).

Figure 4: Mortality: number of survivors at age 40 from 1000 individuals at age 5



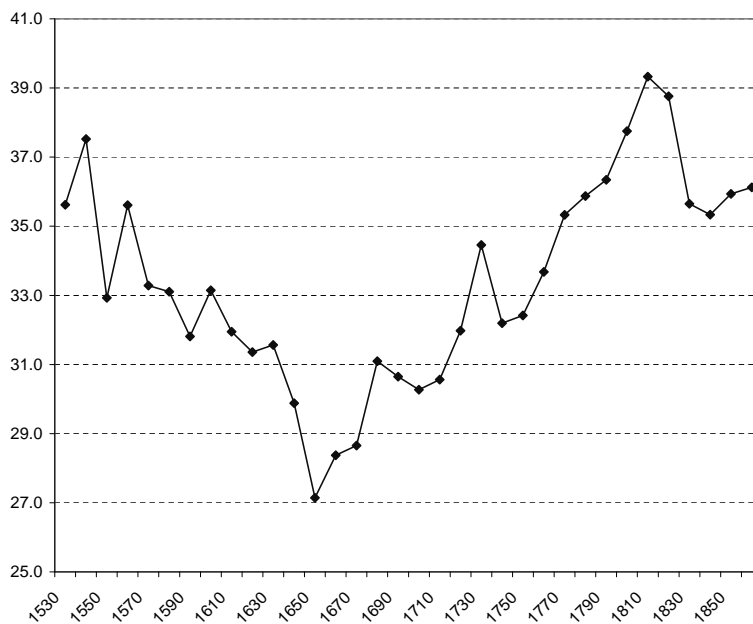
Source: Wrigley et al. (1997).

Figure 5: Population of England aged 5+



Estimation: Wrigley et al. (1997).

Figure 6: Crude birth rate



Estimation: Wrigley et al. (1997).

3.1 The data

Literacy

We borrow the series on the literacy rates from Cressy (1980). Figure 1 shows the evolution over time of literacy rates (average of men and women) for England as estimated by this author. It suggests that improvements in literacy started as early as in the sixteenth century. They also experienced a steady rise from 1580 to 1760.

Technological progress

Technological progress, via the productivity growth process γ_t in our model, increases the attractiveness of the modern sector, and should therefore stimulate schooling. We borrow the data on productivity growth from Clark (2001). As illustrated by Figures 2 and 3, productivity gains in England started to accelerate in the beginning of the nineteenth century. Consequently, the technological factor cannot account for the fact that higher literacy rates were achieved two centuries before any significant gain in productivity. The search for alternative demographic and institutional explanatory factors becomes indispensable.

Demography

The demographic trends in England over the considered period are taken from the detailed historical studies of Wrigley and Schofield (1989) and Wrigley et al. (1997). Recall that we need such a detailed demographic information to be able to identify the processes α_t and β_t of the survival law postulated in Section 2.1, and to estimate the size of the successive generations, ζ_t . These processes are crucial in the schooling decisions taken by the individuals of any cohort in our theoretical set-up.

The survival rates and population size evolution can be extracted from the surveys cited above. Figure 4 presents the survival rate of five years old individuals. It therefore abstracts from infant mortality swings to concentrate on mortality during the active life. We see that adult longevity was first stagnant then declining over the period 1600-1700, probably because of the urban penalty associated with the fast growth of cities. During this period of high mortality, literacy rose continuously, as we have shown on Figure 1.

We consider the population aged 6 and more, because it coincides with the concept of population of our model, disregarding the infants aged 0-5. Looking at the data, Figure 5 shows that population rose rapidly in the sixteenth and nineteenth centuries, while the seventeenth century was one of demographic stagnation. The corresponding swings in crude birth rates are plotted in Figure 6. We notice that rises in population in the sixteenth century correspond to the first wave of improvement in literacy.

School foundations

Concerning school foundations, we collect data from the appendix of the reports of the Schools Inquiry Commission (1868b). Two lists of schools are provided, together with their date of establishment. The “endowed grammar schools” taught a mixture of Latin and practical skills to sons of the middling sort and lesser elite (list in the Schools Inquiry Commission). The “endowed non-classical schools” were products of the Charity School Movement, offering protestant socialization and basic skills to the worthy poor. According to Cressy (1980), although short-lived private schools are omitted from the list, a check against other sources proves the Commission’s work to be reliable. We use these lists to compute the number of school establishments per decade. These data are presented in Table 1.

3.2 Methodology

In order to assess the relative importance of the demographic, technological and institutional factors in the English transition to modern growth, we take the following steps.

Step1

Calibration of a benchmark model: We first bring the data into our theoretical model. To this end, we need to fix the institutional arrangements in the education sector. We choose to start with the scenario **CP**, with a central authority determining the optimal number of classrooms as well as the level of the tuition fee. This will be our benchmark case, we will study in further steps how the results are altered if we switch to other arrangements. Calibration of the benchmark model requires in particular the

Table 1: School establishments

		grammar	non-classical
	<1500	23	2
1500	- 1510	9	2
1510	- 1520	8	2
1520	- 1530	13	0
1530	- 1540	8	0
1540	- 1550	36	0
1550	- 1560	47	10
1560	- 1570	44	7
1570	- 1580	31	2
1580	- 1590	19	14
1590	- 1600	26	13
1600	- 1610	41	18
1610	- 1620	42	27
1620	- 1630	27	33
1630	- 1640	31	38
1640	- 1650	15	33
1650	- 1660	42	45
1660	- 1670	34	58
1670	- 1680	38	67
1680	- 1690	31	75
1690	- 1700	25	81
1700	- 1710	23	188
1710	- 1720	24	287
1720	- 1730	23	279
1730	- 1740	12	152
1740	- 1750	9	91
1750	- 1760	4	93
1760	- 1770	3	128
1770	- 1780	7	84
1780	- 1790	8	99
1790	- 1800	6	106
1800	- 1810	3	87
1810	- 1820	1	125
1820	- 1830	3	64
1830	- 1840	4	36
1840	- 1850	6	9
1850	- 1860	9	4

estimation of the three demographic processes, α_t , β_t , ζ_t , and the productivity growth process, γ_t . It also requires fixing the values of some parameters for which we don't have an accurate information. Of course, an extensive robustness analysis will be conducted later on these parameters. Among them, one can mention the productivity in the traditional sector, the transport costs, and the discount factor.

Step 2

Counterfactual experiments: As we have explained along the way, there are four exogenous processes in our economy: α_t , β_t , ζ_t , and γ_t . The first two processes represent the mortality force, the third one is a measure of birth density force,⁵ and finally the last one is the technological progress force. To evaluate the importance of each of these three forces in accounting for literacy and growth, we run counterfactual experiments: for example if we aim at evaluating the extent to which technological progress can explain the observed historical evolution of literacy and growth, we let this force play alone, which amounts to solving our calibrated model with constant α_t , β_t and ζ_t , and with the estimated γ_t process as the unique active force. Doing the same exercise for all forces permits an assessment of the relative importance of each force in explaining the observed transition to modern growth.

Step 3

Robustness analysis: Obviously the results obtained from the above described counterfactual experiments are conditional to the calibrated model. A minimal requirement to test the scientific validity of the results is to resort to the necessary extensive robustness analysis, which in our case does not only mean performing a sensitivity analysis with respect to some parameters' values but also checking how the results are altered if we move to the alternative institution **MA**.

Before presenting our main findings from the counterfactual experiments and robustness analysis, we will give a brief summary of the calibration procedure of the benchmark model.

⁵We use the label "birth density" for the process ζ_t because the main driving force behind consists in fluctuations of crude birth rate (net of infant mortality). The value of ζ_t should be interpreted as a density, since it depends on the normalization of the circle. If we let the circle shrink, this would have the same effect as increasing ζ_t .

3.3 Calibration

The four exogenous processes α_t , β_t , ζ_t , and γ_t should be made explicit. We assume that all these four processes follow a polynomial function of time. Polynomials of order 3 are sufficient to capture the main trends in the data.

For the survival function processes α_t and β_t , the parameters of the polynomial are chosen by minimizing the distance with the survival functions estimated by Wrigley *et al.* (97). These survival functions apply to the age 5-85, and have been accordingly normalized to 1 at age 5, abstracting from infant mortality. The parameters of the process for ζ_t are chosen so that the distance between total population implied by our model and the observed level of population aged 5 and over is minimized. Finally, the parameter of the exogenous technological progress is set to follow the estimated level of total factor productivity of Figure 2.

In a second step, we select a log-normal distribution for abilities, say function $g(\mu)$ where μ stands for ability, which is commonly used to approach the actual distribution of innate characteristics. We next choose jointly the four following parameters in order to match four endogenous variables. Since we have little information to calibrate those parameters, we chose values which give a reasonable benchmark scenario. The four parameters are: the variance of $g(\mu)$, the transportation cost (in this benchmark calibration, we assume that the transportation cost is indexed on technological progress), the set-up cost, and the productivity or remuneration in the traditional sector. The four endogenous variables are: 10 schools in 1820 (there are 3000 schools in our database in 1820, so the scale of the model is 1/25), the level of literacy in 1820 (55%), the change in literacy over the period 1540-1820, and a skill premium of 60% on average over the period for seven years of education; according to van Zanden (2004) this was the premium received by skilled craftsmen after 7 years of apprenticeship.

4 Findings

We start with a summary of the properties of the benchmark model. We then come to the results of the performed counterfactual experiments and sensitivity exercises.

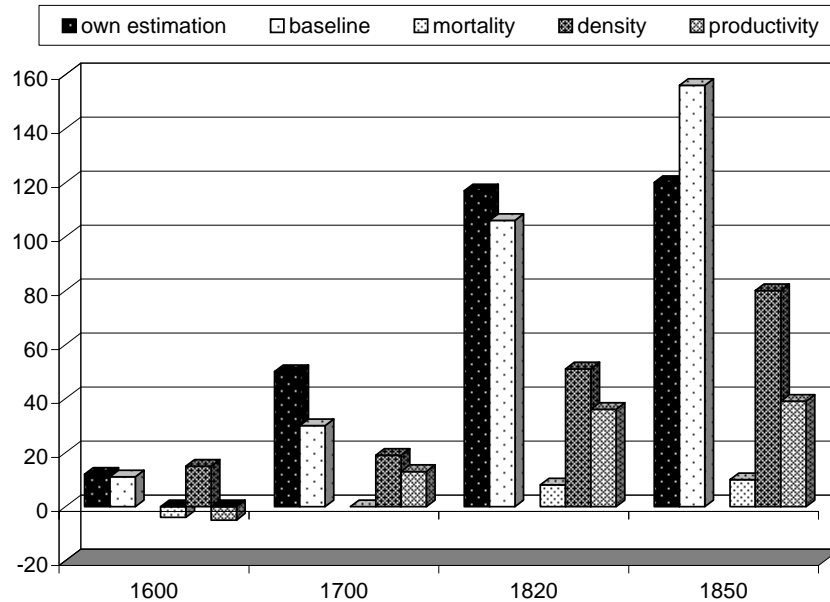
4.1 Benchmark simulation

The two first bars in Figure 7 report measured (“own estimation”) and simulated (“baseline”) school density. Each bar represents the change since 1530. The baseline change in the density of schools is the one which results from the decision process of the central school authority in the institutional arrangement **CP**. Both the measured and the simulated density of schools increase monotonically. The simulation underestimates school creation in the eighteenth century and overestimate it in the nineteenth century, but manages to capture well the overall trend. Literacy presented in Figure 8 follows closely the creation of schools. Estimated literacy rises in a sustained way over the period, while for the baseline simulation, there is a first rise prior to 1600, thanks to the creation of the first schools. It is followed by a period with slower improvements, and, after 1700, by a second period of fast growth.

The density of schools and the level of literacy are globally consistent with the estimated data. Still, a precise mapping is not obtained, but remember that literacy data covers the ability to sign on marriage register, not school attendance. Notice also the role of expectations: the sharp acceleration in the end of the period is related to the anticipation by households of strong productivity gains in the modern sector in the nineteenth century.

Figure 9 displays GDP per capita. The height of the bar is proportional to the change in GDP per capita since 1530. Recall that output in the modern sector is postulated to be the product of the level of technology (or productivity) and aggregate human capital. The former input can be immediately extracted from the already estimated productivity growth process γ_t . The latter input corresponds to the total stock of human capital of all the generations which are currently at work in the modern sector: this implies an exact accounting of all individuals of all the co-existing cohorts, who decide to go to school. Finally we can compute total GDP as the sum of the production of the traditional and modern sector minus the transportation cost minus the set-up cost of schools. According to the baseline there is a period of no growth after 1530, because the economy has to pay the transportation costs of students and the set-up cost of schools, but does not yet benefit from better educated persons. Next, the seventeenth century is characterized by very low growth -too low compared to Maddison data. After this stagnation period, growth starts accelerating after 1700 to reach 0.7% per annum at the end of the eighteenth century.

Figure 7: School Density: Baseline and Counterfactuals



Notice that our GDP numbers should be interpreted as the income generated by the accumulation of human capital and by productivity growth, without any effect from the accumulation of physical capital. The difference between the Maddison estimate and the baseline simulation can be attributed to physical capital accumulation, which is absent from our model.

4.2 Counterfactual experiments

In a first experiment we consider that birth density and technological progress are constant over the period; this allows to isolate the role of mortality. Since mortality drops very late in England (see other data on Geneva and Venice in Boucekkine, de la Croix, and Licandro (2003)), it does not exert a positive influence before the eighteenth century. The bar “mortality” in the figures represents the hypothetical change in school density, literacy and GDP per capita if mortality was the only factor at play. If mortality improvements were the only driving force of the industrial revolution, no school would have been created before 1700 and the literacy rate would have increased by only 6.8% over the period. Compared to the baseline simulation, mortality

Figure 8: Literacy Rate: Baseline and Counterfactuals

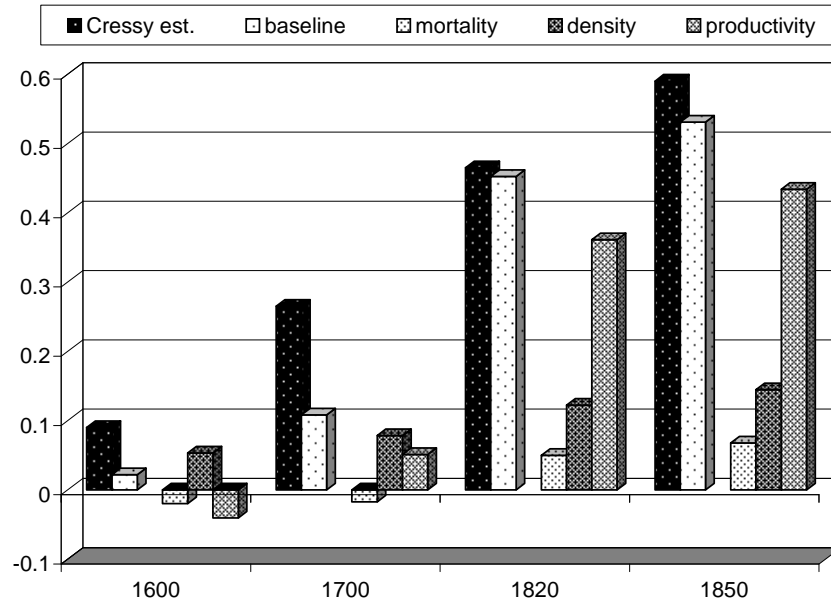
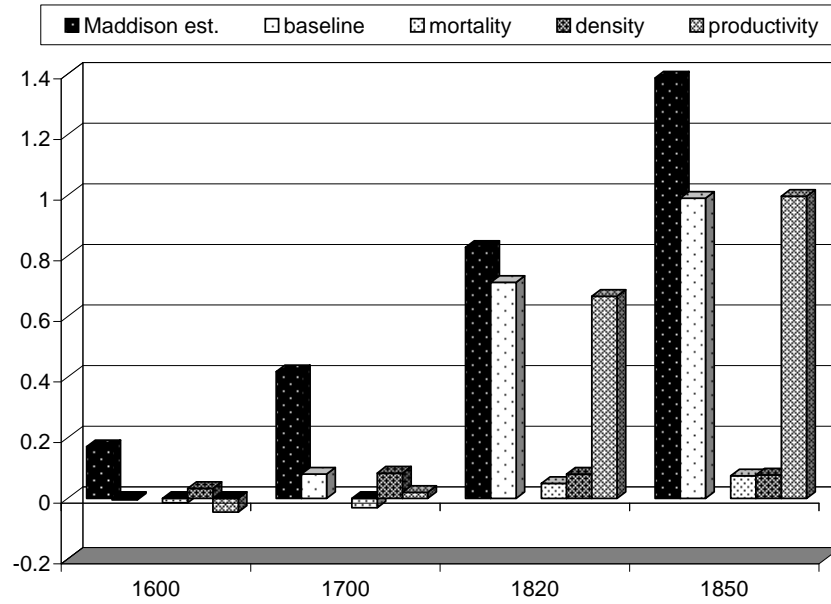


Figure 9: GDP per capita: Baseline and Counterfactuals



improvements explain 6.5% of total school creations over the period 1500-1850, 12.8% of improvements in literacy and 7.5% of growth of income per capita.

In a second step we run a simulation where both mortality and technological progress are constant. Only the birth density ζ_t varies, reflecting all changes in population which are not due to mortality. In this simulation we observe that the rise in population can be held responsible for school creation in the sixteenth century as well as for the early rise in literacy. In the seventeenth century, however, population stagnates, and school creation stops. In the end, the rise in birth density explains a majority of total school creations over the period 1500-1850, 27.5% of improvements in literacy and 7.8% of income growth per capita.

In a third step we run a simulation where both mortality and birth density are constant. Only technological progress is variable. In this simulation we observe that technical progress cannot explain the timing of school creation and literacy improvements, but it nonetheless explains a major part of changes in the end of the period.

The above results display a neat picture of the English transition to modern growth. First of all, the counterfactual analysis conducted highlights the fact that neither productivity increases nor mortality improvements can explain the establishment of schools at a high rate in the sixteenth century documented in Table 1. Only the rise in population density can. Secondly, in terms of growth rate of GDP, technological progress is the predominant engine while increases in longevity play a small role. Of course, these results need to be corroborated by the necessary sensitivity tests, which we conduct in the next section.

4.3 Robustness analysis

We provide a robustness analysis to changes in some of our key hypotheses. For each experiment, we recalibrate the parameters such that the model matches the four moments reported in Section 3.3.

In the benchmark calibration, transportation costs are indexed on technological progress. This assumption is probably too pessimistic because transportation costs relative to other costs have probably been reduced in the eighteenth century.⁶ To evaluate

⁶Culp and Smith (1989) mention that in *The Wealth of Nations*, Smith reviewed 18th-century public attitudes toward two new forms of wealth creation: “forestalling” and “engrossing”. Both activities had become possible only as transportation costs dropped.

the importance of this assumption we have run a simulation where the transportation cost is kept not indexed; as a consequence, the relative importance of this cost will diminish in the eighteenth century and the rise in literacy should be more important. The model is then recalibrated under this assumption. The obtained figures show that the new baseline with non-indexed transport costs yield very similar results, showing that the previous analysis remains valid whether transport costs are indexed on productivity or not.

Another assumption we want to test is the one concerning the evolution of productivity after 1860. In the baseline, we have assumed that households anticipate correctly the evolution of future productivity (1% per year). This creates an incentive to accumulate more human capital in the nineteenth century. To assess the importance of this mechanism, we run a simulation where agents have myopic forecast beyond 1860, i.e. they suppose that productivity will stay at a constant level (they consider that the industrial revolution is a temporary phenomenon). This change in assumption does not require any modification in the calibration. Results show that the effect of lower expectations is quite small.

In another robustness test, we take a lower value of the risk free interest rate, assuming a rate of 3% per year instead of 5%. The other parameters need to be adjusted. A lower interest rate gives an incentive for households to get more education, and so we need higher transportation costs to match the observed education investment. In the figures we observe that the number of schools is very close to the baseline, while literacy increases faster in the beginning of the period. Using 3% as interest rate would bring our simulated literacy closer to the estimate by Cressy in the beginning of the period.

Until now, the robustness analysis indicates that the result on literacy and growth are little affected by changes in the parameters. This conclusion is however not valid when the parameter measuring productivity in the traditional sector, w^h , is concerned. If for example we index w^h on productivity in the modern sector, A_t , there is no way to chose the calibrated parameters so as to match the targeted moments, and in particular the rise in literacy over the period. In fact, the non-indexation of w^h is the main mechanism through which technical progress plays a role in the model. If we shut down this channel by indexing w^h , we reduce drastically the role of technical progress, and we are left with the two other factors, mortality and birth density, which together explain about 40% of the observed rise in literacy.

We finally study the robustness of the results to the assumption on the institutional arrangement. We have run simulations for the model **MA** in which schools are created in a decentralized way as long as there are profit opportunities. Comparing **CP** to **MA** we retain two conclusions.

First, that the timing of the take-off for school creation does not vary across models; in both cases it starts as early as in 1540. Second, with the market solution, the density of schools increases much faster than with a central authority. This very fast rise entails important fixed costs for the economy, which slow growth compared to the central authority case. The model with the market solution is therefore not as good as the model with a central authority to reproduce the acceleration in growth during the early nineteenth century because it would imply too many school establishments.

At last, one issue that is potentially important but very complex involves the model spatial structure. Space is modeled using a circle of unit length, with schools spread evenly over it. This is a one dimensional model of location. In real life, of course, the English countryside is two dimensional. To see whether the predictions of the one dimensional space can be transposed to a more realistic set-up, let us consider the infinite plain. There are basically two ways of covering the plane with regular shapes of the same size: squares and hexagons. It is well-known from the literature on central place areas (see, e.g., Beckmann 1968) that the latter is more efficient than the former, hence we will consider an infinite covering of the plane with hexagons of the same size. The relevant descriptor for our problem is the density of centers, i.e., the number of centers in a unit area. An equivalent descriptor is the edge length of the hexagon. Assume a simpler case where all children have to attend school and consider the free-entry case. Then it can be shown that the density of school is a linear function of the density of population, exactly as it is the case in the uni-dimensional case. It follows that the relationship between the number of schools and the population density is a linear one, whatever the dimensionality of the space.

This result prompts two additional comments. First, in the one dimensional world, the average travel distance is linearly related to the number of schools, implying that the average travel distance falls proportionally to $1/\text{density}$. However, in the two-dimensional world, the average travel time is proportional to $1/(\text{square root of population density})$. Whether this may affect our estimations of the importance of changing population density in explaining rising schooling is an open question. Recent papers in the optimal location literature (see for example Morgan and Bolton (2002))

have provided some estimates in some special cases. Using their results in a simple example, we find in that in the two dimensional world, if the number of schools doubles, then the average distance decreases by a factor of about 0.267, while, in the one dimensional world, doubling the number of schools decreases the average distance by a factor of 0.25. Hence, the difference between $1/\text{density}$ and $1/(\text{square root of density})$ is compensated by a scaling factor which makes the difference between the two worlds acceptable. In the more complex model, the discrepancy will depend, among other things, on whether tuition fees will be set lower to attract students from more distant places in the two dimensional space.

Second, while space is generically two-dimensional, most of human activities at any time use to be organized along some principal routes. Our circular representation of space could well fit such an organization, and that is precisely why it is so frequently adopted in economic geography.

5 Conclusion

In this paper, we build up a theoretical model with the main demographic, economic and institutional factors traditionally considered to crucially matter in the transition to modern growth. In particular, we have provided a formal link between population density and the provision of schools, i.e., due to economies of scales, higher densities allows one to reduce the cost of education per capita and to increase the level of human capital. This is in agreement with a trend of literature about agglomeration economies (see, e.g., Duranton and Puga (2004) and Henderson (2005)).

We apply our theory to the England case over the period 1530-1860. Using a calibrated version of our model, we have measured the impact of mortality, birth density and technological progress on school density, literacy and growth through a set of counterfactual experiments. We find that one third of the rise in literacy over the period 1530-1850 can be directly related to the effect of density, while one sixth is linked to higher longevity and one half to exogenous total factor productivity growth.

Some concluding remarks are in order. First of all, one has to mention the reduced role of mortality declines relative to other factors in explaining England's development over the period studied. This goes at odds with other studies on other countries

(see Boucekkine, de la Croix and Licandro, 2003, for example) but is not that surprising if we have in mind Wrigley and Schofield's study advocated in the introduction of this paper. Since we rely on this study to calibrate the demographics of the model, it is a fortunate outcome of our simulations that mortality declines do not play the major role. Second, it is fair to acknowledge that while the model used is properly calibrated to capture the main observed demographic and technological characteristics of the English transition, it is built on several simplifying assumptions that could be hopefully relaxed in future work to bring the model closer to reality. Including physical capital accumulation and human capital externalities should be the next challenges. Working on a two-dimensional representation of space and determining whether it really matters compared to the one-dimensional space of this paper would also be an interesting and innovative extension.

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A Note: From one dimensional to two dimensional space

One issue that is potentially important but very complex involves the model spatial structure. The story in the paper is that there are fixed costs for creating a school, so higher density allows for creation of more schools, shorter travel time, and more schooling. This is modeled using a circle of unit length, with schools spread evenly over it. This is a one dimensional model of location. In real life, of course, the English countryside is two dimensional. The qualitative predictions of the one dimensional model will carry over to the two dimensional world. But can the quantitative predictions be transposed too? This is a priori not obvious.

A.1 School density and population density

How to implement our model in such a two-dimensional space? Obviously the local demand for education depends only on the transportation cost for its spatial component, thus only on the distance to the school and not on the question whether we are working in one or two dimensions. Next, we should choose the representation of the 2-D space. We chose to work with a circle to avoid the boundary effects which would entail useless and noninformative computational difficulties. The bi-dimensional equivalent of the circle is the sphere. But working on a sphere raises some difficulties. Indeed, whereas it is possible to divide the circle in equal parts for any number of schools, this is not possible on a sphere. For instance, if we can open 5 schools, there is no way to place them so that the market areas are equal. It follows that artifacts are likely to occur and blunder the interpretation of the results. Moreover, the optimal placement of the facilities is a hard geometrical problem. Therefore we decided not to follow this track.

We can consider the infinite plain. There are basically two ways of covering the plane with regular shapes of the same size: squares and hexagons. It is well-known from the literature on central place areas (see, e.g., Beckmann (1968)) that the latter is more efficient than the former, hence we will consider an infinite covering of the plane with hexagons of the same size. The relevant descriptor for our problem is the density of centers, i.e., the number of centers in a unit area. An equivalent descriptor is the edge length of the hexagon.

In this setting the problem is still hard to solve. Assume a simpler case where all children have to attend school and consider the free-entry case.

We assume first a infinite straight line with a uniform density of children ρ . All the children are obliged to go to school. The fixed cost for opening a school is f and the tuition fee is equal to k . In the free-entry process a school will be opened provided it covers its fixed cost. Hence

$$k\rho\ell = f$$

where ℓ is the catchment area of the school, which is thus

$$\ell = \frac{f}{k\rho}$$

Then the density of schools is

$$\delta_1 = \frac{1}{\ell} = \frac{k\rho}{f}$$

which is linear in the density of children.

We now consider the two-dimensional space and we assume it is covered by a grid of hexagons. Let R denote the ray of the circumscribing circle. The area of the hexagon can easily proved to be

$$S = \frac{3\sqrt{3}}{2}R^2$$

It follows that the optimal ray is the solution of the equation

$$k\rho S = k\rho \frac{3\sqrt{3}}{2}R^2 = f$$

hence

$$R = \sqrt{\frac{2f}{k\rho 3\sqrt{3}}}$$

It follows that the measure of the catchment area is

$$S = \frac{f}{k\rho}$$

and the density of schools is

$$\delta_2 = \frac{1}{S} = \frac{k\rho}{f}$$

which is also linear in the density ρ . It follows that the relationship between the number of schools and the population density is a linear one, whatever the dimensionality of the space.

A.2 Distance to nearest school and density

We have shown above that the number of schools on the circle is

$$\delta_1 = \frac{1}{\ell} = \frac{k\rho}{f}$$

It follows that the total distance to the schools equals

$$d_{tot} = \rho\delta_1 2 \int_0^{\frac{1}{\delta_1}} r dr = \rho r^2 \Big|_0^{\frac{1}{\delta_1}} = \frac{\rho}{4\delta_1}$$

so that the average distance of the pupils to the schools in 1-D amounts to

$$\bar{d} = \frac{d_{tot}}{\rho} = \frac{1}{4\delta_1} = \frac{f}{4k\rho}$$

So that doubling the number of schools decreases the average distance by a factor of 0.25.

We can follow Morgan and Bolton (2002) and compute the total distance to the center of an hexagon of radius R :

$$d_{tot} = \rho \frac{\sqrt{3}}{8} (4 + \log 27) R^3$$

Remember that the size of the area of the hexagon is given by

$$S = \frac{3\sqrt{3}}{2} R^2$$

Multiplying this quantity by ρ , we obtain the total population in the hexagon. Hence, the average distance to the center is given by

$$\bar{d} = \frac{d_{tot}}{\rho S} = \frac{4 + \log 27}{12} R \approx 0.608R$$

If we take the optimal radius found above, we find

$$\bar{d} \approx 0.3772 \sqrt{\frac{f}{k\rho}} = 0.3772 \sqrt{\frac{1}{\delta_2}}$$

If the number of schools doubles, then the average distance decreases by a factor of about 0.266718.

This gives some hints on the magnitude of the error we made by computing our results in 1-D rather than in 2-D.