The edge-face choosability of plane graphs

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Abstract

A plane graph \( G \) is said to be \( k \)-edge-face choosable if, for every list \( L \) of colors satisfying \(|L(x)| = k\) for every edge and face \( x \), there exists a coloring which assigns to each edge and each face a color from its list so that any adjacent or incident elements receive different colors. We prove that every plane graph \( G \) with maximum degree \( \Delta(G) \) is \((\Delta(G) + 3)\)-edge-face choosable.

1. Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges unless otherwise stated. A plane graph is a particular drawing in the Euclidean plane of a planar graph. For a plane graph \( G \), we denote its vertex set, edge set, face set, order, maximum vertex degree, and minimum vertex degree by \( V(G) \), \( E(G) \), \( F(G) \), \(|G|\), \( \Delta(G) \), and \( \delta(G) \), respectively.

A plane graph \( G \) is \( k \)-edge-face colorable if the elements of \( E(G) \cup F(G) \) can be colored with \( k \) colors such that any two adjacent or incident elements receive different colors. The edge-face chromatic number \( \chi_{ef}(G) \) of \( G \) is defined to be the least integer \( k \) such that \( G \) is \( k \)-edge-face colorable.

A mapping \( L \) is said to be an assignment for the plane graph \( G \) if it assigns a list \( L(x) \) of possible colors to each element \( x \) in \( E(G) \cup F(G) \). If \( G \) has an edge-face coloring \( \phi \) such that \( \phi(x) \in L(x) \) for all elements \( x \), then we say that \( G \) is \( L \)-edge-face colorable or \( \phi \) is an \( L \)-edge-face coloring of \( G \). \( G \) is \( k \)-edge-face choosable or \( k \)-edge-face list colorable if it is \( L \)-edge-face colorable for every assignment \( L \) satisfying \(|L(x)| = k\) for all elements \( x \) in \( E(G) \cup F(G) \). The list edge-face chromatic number \( \chi_{ef}^L(G) \) of \( G \) is the smallest integer

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k such that \( G \) is \( k \)-edge-face choosable. By considering colorings for \( V(G) \), \( E(G) \), and \( \overline{F}(G) \), we can define analogous notions such as \( k \)-vertex choosability, \( k \)-edge choosability, and \( k \)-face choosability. Let \( \chi'(G) \) and \( \chi_f(G) \) denote the edge chromatic number and the list edge chromatic number of a graph \( G \), respectively.

In 1975, Melnikov [9] conjectured that every plane graph \( G \) is \((\Delta(G) + 3)\)-edge-face colorable. Two similar, yet independent, proofs of this conjecture have been recently published by Waller [13] and Sanders and Zhao [10]. Both proofs made use of the four-color theorem. Without employing the four-color theorem, Wang and Lih [14], and independently Sanders and Zhao [11], gave a new proof of the conjecture. The purpose of this paper is to extend this result to the list edge-face coloring of plane graphs. Our main theorem is the following.

**Theorem 1.** Every plane graph \( G \) is \((\Delta(G) + 3)\)-edge-face choosable.

The organization of this paper is as follows. In Section 2, we will give an example to show that edge-face choosability is different from edge-face colorability. In Section 3, we give structural lemmas and some auxiliary colorings. The proofs of structural lemmas are postponed to Section 5. Section 4 is entirely occupied by the proof of the main theorem. In the final Section 6, we show cases that the upper bound can be reduced. Then we conclude the paper by proposing two open problems.

Now we collect the notation and basic definitions used in the subsequent sections. The unique unbounded face of a plane graph is called its outer face, and other faces its inner faces. Two faces of a plane graph are said to be adjacent if they share at least one common boundary edge. A vertex (or an edge) is said to be incident to a face if it lies on the boundary of the face. For \( f \in F(G) \), we use \( b(f) \) to denote the boundary walk of \( f \) and write \( f = [u_1u_2\ldots u_n] \) if \( u_1, u_2, \ldots, u_n \) are the vertices on \( b(f) \) enumerated clockwise. We also write \( V(f) \) for \( V(b(f)) \). The degree of a face is the number of edge-steps in the boundary walk. Let \( d_G(x) \) denote the degree of a vertex (face) \( x \) in \( G \). A vertex (face) of degree \( k \) is called a \( k \)-vertex (\( k \)-face). The maximum degree of all faces of \( G \) is denoted by \( \Delta'(G) \). For \( v \in V(G) \) and \( k \geq 3 \), let \( F_k(v) \) denote the set of \( k \)-faces that are incident to the vertex \( v \). Let \( F(v) = \bigcup\{F_k(v) \mid k \geq 3\} \). If \( f_1 \) and \( f_2 \) are two faces of \( G \) and \( e \in b(f_1) \cap b(f_2) \), we use \( f_1 \oplus f_2 \) to denote the face enclosed by the boundary \( (b(f_1) \cup b(f_2))\backslash\{e\} \) of the plane graph \( G - e \). A cycle of \( G \) is called a separating cycle if both its interior and exterior contain at least one vertex of \( G \). We use \( N_G(v) \) to denote the neighborhood of the vertex \( v \) in \( G \). For \( S \subseteq V(G) \cup E(G) \), let \( G[S] \) denote the subgraph of \( G \) induced by \( S \).

**2. An example**

For a plane graph \( G \) with \( \Delta(G) \leq 2 \), it is easy to see that \( \chi^L_f(G) = \chi_{ef}(G) \leq 5 \) and \( \chi^L_f(G) = \chi_{ef}(G) = 5 \) if and only if \( G \) contains an odd cycle. The list edge-face chromatic number of a plane graph is clearly the maximum of list edge-face chromatic numbers of its components. Henceforth, we assume that \( G \) is a connected plane graph with \( \Delta(G) \geq 3 \).

For any plane graph \( G \), it is obvious that \( \chi^L_{ef}(G) \geq \chi_{ef}(G) \geq \Delta(G) \). However, the list edge-face chromatic number may be strictly greater than the edge-face chromatic number.
2-edge connected plane graphs

If \( G \) is a plane graph with \( \Delta(G) \leq 4 \), Theorem 1 has already been proved indirectly. An earlier result of Harris [7] asserts that \( \chi'_f(G) \leq 2\Delta(G) - 2 \) if \( G \) is a graph with \( \Delta(G) \geq 3 \). It implies that every graph \( G \) with \( \Delta(G) = 3 \) is 4-edge choosable. It is shown in [8] that every graph \( G \) with \( \Delta(G) = 4 \) is 5-edge choosable. These results together with the following Theorem 4 establish Theorem 1 when \( \Delta(G) \leq 4 \). The proof of Theorem 4 for the class of 2-edge connected plane graphs \( G \) with \( \Delta(G) \leq 4 \) was implicitly worked out in [8]. Here we supply a proof for the general case.

Theorem 4. If \( G \) is a plane graph with \( \Delta(G) \geq 3 \), then \( \chi^L_{ef}(G) \leq \chi'_f(G) + 2 \).

Proof. Let \( L \) be an assignment of \( G \) that satisfies \( |L(x)| = \chi'_f(G) + 2 \) for all \( x \in E(G) \cup F(G) \). It follows from \( \Delta(G) \geq 3 \) that \( |L(x)| = \chi'_f(G) + 2 \geq \Delta(G) + 2 \geq 5 \). Since \( G \) is 5-face choosable (see [12]), we first give an \( L \)-face coloring \( \pi \) of \( G \). Then for
every edge $e$ we define a new list $L'(e) = L(e) \setminus \{\pi(y) \mid y \text{ is a face incident to } e\}$. Since $e$ is incident to at most two faces of $G$, we have $|L'(e)| \geq \chi'_f(G)$. Thus $E(G)$ has an $L'$-edge coloring. Therefore an $L$-edge-face coloring of $G$ is obtained. □

Now we only need to show Theorem 1 for the case $\Delta(G) \geq 5$. An inductive proof will proceed according to some structural features of the graph. In this section, we will complete the preparation work, while the proof of Theorem 1 will occupy the next section.

We first introduce some special configurations. We will state three lemmas asserting their existence. We postpone the proofs to Section 5. Then we establish three auxiliary lemmas to be used in the proof of our main theorem.

In order to describe the special configurations, we introduce a few terms as follows. Let $G$ be a plane graph. A face $f$ of $G$ is said to be minor if $d_G(f) \leq 4$, and feasible if $d_G(f) = 5$ and it is adjacent to at least two minor faces. Moreover, a face is said to be good if it is either minor or feasible. For $e \in E(G)$, let $t(e), q(e)$, and $p'(e)$ denote, respectively, the number of 3-faces, the number of 4-faces, and the number of feasible 5-faces which are incident to the edge $e$. If an edge $e = xy$ satisfies $t(e) \geq 1$, then $e$ is called a strong edge of $G$ with its strong weight $\sigma_s(e)$ defined to be $d_G(x) + d_G(y) + 2 - t(e)$. Similarly, if $t(e) + q(e) + p'(e) \geq 1$, $e = xy$ is called a weak edge of $G$ with its weak weight $\sigma_w(e)$ defined to be $d_G(x) + d_G(y) + 2 - t(e) - q(e) - p'(e)$. Obviously, a strong edge is a weak edge, but not vice versa. A 5-vertex $v$ of $G$ is called full if it is incident to five 3-faces, and nearly-full if it is incident to four 3-faces. A 3-face of $G$ is called full if it is incident to a full 5-vertex. A subgraph $H$ of a plane graph $G$ is said to be conformable if each of the inner faces of $H$ is a face of $G$.

Let $H_1$ denote the plane graph obtained from a path $u_1u_2u_3u_4u_5$ of length 4 by joining all $u_i$ to a new vertex $u$. Let $H_2$ denote the plane graph obtained from a cycle $C_5 = v_1v_2v_3v_4v_5v_1$ of length 5 by adding a vertex $v$ to the interior and a vertex $w$ to the exterior, and then joining $v$ to $v_i, i = 1, 2, \ldots, 5$, and $w$ to $v_j, j = 1$ and 2.

To prove our main result by induction, we need to consider the following special configurations.

(C1) A 2-vertex not lying on a separating 3-cycle;
(C2) A 3-vertex incident to a minor face;
(C3) A 5-face incident to two adjacent 3-vertices;
(C4) A strong edge $e$ with $\sigma_s(e) \leq 11$;
(C4.1) A strong edge $e$ with either $\sigma_s(e) \leq 9$, or $\sigma_s(e) = 10$ and $q(e) > 0$;
(C4.2) A weak edge $e$ with $\sigma_w(e) \leq 10$;
(C5.1) A nearly-full 5-vertex $v$ so that the boundary edges of the four 3-faces of $F(v)$ induce a conformable $H_1$;
(C5.2) A full 5-vertex $v$ such that some face of $F(v)$ is adjacent to a 3-face $f'$ of $F(G) \setminus F(v)$ satisfying $V(f') \not\subseteq N_G(v)$. In other words, the boundary edges of faces of $F(v) \cup \{f'\}$ induce a conformable $H_2$.

Borodin [3] established a structural theorem for plane graphs. For our purposes, we only need the following simplified version of that theorem.

**Lemma 5.** Every plane graph $G$ with $\delta(G) \geq 2$ contains at least one of (C1), (C2), (C3) or (C4).
Lemma 5 will be applied in Section 4 to prove Theorem 1 when \( \Delta(G) \geq 7 \). However, we need the following two lemmas to settle the case for \( 5 \leq \Delta(G) \leq 6 \).

**Lemma 6.** Let \( G \) be a plane graph with \( \Delta(G) = 5 \) and \( \delta(G) \geq 2 \). Then \( G \) contains at least one of \((C1), (C2), (C3), (C4.1), or (C5.1)\).

**Lemma 7.** Let \( G \) be a plane graph with \( \Delta(G) = 6 \) and \( \delta(G) \geq 2 \). Then \( G \) contains at least one of \((C1), (C2), (C3), (C4.2), or (C5.2)\).

The next three lemmas give auxiliary colorings. Lemma 8 is an obvious fact.

**Lemma 8.** Let \( C \) be a cycle of length \( 3 \) or more. Let \( L \) be an assignment of the vertex set of \( C \) such that \( |L(v)| = 2 \) for each vertex \( v \) and \( L(v_1) \neq L(v_2) \) for some pair of consecutive vertices \( v_1 \) and \( v_2 \). Then \( C \) is \( L \)-vertex colorable.

**Lemma 9.** Let \( L \) be an assignment of the edge set of the graph \( H_1 \) such that \( |L(u_1u_2)| = |L(u_2u_3)| = 2, |L(u_3u_4)| = 3, |L(u_1)| = |L(u_5)| = 4, \) and \( |L(u_i)| = 6 \) for \( i = 2, 3, 4 \). Then \( H_1 \) is \( L \)-edge colorable.

**Proof.** If \( L(u_5) \cap L(u_3u_4) = \emptyset \), then we first color \( uu_5 \) and \( uu_3u_4 \) with some color from \( L(u_5) \cap L(u_3u_4) \), then color \( uu_5, uu_3u_2, uu_1, uu_2u_3, uu_3u_4 \) in succession. If \( L(u_5) \cap L(u_3u_4) = \emptyset \), then there is \( \alpha \in (L(u_5) \cup L(u_3u_4)) \setminus L(u_1u_2) \) such that \( |L(u_5)| + |L(u_3u_4)| = 7 > 6 = |L(u_1u_2)| \). If \( \alpha \in L(u_3u_4) \), we color \( uu_3u_4 \) with \( \alpha \), then color \( uu_5, uu_3u_2, uu_1, uu_2u_3, uu_3u_4 \) in succession. If \( \alpha \in L(u_5) \), we color \( uu_5 \) with \( \alpha \), then color \( uu_3u_4, uu_5u_3, uu_2u_3, uu_1, uu_2, uu_3, uu_4 \) in succession. \( \square \)

**Lemma 10.** Let \( L \) be an assignment of the edge set of the graph \( H_2 \) such that \( |L(v_3v_4)| = |L(v_4v_5)| = |L(v_5v_1)| = |L(v_2v_3)| = 2, |L(v_2v_3)| = |L(v_5v_1)| = 3, |L(v_2v_3)| = 5, |L(v_2v_3)| = |L(v_5v_1)| = |L(v_2v_3)| = 6, \) and \( |L(v_2v_3)| = |L(v_5v_1)| = |L(v_2v_3)| = 7 \). Then \( H_2 \) is \( L \)-edge colorable.

**Proof.** If \( L(v_3) \cap L(v_2) = \emptyset \), then we first color \( v_3v_4 \) and \( v_2v_3 \) with some color from \( L(v_3v_4) \cap L(v_2v_3) \), then color \( v_3v_4, v_4v_5, vv_1v_2, vv_3, vv_4, vv_5, vv_1, vv_2 \) in succession. If \( L(v_3) \cap L(v_2) = \emptyset \), then there is \( \beta \in (L(v_3) \cup L(v_2)) \setminus L(v_2) \) such that \( |L(v_3)| + |L(v_2)| = 8 > 7 = |L(v_2)| \). If \( \beta \in L(v_2) \), we color \( vv_2 \) with \( \beta \), then color \( vv_1, vv_5, vv_1, vv_4, vv_5, vv_1, vv_2 \) in succession. If \( \beta \in L(v_3) \), we color \( vv_3 \) with \( \beta \), then color \( v_3v_4, v_4v_5, vv_2, vv_1, vv_5v_1, vv_1, vv_2, vv_4, vv_5 \) in succession. \( \square \)

4. Proof of Theorem 1

We continue the proof of Theorem 1 for the case \( \Delta(G) \geq 5 \). We proceed by induction on \( |G| + |E(G)| \). When \( |G| + |E(G)| \leq 11 \), the theorem holds trivially. Let \( G \) be a plane graph with \( \Delta(G) \geq 5 \) and \( |G| + |E(G)| \geq 12 \). Let \( L \) be an assignment of \( G \) that satisfies \( |L(x)| = \Delta(G) + 3 \geq 8 \) for all \( x \in E(G) \cup F(G) \). If \( G \) contains a 1-vertex \( v \), then \( G - v \) is \( L \)-edge-face choosable by the induction hypothesis. An \( L \)-edge-face coloring of \( G \) can be derived easily from an \( L \)-edge-face coloring of \( G - v \). Thus we may suppose that \( \delta(G) \geq 2 \). By Lemmas 5–7, we need to consider the following eight cases. In the subsequent proof,
if $e$ is one of the boundary edges of a face $f$, we use $f_e$ to denote the face adjacent to $f$ that shares the edge $e$ with $f$.

**Case 1.** $G$ contains a 2-vertex $v$ incident to edges $vw$ and $vw$ and faces $f_1$ and $f_2$ such that no separating 3-cycle passes through $v$.

If neither $f_1$ nor $f_2$ is a 3-face, then set $H = G - v + uw$. Let $f'_i$ denote the face incident to $uw$ in $H$ that corresponds to $f_i$ in $G$ for $i = 1$ and 2. Define $L(uw) = L(uv)$ and $L(f'_1) = L(f_1)$ for $i = 1$ and 2. By the induction hypothesis, $H$ has an $L$-edge-face coloring $\phi$ that can be regarded as a partial coloring of $G$. Then we color the edge $uv$ with $\phi(uw)$, and the face $f_i$ with $\phi(f'_i)$ for $i = 1$ and 2. Now there exist at most $\Delta(G) + 2$ forbidden colors for the edge $vw$, while $|L(vw)| = \Delta(G) + 3$, hence $vw$ can be properly colored.

We next assume that $f_1$ is a 3-face and let $H = G - vu$. Define $L(f_1 \oplus f_2)$ to be the set $L(f_2)$. We use the color of $f_1 \oplus f_2$ of an $L$-edge-face coloring of $H$ to color $f_2$. Thus there exist at most $\Delta(G) + 1$ forbidden colors for $uu$. Once $uu$ is colored, there are at most 7 forbidden colors for $f_1$ and $|L(f_1)| = \Delta(G) + 3 \geq 8$, hence $f_1$ can be properly colored.

**Case 2.** A 3-vertex $v_1$ is incident to a face $f = \{v_1v_2 \cdots v_k\}$, where $3 \leq k \leq 4$.

Let $H = G - v_1v_2$ and define $L(f \oplus f_{v_1v_2})$ to be the set $L(f_{v_1v_2})$. By the induction hypothesis, $H$ has an $L$-edge-face coloring $\phi$. We first color $f_{v_1v_2}$ with $\phi(f \oplus f_{v_1v_2})$. If $\Delta(G) \geq 6$, we then color the edge $v_1v_2$ and the face $f$ in succession. One can check that $v_1v_2$ has, respectively, at most $\Delta(G) + 2$ and $|f|$ at most 8 forbidden colors, whereas $|L(f)| \geq 9$. An $L$-edge-face coloring is thus constructed from $\phi$. If $\Delta(G) = 5$ and $k = 3$, we can give a similar coloring since $f$ has at most 6 forbidden colors while $|L(f)| = 8$.

Now suppose that $\Delta(G) = 5$ and $k = 4$. Let $x \in N_G(v_1) \setminus \{v_2, v_4\}$. We remove the color of $v_1v_4$. Since both $v_1v_2$ and $v_1v_4$ are yet to be colored, there are at least two colors available for $v_1x$. Once a color of $v_1x$, say $\alpha$, is chosen, each of $v_1v_2$ and $v_1v_4$ has at least two colors to choose. Moreover there are at least two colors available for the face $f$. The reduced lists of $v_1v_2$, $v_1v_4$, and $f$ can be made non-identical by choosing $\alpha$ appropriately. Since the adjacency and incidence relations among $v_1v_2$, $v_1v_4$, and $f$ form a cycle of length 3, they can be properly colored by Lemma 8.

**Case 3.** There is a 5-face $f = \{v_1v_2v_3v_4v_5\}$ such that $d_G(v_1) = d_G(v_2) = 3$.

Let $H = G - v_1v_2$ and define $L(f \oplus f_{v_1v_2})$ to be the set $L(f_{v_1v_2})$. By the induction hypothesis, there is an $L$-edge-face coloring $\phi$ of $H$. We remove the colors of $v_3v_1$ and $v_5v_3$. We first color $f_{v_1v_2}$ with $\phi(f \oplus f_{v_1v_2})$ in $G$, then color $f$, $v_3v_1$, $v_2v_3$, and $v_1v_2$ in succession. It is easy to check that $f$ has at most 7, $v_3v_1$ at most $\Delta(G) + 2$, $v_2v_3$ at most $\Delta(G) + 2$, and $v_1v_2$ at most 6 forbidden colors, respectively.

**Case 4.** $\Delta(G) = 5$ and there is a strong edge $e = xy$ such that either $\sigma_5(e) \leq 9$ or $\sigma_5(e) = 10$ and $q(e) > 0$.

First note that $|L(t)| = 8$ for all $t \in E(G) \cup F(G)$. Let $f_1$ and $f_2$ be the strong faces of $e$ in $G$. Since $e$ is strong, at least one of $f_1$ and $f_2$ is of degree 3. We suppose that $d_G(f_1) = 3$. Let $H = G - e$ and define $L(f_1 \oplus f_2)$ to be the set $L(f_2)$. By the induction
hypothesis, \( H \) has an \( L \)-edge-face coloring \( \phi \). In order to modify \( \phi \) to an \( L \)-edge-face coloring of \( G \), we delete the color of the face \( f_1 \oplus f_2 \) and consider two subcases.

Assume that \( \sigma_5(e) \leq 9 \). If \( f_2 \) is a 3-face, it follows that \( d_G(x) + d_G(y) \leq 9 - 2 + t(e) = 9 \). Coloring \( e \), \( f_1 \), and \( f_2 \) in succession, one can check that \( e \) has at most 7, \( f_1 \) at most 5, and \( f_2 \) at most 6 forbidden colors, respectively. If \( f_2 \) is not a 3-face, then \( d_G(x) + d_G(y) \leq 9 - 2 + 1 = 8 \). We first color \( f_2 \) with the color \( \phi(f_1 \oplus f_2) \), then color \( e \) and \( f_1 \) in succession. It is easy to see that \( e \) has at most 7 and \( f_1 \) at most 6 forbidden colors, respectively. Assume that \( \sigma_5(e) = 10 \) and \( q(e) > 0 \). Then \( e \) is incident to a 3-face \( f_1 \) and a 4-face \( f_2 \) in \( G \) such that \( d_G(x) + d_G(y) = 10 + 1 - 2 = 9 \). We can color \( e \), \( f_1 \), and \( f_2 \) in succession because \( e \) has at most 7, \( f_2 \) at most 7, and \( f_1 \) at most 6 forbidden colors, respectively.

**Case 5.** \( \Delta(G) = 5 \) and \( G \) contains a conformable subgraph \( H_1 \).

Let \( H = G - uu_3 \) and define \( L([uu_2u_3] \oplus [uu_3u_4]) \) to be the set \( L([uu_2u_3]) \). By the induction hypothesis, \( H \) has an \( L \)-edge-face coloring \( \phi \). For convenience, we write \( A = ([uu_1u_2], [uu_1u_3], [uu_2u_3]\oplus[uu_3u_4]) \). Now we remove the colors of edges in \( E(H_1) \) and faces in \( A \). For every \( e \in E(H_1) \), let \( L'(e) = L(e) \setminus \{ \phi(z) : z \text{ belongs to } (E(H) \cup F(H)) \setminus (E(H_1) \cup A) \text{ and is adjacent or incident to } e \} \). It follows from \( \Delta(G) = 5 \) that \( |L'(uu_1)| \geq |L(uu_1)| - 2 = 6 \) for \( i = 2, 3, 4, |L'(uu_3)| \geq 4 \) for \( i = 1, 5, |L'(uu_2u_3)| \geq 3, |L'(uu_3u_4)| \geq 3, |L'(uu_1u_2)| \geq 2, \) and \( |L'(uu_4u_5)| \geq 2 \). By Lemma 9, \( E(H_1) \) is \( L \)-edge colorable. Once \( E(H_1) \) is colored, we can color [\( uu_1u_2 \), [uu_2u_3]], [uu_3u_4], [uu_4u_5] in succession.

**Case 6.** \( \Delta(G) = 6 \) and there is a weak edge \( e = xy \) with \( \sigma_5(e) \leq 10 \).

Note that \( |L(t)| = 9 \) for all \( t \in E(G) \cup F(G) \). Let \( f_1 \) and \( f_2 \) denote the incident faces of \( e \) in \( G \). Without loss of generality, we may assume that \( f_1 \) is a good face. So \( f_1 \) is either a minor face or a feasible 5-face. Let \( H = G - e \) and define \( L([f_1 \oplus f_2]) \) to be the set \( L(f_2) \). By the induction hypothesis, \( H \) has an \( L \)-edge-face coloring \( \phi \). For a face \( f' \in F(G) \), we use \( M(f') \) to denote the set of minor faces adjacent to \( f' \). We first remove the colors of all faces in \( M(f_1) \cup M(f_2) \). If \( f_2 \) is not a good face, then \( d_G(x) + d_G(y) \leq 9 \). We color \( f_2 \) with \( \phi(f_1 \oplus f_2) \), then color \( e, f_1 \), and all faces in \( M(f_1) \cup M(f_2) \) in succession. If \( f_2 \) is a good face, then \( d_G(x) + d_G(y) \leq 10 - 2 + t(e) + q(e) + p'(e) = 10 \). When \( f_1 \) is a minor face, we color \( e, f_2, f_1 \), then recolor all faces in \( M(f_1) \cup M(f_2) \) in succession. When \( f_1 \) is a feasible 5-face, we color \( e, f_1, f_2, \) and all faces in \( M(f_1) \cup M(f_2) \) in succession. It is easy to check that every element has at most 8 forbidden colors when it comes to be colored.

**Case 7.** \( \Delta(G) = 6 \) and \( G \) contains a conformable subgraph \( H_2 \).

Let \( H = G - v_1v_2 \). We define \( L([v_1v_2] \oplus [v_1v_2]) \) to be the set \( L([v_1v_2]) \). We write \( B = ([v_1v_2] \oplus [v_1v_2], [v_2v_3], [v_1v_2], [v_1v_2v_3], [v_1v_2v_3]) \). By the induction hypothesis, \( H \) has an \( L \)-edge-face coloring \( \phi \). We remove the colors of all faces in \( B \) and all edges in \( E(H_2) \cup [v_1v_2] \). For every \( e \in E(H_2) \), let \( L'(e) = L(e) \setminus \{ \phi(z) : z \text{ belongs to } (E(H) \cup F(H)) \setminus (E(H_2) \cup B) \text{ and is adjacent or incident to } e \} \). Since \( \Delta(G) = 6 \), we see that \( |L'(vu_i)| \geq |L(uv_i)| - 2 = 7 \) for \( i = 1, 2, |L'(vu_i)| \geq 6 \) for \( i = 3, 4, 5, |L'(v_1v_2)| \geq 5, |L'(v_2v_3)| \geq 3, |L'(v_3v_4)| \geq 3, \) and \( |L'(e)| \geq 2 \) for each
The new weight \( w \). Lemma 10 asserts that \( E(H_2) \) is \( L' \)-edge colorable. We then color \([w_1v_2], [v_1v_2], [v_2v_3], [v_3v_4], [v_4v_5], \) and \([v_5v_1]\) in succession to form an \( L \)-edge-face coloring of \( G \).

**Case 8.** \( \Delta(G) \geq 7 \) and there is a strong edge \( e = xy \) with \( \sigma_v(e) \leq 11 \).

Note that \( |L(t)| = \Delta(G) + 3 \geq 10 \) for all \( t \in E(G) \cup F(G) \). The argument is similar to Case 4. □

5. Proofs of Lemmas 6 and 7

**Proof of Lemma 6.** Suppose that the lemma is false and \( G \) is a counterexample. Therefore \( G \) is a plane graph, \( \Delta(G) = 5 \), \( \delta(G) \geq 2 \), and containing none of (C1)–(C3), (C4.1), and (C5.1). We define a subgraph \( H \) in the following way. If \( G \) contains a separating 3-cycle, then we choose a separating 3-cycle \( T^* \) with fewest interior vertices and define \( H = G[V^*(T^*) \cup V(T^*)] \), where \( V^*(T^*) \) denotes the set of vertices inside \( T^* \); otherwise, we let \( H = G \). In the sequel, we write \( V^*(H) \) for the set \( V(H) \setminus V(T^*) \). It is easy to see that \( H \) is a conformable induced subgraph of \( G \) satisfying \( d_H(v) = d_G(v) \) for each \( v \in V(H) \). By the definition of \( G \) and the choice of \( T^* \), the following configurations are excluded from \( H \).

(P1) A vertex \( v \in V(H) \) with \( d_H(v) = 2 \);
(P2) A 3-vertex \( v \in V(H) \) incident to a minor face;
(P3) A 5-face \( f \in F(H) \) incident to an edge \( xy \) such that \( x, y \in V^*(H) \) and \( d_H(x) = d_H(y) = 3 \);
(P4) A 5-vertex \( v \in V^*(H) \) incident to at least four 3-faces;
(P5) An edge \( xy \) incident to a 3-face such that \( x, y \in V^*(H) \) and \( d_H(x) = d_H(y) = 4 \);
(P6) An edge \( xy \) incident to a 3-face and another minor face such that \( x \in V^*(H) \) and \( d_H(x) = 4 \).

Since \( G \) is connected, so is \( H \). We have the following identity by rewriting Euler’s formula \(|H| - |E(H)| + |F(H)| = 2\).

$$\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12. \tag{1}$$

Let \( w \) denote the weight function defined on \( V(H) \cup F(H) \) by \( w(x) = 2d_H(x) - 6 \) if \( x \in V(H) \) and \( w(x) = d_H(x) - 6 \) if \( x \in F(H) \). Thus \( \sum \{w(x) \mid x \in V(H) \cup F(H)\} = -12 \). We are about to redistribute the vertex weight \( w(x) \) to its incident faces so that the new weight \( w'(x) \) is non-negative for all \( x \in V^*(H) \cup F(H) \) and \( \sum \{w'(x) \mid x \in V^*(H) \cup F(H)\} \geq -11 \). During the redistribution process, the sum of all weights is kept fixed. Then an obvious contradiction arises as follows.

$$-11 \leq \sum \{w'(x) \mid x \in V^*(H) \cup F(H)\} + \sum \{w'(x) \mid x \in V(T^*)\} = \sum \{w'(x) \mid x \in V(H) \cup F(H)\} = -12.$$
For \( v \in V(G) \) and \( f \in F(v) \), we use \( W(v \to f) \) to represent the weight transferred from \( v \) to \( f \) in the following discharging rules.

(R0) If \( v \in V(T^*) \), we let \( W(v \to f) = 1 \) for every incident face \( f \) of \( v \) in \( H \).

(R1) If \( v \in V^6(H) \) with \( d_H(v) = 4 \), we let \( W(v \to f) = 1 \) for every 3-face \( f \in F_3(v) \), \( W(v \to f) = \frac{1}{2} \) for every 4-face \( f \in F_4(v) \), and \( W(v \to f) = (2 - |F_2(v)| - \frac{1}{2}|F_4(v)|)/|F_3(v)| \) for every 5-face \( f \in F_5(v) \) if \( |F_3(v)| \geq 1 \).

(R2) If \( v \in V^6(H) \) with \( d_H(v) = 5 \), we let \( W(v \to f) = 1 \) for every 3-face \( f \in F_3(v) \), and \( W(v \to f) = \frac{1}{2} \) for every face \( f \in F(v) \setminus F_3(v) \).

Claim 1. For a 3-face \( f \in F(H) \) and a vertex \( v \in V(f) \), we have \( W(v \to f) = 1 \).

Claim 2. For a 4-face \( f \in F(H) \) and a vertex \( v \in V(f) \), we have \( W(v \to f) \geq \frac{1}{2} \).

Claim 1 holds by inspection. Claim 2 holds clearly when \( v \in V(T^*) \) by (R0). Suppose \( v \in V^6(H) \). (P1) and (P2) imply that \( d_H(v) \geq 4 \). Thus \( W(v \to f) \geq \frac{1}{2} \) by (R1) and (R2).

Now we want to show that conditions on the resultant weight function \( w'(x) \) are satisfied. First suppose that \( v \in V(H) \). If \( v \in V(T^*) \), then \( w'(v) \geq w(v) - d_H(v) = d_H(v) - 6 \) by (R0). Since \( \delta(G) \geq 2 \), it follows that \( \delta(H) \geq 2 \) and \( V(T^*) \) contains at most two 2-vertices in \( H \). Thus \( \sum_{v \in V(T^*)} w'(v) = \sum_{v \in V(T^*)} w(v) - 18 \geq 2 + 2 + 3 - 18 = -11 \).

Suppose \( v \in V^6(H) \). Then \( d_H(v) \geq 3 \) by (P1). If \( d_H(v) = 3 \), then \( w'(v) = w(v) = 0 \). Assume that \( d_H(v) = 4 \), then \( w'(v) = 2 \). Let \( A(v) \) denote the total weight transferred from \( v \) to its incident 3-faces and 4-faces. If \( v \) is not incident to any 3-face, then (R1) implies that \( A(v) \leq 4 \times \frac{1}{2} = 2 \). If \( v \) is incident to at least one 3-face, then \( |F_3(v)| + |F_4(v)| \leq 2 \) by (P6). Hence \( A(v) \leq 1 + 1 = 2 \). By (R1), the total weight transferred from \( v \) to its incident 5-faces (if any) is \( 2 - A(v) \), and \( w'(v) \geq 0 \) follows. Assume \( d_H(v) = 5 \), then \( w'(v) = 4 \). Since \( G \) does not contain \( P_4 \), \( v \) is incident to at most three 3-faces. Therefore \( w'(v) \geq 4 - 3 - \frac{1}{2} - \frac{1}{2} = 0 \) by (R2).

Suppose \( f \in F(H) \). If \( d_H(f) \geq 6 \), it is evident that \( w'(f) \geq w(f) = d_H(f) - 6 \geq 0 \). If \( d_H(f) = 3 \), then \( w'(f) = -3 \) and \( w'(f) \geq w(f) + 1 + 1 + 1 = 0 \) by Claim 1. If \( d_H(f) = 4 \), then \( w'(f) = -2 \) and \( w'(f) \geq w(f) + 4 \times \frac{1}{2} = 0 \) by Claim 2. Now assume that \( d_H(f) = 5 \), so \( w'(f) = -1 \). Let \( f = [v_1v_2v_3v_4v_5] \). If \( f \) is incident to at least one vertex in \( V(T^*) \), then \( w'(f) \geq -1 + 1 + 1 = 0 \) by (R0). Thus we suppose \( V(f) \subseteq V^6(H) \).

(P1) and (P3) imply that \( d_H(u_i) \geq 3 \) for all \( 1 \leq i \leq 5 \) and at least three \( u_i \)'s are of degree at least 4.

If \( V(f) \) contains at least two 5-vertices, then \( w'(f) \geq -1 + \frac{1}{2} + \frac{1}{2} = 0 \) by (R2). Assume that \( V(f) \) contains exactly one 5-vertex, say \( d_H(v_1) = 5 \). We see that \( W(v_1 \to f) = \frac{1}{2} \) by (R2). So \( V(f) \) contains at least two more 4-vertices. Suppose \( d_H(v_k) = 4 \) for some \( k \neq 1 \). If \( |F_3(v_k)| = 0 \), then \( W(v_k \to f) = \frac{1}{2} \) by (R1). If \( |F_3(v_k)| = 1 \), then \( v_k \) is incident to at most one 4-face by (P6), and hence \( W(v_k \to f) = \frac{1}{2} \). We claim now that \( |F_3(v_k)| \leq 1 \). Suppose on the contrary that \( v_k \) is incident to two 3-faces \( f_1 \) and \( f_2 \). Then \( f_1 \) is not adjacent to \( f_2 \) by (P6). It follows that both \( f_1 \) and \( f_2 \) are adjacent to the face \( f \). Suppose that \( v_{k+1} \neq v_1 \), otherwise we consider the case \( v_{k-1} \neq v_1 \). Then \( d_H(v_{k+1}) \leq 4 \) and \( v_kv_{k+1} \in b(f_i) \) for \( i = 1 \) or 2, this contradicts (P5). We therefore have \( w'(f) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{2} = 0 \). Finally, suppose that \( V(f) \) does not contain 5-vertices.
By (P5), $f$ is not adjacent to any 3-face. Thus every 4-vertex $v_k$ of $V(f)$ is incident to at most one 3-face. Similar to the previous argument, we always have $W(v_k \rightarrow f) \geq \frac{1}{3}$. Again, since $V(f)$ contains at least three 4-vertices, we obtain $w'(f) \geq -1 + 3 \times \frac{1}{3} = 0$. □

**Proof of Lemma 7.** Suppose that the lemma is false and $G$ is a counterexample. Therefore $G$ is a connected plane graph, $\Delta(G) = 6$, $\delta(G) \geq 2$, containing none of (C1)–(C3), (C4.2), and (C5.2). We proceed as in the previous proof. We define a subgraph $H$ in the following way. If $G$ contains a separating 3-cycle, then we choose a separating 3-cycle $T^*$ with fewest interior vertices and define $H = G[V^\circ(T^*) \cup V(T^*)]$; otherwise, we let $H = G$. We again write $V^\circ(H)$ for the set $V(H) \setminus V(T^*)$.

We say that a 5-face $f$ of $H$ is inner feasible if $V(f) \subseteq V^\circ(H)$ and it is adjacent to at least two minor faces of $H$. A 3-face of $H$ is said to be inner full if it is incident to a full 5-vertex of $V^\circ(H)$. For $v \in V(H)$, let $F^+_4(v)$ denote the set of the inner full 3-faces in $H$ each of which is incident to $v$. Thus (P1)–(P3), and the following configurations are excluded from $H$.

(Q1) A 3-vertex $v \in V^\circ(H)$ incident to an inner feasible 5-face;
(Q2) An edge $xy$ incident to a minor face or an inner feasible 5-face with $x, y \in V^\circ(H)$, $d_H(x) \leq 4$, and $d_H(y) \leq 5$;
(Q3) An edge $xy$ incident to two minor faces or inner feasible 5-faces such that either $x \in V^\circ(H)$ and $d_H(x) = 4$, or $x, y \in V^\circ(H)$ and $d_H(x) = d_H(y) = 5$;
(Q4) A conformable subgraph $H_2$ such that $v \in V^\circ(H)$ and $[wv_1v_2]$ is not the outer face of $H$ (cf. the definition of $H_2$ in Section 3).

We use identity $(1)$ and the same weight function $w$ on $V(H) \cup F(H)$ again. Let $v \in V(H)$. We carry out the following discharging rules.

(R0) If $v \in V(T^*)$, we let $W(v \rightarrow f) = 1.1$ for every incident face $f$ in $H$.

(R1) If $v \in V^\circ(H)$ with $d_H(v) = 4$, we let $W(v \rightarrow f) = 1$ for every 3-face $f \in F_3(v)$, $W(v \rightarrow f) = \frac{1}{2}$ for every 4-face $f \in F_4(v)$, and $W(v \rightarrow f) = (2 - |F_3(v)| - \frac{1}{4}|F_4(v)|)/|F_3(v)|$ for every 5-face $f \in F_5(v)$ if $|F_3(v)| \geq 1$.

(R2) If $v \in V^\circ(H)$ with $d_H(v) = 5$, we have the following subrules.

- If $|F_3(v)| \leq 3$, we let $W(v \rightarrow f) = 1$ for every 3-face $f \in F_3(v)$ and $W(v \rightarrow f) = \frac{1}{2}$ for every face $f \in F(v) \setminus F_3(v)$.
- If $|F_3(v)| = 4$, we let $W(v \rightarrow f) = 1$ for every 3-face $f \in F_3(v)$.
- If $|F_3(v)| = 5$, we let $W(v \rightarrow f) = \frac{1}{2}$ for every 3-face $f \in F_3(v)$.

(R3) If $v \in V^\circ(H)$ with $d_H(v) = 6$, we let $W(v \rightarrow f) = 1.1$ for every inner full 3-face $f \in F^+_3(v)$, $W(v \rightarrow f) = 1$ for every 3-face $f \in F_3(v) \setminus F^+_3(v)$, and $W(v \rightarrow f) = (6 - 1.1|F^+_3(v)| - |F_3(v) \setminus F^+_3(v)|)/|F(v) \setminus F_3(v)|$ for every face $f \in F(v) \setminus F_3(v)$ if $|F(v) \setminus F_3(v)| \geq 1$.

**Claim 1.** If a 6-vertex $u \in V^\circ(H)$ is adjacent to a full 5-vertex $v \in V^\circ(H)$, then $|F_3(u)| \leq 4$.

Let $f_1, f_2, f_3, f_4, f_5$ denote the incident faces of $v$ enumerated clockwise, and $f_1, f_2, f_3', f_4', f_5'$, and $f_6'$ denote the incident faces of $u$ enumerated counterclockwise.


Thus the edge $uv$ is incident to both $f_1$ and $f_2$ and all $f_i$ are of degree 3. Moreover, every $f_i$ and every $f'_j$ are inner faces of $H$ since $u, v \in V^o(H)$. Suppose on the contrary that $|F_3(u)| \geq 5$. It follows that at least one of $f'_j$ and $f_6$ is of degree 3. Suppose $f'_6 = [uxy]$ is a 3-face satisfying $x, u \in V(f_1)$. If $y \notin N_H(u)$, then $H$ would contain a conformable subgraph $H_2$, contradicting (Q4). If $y \in N_H(u)$, it is easy to see that $G$ contains a separating 3-cycle $T'$ with fewer interior vertices than $T^*$, which contradicts the choice of $T^*$. Thus we have $|F_3(u)| \leq 4$.

Claim 2. For every 6-vertex $v \in V^o(H)$ and every face $f \in F(v)$ with $d_H(f) \geq 4$, we have $W(v \to f) \geq \frac{4}{5}$.

Claim 2 is obvious by (R’3) when $v$ is not incident to any full 3-face. If $v$ is incident to some full 3-face, we see that $|F_3^*(v)| \leq |F_3(v)| \leq 4$ by Claim 1. Thus $W(v \to f) \geq (6 - 4 \times 1.1)/2 = \frac{4}{5}$.

Claim 3. For every 3-face $f \in F(H)$ and every vertex $u \in V(f)$, we have $W(u \to f) \geq \frac{4}{5}$; and $W(u \to f) = \frac{4}{5}$ if and only if $u \in V^o(H)$ is a full 5-vertex.

This follows immediately from (R’0)–(R’3) since $u \in V^o(H)$ implies $d_H(u) \geq 4$ by (P1) and (P2).

Claim 4. Let $f$ be a full 3-face and $u \in V^o(H)$ a full 5-vertex on the boundary of $f$. If $x \in (V(f) \setminus \{u\}) \cap V^o(H)$, then $d_H(x) = 6$.

Suppose on the contrary that there exists a vertex $x \in (V(f) \setminus \{u\}) \cap V^o(H)$ such that $d_H(x) \leq 5$. Then the edge $ux$ is incident to two 3-faces, contradicting (Q3).

Now let $w'$ denote the resultant weight function when the discharging is complete. We will prove that $w'(x) \geq 0$ for all $x \in V^o(H) \cup F(H)$.

Let $v \in V^o(H)$. If $d_H(v) \leq 5$, the proof of $w'(v) \geq 0$ is similar to that of Lemma 6. If $d_H(v) = 6$, then $w(v) = 6$. By (R’3) and Claim 1, we see that the weight transferred from $v$ to its incident faces is at most 6. Thus $w'(v) \geq 0$.

Suppose $f \in F(H)$. If $d_H(f) \geq 6$, it is evident that $w'(f) \geq w(f) \geq 0$. Assume that $d_H(f) \leq 5$ and let $f = [u_1u_2\ldots u_{d_H(f)}]$. Let $f_{i,i+1}$ denote the adjacent face of $f$ in $H$ sharing the common boundary edge $u_iu_{i+1}$ with $f$ (indices modulo $d_H(f)$) for $i = 1, 2, \ldots, d_H(f)$.

Assume that $d_H(f) = 3$. Thus $w(f) = -3$. If $V(f) \subseteq V(T^*)$, i.e., $f$ is the outer face of $H$, then $w'(f) \geq -3 + 3 \times 1.1 = 0.3$ by (R’0). If $|V(f) \cap V(T^*)| = 2$, then $w'(f) \geq -3 + 0.8 + 2 \times 1.1 = 0$ by Claim 3. If $|V(f) \cap V(T^*)| = 1$, we suppose $u_1 \in V(T^*)$. If $f$ is not an inner full 3-face, then $w'(f) \geq -3 + 1 + 1 + 1.1 = 0.1$ by (R’1) to (R’3). If $f$ is an inner full 3-face, we see that $W(u_2 \to f) + W(u_3 \to f) \geq 1.1 + 0.8 = 1.9$ by Claim 4. Hence $w'(f) \geq -3 + 1.1 + 1.9 = 0$. Suppose now that $V(f) \subseteq V^o(H)$. If $f$ is not a full 3-face, then $w'(f) \geq -3 + 1 + 1 + 1 = 0$. If $f$ is a full 3-face, then $w'(f) \geq -3 + 2 \times 1.1 + 0.8 = 0$ by Claim 4.

Assume that $d_H(f) = 4$. Then $w(f) = -2$. Obviously, $V(f)$ does not contain any full 5-vertex. If $|V(f) \cap V(T^*)| \geq 2$, then $w'(f) \geq -2 + 1.1 + 1.1 = 0.2$. Thus we suppose that $|V(f) \cap V(T^*)| \leq 1$. 
First consider the case $|V(f) \cap V(T^*)| = 0$. We see that $d_H(u_i) \geq 4$ for $i = 1, 2, 3, 4$ by (P1) and (P2). Without loss of generality, we suppose that $d_H(u_1) = \min_{1 \leq i \leq 4}(d_H(u_i))$.

If $d_H(u_1) = 4$, then $d_H(u_2) = d_H(u_4) = 6$ by (Q2), and hence $w'(f) \geq -2 + 2 \times 0.8 + 0.5 = 0.1$ by (R'1) and Claim 2. So we suppose that $d_H(u_1) \geq 5$. If $V(f)$ contains at least three 6-vertices, then $w'(f) \geq -2 + 3 \times 0.8 = 0.4$.

If $V(f)$ contains at most one 6-vertex, we suppose that $d_H(u_1) = d_H(u_2) = d_H(u_3) = 5$. Then $d_H(f_{1,2}) \geq 5$ and $d_H(f_{2,3}) \geq 5$ by (Q3). This implies that each $u_i, i = 1, 2, 3$, is not a nearly-full 5-vertex, and thus $W(u_i \rightarrow f) = \frac{1}{2}$ by (R'2). If $u_4$ is a 6-vertex, then $w'(f) \geq -2 + 3 \times 0.5 + 0.8 = 0.3$ by Claim 2. If $u_4$ is a 5-vertex, we can reason as before that $u_4$ is not a nearly-full 5-vertex. Consequently, $w'(f) \geq -2 + 4 \times 0.5 = 0$ by (R'2).

If $V(f)$ contains exactly two 6-vertices, we need to handle the following two cases. If $d_H(u_2) = d_H(u_3) = 6$, then by (Q3) neither $u_1$ nor $u_4$ is nearly-full 5-vertices, hence $w'(f) \geq -2 + 2 \times 0.8 + 2 \times 0.5 = 0.6$ by Claim 2. Assume that $d_H(u_2) = d_H(u_4) = 6$. If either $u_1$ or $u_3$ is not a nearly-full 5-vertex, then $w'(f) \geq -2 + 2 \times 0.8 + 0.5 = 0.1$ by (R'1) and (R'2). If both $u_1$ and $u_3$ are nearly-full 5-vertices, we claim that neither $u_2$ nor $u_4$ is incident to a full 3-face whose boundary contains a full 5-vertex of $V^o(H)$. We can show that $|V(f) \cap V(T^*)| = 1$. It suffices to prove this for $u_2$ since $u_4$ can be treated similarly. Let $u_1, x_1, x_2, x_3, x_4, u_3$ be the neighbors of $u_2$ in $H$ enumerated clockwise. Assume $i = 1$ or 4. The edge $u_1 x_i$ is incident to two 3-faces and $x_i \in V^o(H)$, so $d_H(x_i) = 6$ by (Q3). Next, if $x_2 \in V^o(H)$ is a full 5-vertex, then $H$ contains a conformable subgraph $H_2$ on the vertex set $N_H(x_2) \cup \{x_2, u_1\}$, contradicting (Q4). Similarly, we may prove that $x_3$ is not a full 5-vertex if it belongs to $V^o(H)$.

Now suppose that $|V(f) \cap V(T^*)| = 1$ and $u_1 \in V(T^*)$. Then by (R'0) the amount 1.1 is transferred from $u_1$ to the face $f$. We can reason as before to show that $W(u_2 \rightarrow f) + W(u_3 \rightarrow f) + W(u_4 \rightarrow f) \geq 1$, and hence $w'(f) \geq 0.1$.

Assume $d_H(f) = 5$. Then $w(f) = -1$. If $|V(f) \cap V(T^*)| \neq 0$, then $w'(f) \geq -1 + 1.1 = 0.1$ by (R'0). So suppose that $V(f) \subseteq V^o(H)$. By (P1), (P3), and the choice of $T^*$, we see that all $u_i$ are of degree at least 3 and at least three of them are of degree at least 4.

If $V(f)$ contains at least two 6-vertices, then $w'(f) \geq -1 + 2 \times 0.8 = 0.6$ by Claim 2.

If $V(f)$ contains exactly one 6-vertex, say $u_1$, then (Q3) implies that either $f_{2,3}$ or $f_{3,4}$ is of degree at least 5. Without loss of generality, we suppose $d_H(f_{2,3}) \geq 5$. By (P3), at least one of $u_2$ and $u_3$ is of degree at least 4, say $d_H(u_2) \geq 4$. If $d_H(u_2) = 4$, then $u_2$ is incident to at most one minor face by (Q3), hence $W(u_2 \rightarrow f) \geq \frac{1}{2}$ by (R'1). We therefore have $w'(f) \geq -1 + 0.8 + 0.25 = 0.05$. If $d_H(u_2) = 5$, then it is obvious that $W(u_2 \rightarrow f) \geq \frac{1}{2}$ since $u_2$ is not a nearly-full 5-vertex in this case. We also have $w'(f) \geq -1 + 0.8 + 0.5 = 0.3$.

If $V(f)$ does not contain 6-vertices, then $f$ is adjacent to at most one minor face by (Q3). If $f$ is not adjacent to any minor face, then $W(u_1 \rightarrow f) \geq \frac{1}{2}$ when $d_H(u_1) \geq 4$. Note that $b(f)$ contains at least three vertices of degree at least 4. Thus $w'(f) \geq -1 + 3 \times \frac{1}{2} = 0$. If $f$ is adjacent to exactly one minor face, say $d_H(f_{1,2}) \leq 4$, then $d_H(u_1) = d_H(u_2) = 5$ by (Q2) and thus $w'(f) \geq -1 + 2 \times \frac{1}{2} = 0$. 


For every vertex \( v \in V(T^*) \), we have that \( w'(v) \geq w(v) - 1.1d_H(v) = 2d_H(v) - 6 - 1.1d_H(v) = 0.9d_H(v) - 6 \) by (R0). Since \( \delta(H) \geq 2 \) and \( H \) is connected, \( V(T^*) \) contains a vertex of degree at least 3. Thus \( \sum_{v \in V(T^*)} w'(v) \geq -3.3 - 4.2 - 4.2 = -11.7 \). Finally, the following contradiction completes the proof.

\[
-11.7 \leq \sum_{x \in V(T^*)} w'(x) + \sum_{x \in V(H) \cup F(H)} w'(x) = \sum_{x \in V(H) \cup F(H)} w(x) = -12. \quad \square
\]

6. Smaller upper bounds

The well-known list coloring conjecture asserts that every multigraph \( G \) satisfies \( \chi'_c(G) = \chi'(G) \). Galvin [6] established this conjecture for bipartite multigraphs. It implies that \( \chi'_c(G) = \Delta(G) \) for a simple bipartite graph \( G \). For a planar graph \( G \), it is known that \( \chi'_c(G) \leq \Delta(G) + 1 \) if \( \Delta(G) \geq 9 \) [1], and \( \chi'_c(G) = \Delta(G) \) if \( \Delta(G) \geq 12 \) [4]. The following result follows immediately from these facts and Theorem 4.

**Theorem 11.** Let \( G \) be a plane graph with \( \Delta(G) \geq 3 \). If either \( \Delta(G) \geq 12 \) or \( G \) is bipartite, then \( \chi'_{cl}(G) \leq \Delta(G) + 2 \).

**Theorem 12.** If \( G \) is a plane graph with \( \Delta(G) \geq 2\Delta^*(G) \), then \( \chi'_{cl}(G) = \chi'_c(G) \).

**Proof.** It is obvious that \( \chi'_{cl}(G) \geq \chi'_c(G) \geq \chi'(G) \geq \Delta(G) \). Let \( L \) be an assignment of \( G \) satisfying \( |L(x)| = \chi'_c(G) \) for every \( x \in E(G) \cup F(G) \). Thus \( G \) admits a partial \( L \)-edge coloring \( \phi \). For every \( f \in F(G) \), we define a new list \( L'(f) = L(f) \setminus \{\phi(e) \mid e \text{ is incident to } f\} \). Clearly, \( |L'(f)| \geq |L(f)| - d_G(f) \geq \Delta(G) - \Delta^*(G) \geq \Delta^*(G) \). Since \( \Delta^*(G) \geq 3 \), we have \( \Delta(G) \geq 6 \). It follows that \( G \) and its dual graph \( G^* \) are neither complete graphs nor odd cycles. If \( v^* \) is the vertex in \( G^* \) corresponding to the face \( f \) in \( G \), then we assign the list \( L'(f) \) to \( v^* \). Thus \( |L'(v^*)| \geq \Delta^*(G) = \Delta(G^*) \) for all \( v^* \in V(G^*) \).

By a well-known generalization of Brooks’ theorem to choosability [5], \( G^* \) is \( L^* \)-vertex colorable. Hence \( G \) is \( L^* \)-face colorable. The construction of an \( L^* \)-edge-face coloring of \( G \) can be accomplished. Hence \( \chi'_{cl}(G) \leq \chi'_c(G) \), and therefore \( \chi'_{cl}(G) = \chi'_c(G) \).

**Corollary 13.** Let \( G \) be a plane graph with \( \Delta(G) \geq 12 \). If \( \Delta^*(G) \leq 6 \), then \( \chi'_{cl}(G) = \Delta(G) \).

Borodin [2] proved that every plane graph \( G \) with \( \Delta(G) \geq 10 \) is \( (\Delta(G) + 1) \)-edge-face colorable. In fact, his proof implies the following stronger result.

**Theorem 14.** If \( G \) is a plane graph with \( \Delta(G) \geq 10 \), then \( \chi'_{cl}(G) \leq \chi_{cl}(G) \leq \Delta(G) + 1 \).

Both the upper and the lower bounds in Theorem 14 are best possible. For instance, a wheel \( W_n \) of order \( n > 10 \) satisfies \( \chi'_{cl}(W_n) = \chi_{cl}(W_n) = \Delta(W_n) \). A star \( K_{1,n} \) satisfies \( \chi'_{cl}(K_{1,n}) = \chi_{cl}(K_{1,n}) = \Delta(K_{1,n}) + 1 \) if \( n \geq 2 \).

We conclude this paper with the following open problems.

**Problem 15.** Characterize plane graphs \( G \) that satisfy \( \chi'_{cl}(G) = \chi_{cl}(G) \).
Problem 16. Determine a sharp upper bound for $\chi^L_G(G)$ if the plane graph $G$ satisfies $3 \leq \Delta(G) \leq 9$.

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