

Note on the Intermediate Field Representation of ϕ^{2k} Theory in Zero Dimension

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Abstract

This note is a sequel to [1]. We simplify and make more systematic the intermediate field representation for the stable ϕ^{2k} field theory in zero dimension introduced there and we extend it to the case of complex conjugate fields. We also correct some mistakes of [1]: in particular in section 4 of for $k > 3$ Lemma 4.1 as stated there is wrong but becomes correct by adding a single last step of intermediate field decomposition.

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I Introduction

The intermediate field representation and the associated constructive loop vertex expansion (LVE) [2]-[5] have been increasingly used in recent years [6]-[13] for models with quartic interactions. It is important to extend such techniques to models with higher order stable interactions, as first attempted in [1]. The case of a ϕ^6 interaction is treated in sections 2 and 3 of [1], using imaginary Gaussian measures with a small contour deformation, and the case of a general ϕ^{2k} interaction is sketched in section 4 of [1]. Unfortunately Lemma 4.1 as stated there is not correct and requires a slight modification. Also the number of intermediate fields introduced in [1] is not optimal. Finally we found that the loop vertex expansion in [1] is not correct since the interpolation of imaginary Gaussian covariances through forest formulas is not fully justified.

The purpose of this note is to give a new, correct version of the intermediate field representation of [1]. for the ϕ^{2k} stable interaction for general k in dimension 0, which is inspired by, but not identical to that of [1], and to extend it to

the case of complex conjugate fields. The problem is subtle because imaginary Gaussian integrals with deformed contours are not easy to manipulate and in particular may not always commute. So the strategy we use to derive our intermediate field representation is the following. In typical theoretical physicist manner, we first guess the form of our intermediate field representation by commuting functional integrals in a way not necessarily justified non-perturbatively but which respects every order of perturbation. Then, and this is the main mathematical result of this paper, having guessed the desired form of our intermediate field representation, we check that it is an absolutely convergent integral and establish a posteriori that it is equal (non-perturbatively!) to the initial integral. This is done by proving Borel-Le Roy summability both of the initial partition function and of the intermediate field partition function. Since these two partition functions have the same asymptotic power series at the origin, by unicity of the Borel sum we conclude that the two functions are the same. This justifies a posteriori our heuristic derivation.

However in spite of several attempts, we have not been able yet to define a convergent LVE for this intermediate field representation, hence correct the last problem in [1]. Although we feel that it would be a necessary step for the constructive study of matrix and tensor [14] models with higher than quartic interactions, we have therefore unfortunately to leave this difficult point to a future study.

II ϕ^{2k} theory in zero dimension

II.1 Imaginary Gaussian Measures

Consider a function $f(z)$ which is analytic in the strip $\Im z \leq \delta$ and exponentially bounded in that domain by $Ke^{\eta|z|}$ for some $0 \leq \eta < \delta$, where K is some constant.

The imaginary Gaussian integral of f with covariance $\pm iC$, where $C > 0$, is then defined as

$$\int d\mu_{\pm iC}(x)f(x) := \int_{C_{\pm,\epsilon}} \frac{e^{-z^2/\pm 2iC} dz}{\sqrt{\pm 2\pi iC}} f(z) = \int_{C_{\pm,\epsilon}} \frac{e^{\pm iz^2/2C} dz}{\sqrt{\pm 2\pi iC}} f(z) \quad (\text{II-1})$$

where the contour $C_{\pm,\epsilon}$ can be for instance chosen as the graph in the complex plane (identifying \mathbb{C} to \mathbb{R}^2) of the real function $\Re z = x \rightarrow \Im z = y = \pm \epsilon \tanh(x)$ for any $\epsilon \in]C\eta, \delta[$. Remark indeed that from our hypotheses on f , the integral (II-1) is well defined and absolutely convergent for $C\eta < \epsilon < \delta$, and by Cauchy theorem, independent of $\epsilon \in]C\eta, \delta[$. The contour $C_{+,\epsilon}$ is shown in Figure 1.

Although the result of integration does not depend on the contour, actual bounds on the result typically depend on choosing particular contours in which ϵ is not too small, see Appendix.

Remark also that if f is a polynomial, the Gaussian rules of integration apply, e.g. defining $(2n-1)!! := (2n-1)(2n-3)\cdots 5.3.1$

$$\int d\mu_{\pm iC}(x)x^{2n} = (\pm iC)^n (2n-1)!! . \quad (\text{II-2})$$

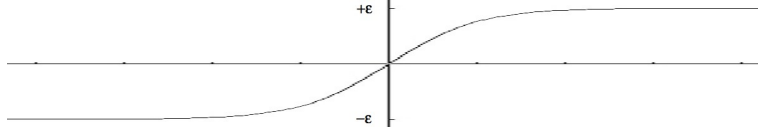


Figure 1: The integration contour $C_{+,\epsilon}$.

This is easy to check since a polynomial is an entire function and we can deform the contour into $z = x + ix$, in which case we recover an ordinary Gaussian integration. Similarly

$$\int d\mu_{\pm iC}(x)e^{ax} = e^{\pm iCa^2/2}, \quad (\text{II-3})$$

the integral being absolutely convergent for any contour such that $C|a| < \epsilon$ ¹.

We can also define imaginary complex normalized Gaussian measures $d\mu_{\pm iC}^c(x)$ of covariance $\pm i$ for a pair of complex conjugate variables z and \bar{z} for which the computation becomes

$$\int d\mu_{\pm iC}^c(x)(z\bar{z})^n = (\pm iC)^n n! \quad (\text{II-4})$$

and

$$\int d\mu_{\pm iC}^c(x)e^{az+b\bar{z}} = e^{\pm iabC}, \quad (\text{II-5})$$

again these integrals being absolutely convergent if $C \sup\{|a|, |b|\} < \epsilon$.

II.2 ϕ^{2k} theory in 0 dimension

The partition function of the ϕ^{2k} scalar theory in zero dimension for $k \geq 2$ is the one dimensional integral

$$Z_k(\lambda) = \int_{-\infty}^{+\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} e^{-\lambda\phi^{2k}/2} = \int d\mu(\phi)e^{-\lambda\phi^{2k}/2}, \quad (\text{II-6})$$

where $d\mu$ is the normalized one-dimensional Gaussian measure of covariance 1. Its “free energy” is simply $\log Z_k(\lambda)$. The factor 1/2 in front of λ is a suitable normalization to simplify the intermediate field representation below.

The complex version of this integral is

$$Z_{k,c}(\lambda) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\phi d\bar{\phi}}{\pi} e^{-\bar{\phi}\phi} e^{-\lambda(\bar{\phi}\phi)^k} = \int d\mu^c(\phi)e^{-\lambda(\bar{\phi}\phi)^k}, \quad (\text{II-7})$$

where $d\mu^c$ is the normalized one-dimensional complex Gaussian measure of covariance 1.

¹As usual we can extend these formulas to the case $C = 0$ by defining $d\mu$ in this case to be the Dirac measure at the origin.

The goal is to prove in an explicitly convergent intermediate field representation the following standard result

Theorem II.1. *The partition functions $Z_k(\lambda)$ and $Z_{k,c}(\lambda)$ and the free energies $\log Z_k(\lambda)$ and $\log Z_{k,c}(\lambda)$ are Borel-Le Roy summable of order $k - 1$.*

For completeness we recall Nevanlinna's theorem for Borel-Le Roy summability of order k in the Appendix, and a proof of Theorem II.1 in the standard representation.

II.3 Intermediate field representation for ϕ^{2k}

We first split the interaction in two using an intermediate field σ with normalized Gaussian measure $d\mu(\sigma)$ of covariance 1. The result is:

$$e^{-\lambda\phi^{2k}/2} = \int d\mu(\sigma) e^{i\sqrt{\lambda}\phi^k\sigma}. \quad (\text{II-8})$$

We define $g_k = \lambda^{\frac{1}{2k}}$, and as next step we decompose

$$i\sqrt{\lambda}\phi^k\sigma = \frac{i}{4}[(g_k\phi\sigma + (g_k\phi)^{k-1})^2 - (g_k\phi\sigma - (g_k\phi)^{k-1})^2]. \quad (\text{II-9})$$

We introduce a pair of intermediate fields a_1 and b_1 with imaginary covariances $-i$ and $+i$, hence the Gaussian measure $d\mu_{\pm i}(a_1, b_1) = d\mu_{-i}(a_1)d\mu_i(b_1)$ so that

$$e^{i\sqrt{\lambda}\phi^k\sigma} = \int d\mu_{\pm i}(a_1, b_1) e^{\frac{i}{\sqrt{2}}[(g_k\phi\sigma + (g_k\phi)^{k-1})a_1 + (g_k\phi\sigma - (g_k\phi)^{k-1})b_1]} \quad (\text{II-10})$$

$$= \int d\mu_{\pm i}(a_1, b_1) e^{i[g_k\phi\sigma\frac{a_1+b_1}{\sqrt{2}} + (g_k\phi)^{k-1}\frac{a_1-b_1}{\sqrt{2}}]}. \quad (\text{II-11})$$

We now change variables for

$$\alpha_1 = \frac{a_1 + b_1}{\sqrt{2}}, \quad \beta_1 = \frac{a_1 - b_1}{\sqrt{2}}, \quad (\text{II-12})$$

so that

$$e^{i\sqrt{\lambda}\phi^k\sigma} = \int d\mu_X(\alpha_1, \beta_1) e^{ig_k\phi\sigma\alpha_1 + i(g_k\phi)^{k-1}\beta_1}, \quad (\text{II-13})$$

where the measure $d\mu_X$ is defined by its moments

$$\langle \alpha_1\beta_1 \rangle_X = -i, \quad \langle \alpha_1^2 \rangle_X = 0, \quad \langle \beta_1^2 \rangle_X = 0. \quad (\text{II-14})$$

We keep the term $ig_k\phi\sigma\alpha_1$ and decompose the $ig_k^{k-1}\phi^{k-1}\beta_1$ term as

$$e^{ig_k^{k-1}\phi^{k-1}\beta_1} = \int d\mu_X(\alpha_2, \beta_2) e^{ig_k\phi\beta_1\alpha_2 + i(g_k\phi)^{k-2}\beta_2}. \quad (\text{II-15})$$

Continuing in this way we prove inductively the following representation:

$$e^{-\lambda\phi^{2k}/2} = \int d\mu(\sigma) \prod_{j=1}^{k-1} d\mu_X(\alpha_j, \beta_j) e^{ig_k[\phi\sigma\alpha_1 + \sum_{j=1}^{k-2} \phi\beta_j\alpha_{j+1} + \phi\beta_{k-1}]}, \quad (\text{II-16})$$

where the α_j, β_j and the measure $d\mu_X$ are respectively defined as in (II-12) and (II-14).

We now integrate²

- for k odd, over ϕ, σ and all even α_{2j}, β_{2j} , for $j \in \{1, \dots, \frac{k-1}{2}\}$. In that case we denote $\Phi = (\phi, \sigma, \alpha_2, \beta_2, \dots, \alpha_{k-1}, \beta_{k-1})$ the $k+1$ integrated variables and $\Psi = (\alpha_1, \beta_1, \dots, \alpha_{k-2}, \beta_{k-2})$ the $k-1$ remaining ones. The Gaussian measure $d\mu(\sigma) \prod_{j=1}^{k-2} d\mu_X(\alpha_j, \beta_j)$ factorizes as $d\nu(\Phi)d\chi(\Psi)$.
- for k even, over ϕ and all odd $\alpha_{2j-1}, \beta_{2j-1}$, for $j \in \{1, \dots, \frac{k}{2}\}$. In that case we denote $\Phi = (\phi, \alpha_1, \beta_1, \dots, \alpha_{k-1}, \beta_{k-1})$ the $k+1$ integrated variables and $\Psi = (\sigma, \alpha_2, \beta_2, \dots, \alpha_{k-2}, \beta_{k-2})$ the $k-1$ remaining ones. The Gaussian measure $d\mu(\sigma) \prod_{j=1}^{k-2} d\mu_X(\alpha_j, \beta_j)$ factorizes again as $d\nu(\Phi)d\chi(\Psi)$.

The partition function therefore writes

$$Z_k(\lambda) = \int d\chi(\Psi) \left[d\nu(\Phi) \exp\left[\frac{ig_k}{2} \langle \Phi, H_k(\Psi) \cdot \Phi \rangle \right], \quad (\text{II-17})$$

where H_k is a $(k+1) \times (k+1)$ real symmetric matrix. More precisely :

- if $k = 2p + 1$ is odd, H_k is

$$H_k = \left(\begin{array}{c|cccc} 0 & \alpha_1 & \beta_1 & \cdots & \beta_{k-2} & 1 \\ \alpha_1 & & & & & \\ \beta_1 & & & & & \\ \vdots & & & & & \\ \beta_{k-2} & & & & 0 & \\ 1 & & & & & \end{array} \right), \quad (\text{II-18})$$

- if $k = 2p$ is even, H_k is

$$H_k = \left(\begin{array}{c|cccc} 0 & \sigma & \alpha_2 & \cdots & \beta_{k-2} & 1 \\ \sigma & & & & & \\ \alpha_2 & & & & & \\ \vdots & & & & & \\ \beta_{k-2} & & & & 0 & \\ 1 & & & & & \end{array} \right). \quad (\text{II-19})$$

²The careful reader might worry whether exchange in the order of integrals is allowed, in particular in view of the contour conditions in (II-3). However, following the strategy explained in the Introduction, we consider the computations of this Section II just as a heuristic way to arrive at the integral representation (II-20); That this integral representation is absolutely convergent and in fact equal to the initial $Z(\lambda)$ is proved only in Section III.

The Gaussian integration over Φ gives a determinant, which, rewritten in the following way, gives a new representation of the partition function.

Theorem II.2 (Intermediate Field Representation of ϕ^{2k}).

$$Z_k(\lambda) = \int d\chi(\Psi) \exp\left[-\frac{1}{2}\text{Tr} \ln(1 - g_k M(\Psi))\right], \quad (\text{II-20})$$

where $M_k(\Psi) = iC_k \cdot H_k(\Psi)$ and C_k , the covariance for the Φ variables, is

$$C_{\text{odd}} = \left(\begin{array}{c|cc|cc|c} 1 & 0 & & & & \\ \hline 0 & 1 & & & & \\ \hline & & \boxed{0 & -i} & & 0 \\ & & \boxed{-i & 0} & & \\ \hline & & & & \boxed{0 & -i} & \\ & & & & \boxed{-i & 0} & \\ \hline & & & & & & \ddots \end{array} \right), \quad (\text{II-21})$$

$$C_{\text{even}} = \left(\begin{array}{c|cc|cc|c} 1 & & & & & \\ \hline & \boxed{0 & -i} & & & 0 \\ & \boxed{-i & 0} & & & \\ \hline & & & & \boxed{0 & -i} & \\ & & & & \boxed{-i & 0} & \\ \hline & & & & & & \ddots \end{array} \right). \quad (\text{II-22})$$

The proof of this Theorem and of the convergence of the integral representation (II-20) is given in Section III. In the simplest cases $k = 3, 4$, hence for the $e^{-\lambda\phi^6/2}$ and $e^{-\lambda\phi^8/2}$ models, we obtain the representations :

$$Z_3(\lambda) = \int d\chi(\alpha_1, \beta_1) e^{-\frac{1}{2}\text{Tr} \ln[1 - \lambda^{1/6} M_3]} \quad (\text{II-23})$$

with

$$M_3 = \begin{pmatrix} 0 & i\alpha_1 & i\beta_1 & i \\ i\alpha_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \quad (\text{II-24})$$

and

$$Z_4(\lambda) = \int d\chi(\sigma, \alpha_2, \beta_2) e^{-\frac{1}{2}\text{Tr} \ln[1 - \lambda^{1/8} M_4]}, \quad (\text{II-25})$$

with

$$M_4 = \begin{pmatrix} 0 & i\sigma & i\alpha_2 & i\beta_2 & i \\ \alpha_2 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{II-26})$$

II.4 Intermediate field representation for $(\phi\bar{\phi})^k$

As in the previous section, we first split the interaction in two using a complex intermediate field σ with normalized Gaussian measure $d\mu(\sigma)$ of covariance 1. We obtain

$$e^{-\lambda(\phi\bar{\phi})^{2p+1}} = \int d\mu^c(\sigma) e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\bar{\phi}\sigma+\phi\bar{\sigma})}, \quad (\text{II-27})$$

$$e^{-\lambda(\phi\bar{\phi})^{2p}} = \int d\mu^c(\sigma) e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\sigma+\bar{\sigma})}. \quad (\text{II-28})$$

We define $g_k = \lambda^{\frac{1}{2k}}$, and as next step we decompose

$$i\sqrt{\lambda}(\phi\bar{\phi})^p(\bar{\phi}\sigma+\phi\bar{\sigma}) = \frac{i}{2}[|g_k\bar{\phi}\sigma + (g_k^2\phi\bar{\phi})^p|^2 - |g_k\bar{\phi}\sigma - (g_k^2\phi\bar{\phi})^p|^2], \quad (\text{II-29})$$

$$i\sqrt{\lambda}(\phi\bar{\phi})^p(\sigma+\bar{\sigma}) = \frac{i}{2}[|g_k\bar{\phi}\sigma + (g_k^2\phi\bar{\phi})^{p-1}g_k\bar{\phi}|^2 - |g_k\bar{\phi}\sigma - (g_k^2\phi\bar{\phi})^{p-1}g_k\bar{\phi}|^2].$$

We introduce a pair of complex intermediate fields a_1 and b_1 with imaginary covariances $-i$ and $+i$, hence the Gaussian measure

$$d\mu_{\pm i}^c(a_1, b_1) = d\mu_{-i}^c(a_1, \bar{a}_1) d\mu_i^c(b_1, \bar{b}_1) \quad (\text{II-30})$$

so that

$$e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\bar{\phi}\sigma+\phi\bar{\sigma})} = \int d\mu_{\pm i}^c(a_1, b_1) e^{\frac{i}{\sqrt{2}}[(g_k\bar{\phi}\sigma+(g_k^2\phi\bar{\phi})^p)a_1 + (g_k\bar{\phi}\sigma-(g_k^2\phi\bar{\phi})^p)b_1 + c.c.]} \quad (\text{II-31})$$

$$= \int d\mu_{\pm i}^c(a_1, b_1) e^{i[g_k\bar{\phi}\sigma\frac{a_1+b_1}{\sqrt{2}}+(g_k^2\phi\bar{\phi})^p\frac{a_1-b_1}{\sqrt{2}}+c.c.]},$$

$$e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\sigma+\bar{\sigma})} = \int d\mu_{\pm i}^c(a_1, b_1) e^{i[g_k\bar{\phi}\sigma\frac{a_1+b_1}{\sqrt{2}}+(g_k^2\phi\bar{\phi})^p g_k\bar{\phi}\frac{a_1-b_1}{\sqrt{2}}+c.c.]},$$

where $c.c.$ means complex conjugate. Then we change variables as in the real case for

$$\alpha_1 = \frac{a_1 + b_1}{\sqrt{2}}, \quad \beta_1 = \frac{a_1 - b_1}{\sqrt{2}}, \quad (\text{II-32})$$

and complex conjugates, so that

$$e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\bar{\phi}\sigma+\phi\bar{\sigma})} = \int d\mu_X^c(\alpha_1, \beta_1) e^{i[g_k\bar{\phi}\sigma\alpha_1+(g_k^2\phi\bar{\phi})^p\beta_1+c.c.]}, \quad (\text{II-33})$$

$$e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\sigma+\bar{\sigma})} = \int d\mu_X^c(\alpha_1, \beta_1) e^{i[g_k\bar{\phi}\sigma\alpha_1+(g_k^2\phi\bar{\phi})^p g_k\bar{\phi}\beta_1+c.c.]}, \quad (\text{II-34})$$

where the measure $d\mu_X^c(\alpha_1, \beta_1)$ is defined by its moments

$$\begin{aligned} \langle \alpha_1 \bar{\beta}_1 \rangle_X &= \langle \bar{\alpha}_1 \beta_1 \rangle_X = -i, \\ \langle \alpha_1^2 \rangle_X &= \langle \bar{\alpha}_1^2 \rangle_X = \langle \beta_1^2 \rangle_X = \langle \bar{\beta}_1^2 \rangle_X = 0. \end{aligned} \quad (\text{II-35})$$

This reasoning is inductive and leads to the following expression :

$$e^{-\lambda(\phi\bar{\phi})^k} = \int d\mu^c(\sigma) \prod_{j=1}^{k-1} d\mu_X^c(\alpha_j, \beta_j) e^{ig_k[\bar{\phi}\sigma\alpha_1 + \sum_{j=1}^{k-2} \bar{\phi}\beta_j\alpha_{j+1} + \bar{\phi}\beta_{k-1} + c.c.]}, \quad (\text{II-36})$$

where the α_j, β_j and the measure $d\mu_X^c$ are respectively defined as in (II-32) and (II-35).

We now integrate

- for k odd, over ϕ, σ , all even α_{2j}, β_{2j} , for $j \in \{1, \dots, \frac{k-1}{2}\}$ and complex conjugates. In that case we denote $\Phi = (\phi, \sigma, \alpha_2, \beta_2, \dots, \alpha_{k-1}, \beta_{k-1})$ the $k+1$ integrated variables and $\Psi = (\alpha_1, \beta_1, \dots, \alpha_{k-2}, \beta_{k-2})$ the $k-1$ remaining ones. The Gaussian measure $d\mu^c(\sigma) \prod_{j=1}^{k-2} d\mu_X^c(\alpha_j, \beta_j)$ factorizes as $d\chi(\Psi, \bar{\Psi})d\nu(\Phi, \bar{\Phi})$.
- for k even, over ϕ , all odd $\alpha_{2j-1}, \beta_{2j-1}$, for $j \in \{1, \dots, \frac{k}{2}\}$ and complex conjugates. In that case we denote $\Phi = (\phi, \alpha_1, \beta_1, \dots, \alpha_{k-1}, \beta_{k-1})$ the $k+1$ integrated variables and $\Psi = (\sigma, \alpha_2, \beta_2, \dots, \alpha_{k-2}, \beta_{k-2})$ the $k-1$ remaining ones. The Gaussian measure $d\mu^c(\sigma) \prod_{j=1}^{k-2} d\mu_X^c(\alpha_j, \beta_j)$ factorizes again as $d\chi(\Psi, \bar{\Psi})d\nu(\Phi, \bar{\Phi})$.

The partition function therefore writes

$$Z_{k,c}(\lambda) = \int d\chi(\Psi, \bar{\Psi}) \left[d\nu(\Phi, \bar{\Phi}) \exp[ig_k \langle \Phi, H_k(\Psi, \bar{\Psi}) \cdot \Phi \rangle], \quad (\text{II-37}) \right]$$

where H_k is a $(k+1) \times (k+1)$ Hermitian matrix. More precisely :

- if $k = 2p + 1$ is odd, H_k is

$$H_k = \left(\begin{array}{c|cccc} 0 & \alpha_1 & \beta_1 & \cdots & \beta_{k-2} & 1 \\ \hline \bar{\alpha}_1 & & & & & \\ \bar{\beta}_1 & & & & & \\ \vdots & & & & & \\ \bar{\beta}_{k-2} & & & & & \\ 1 & & & & & \end{array} \right), \quad (\text{II-38})$$

- if $k = 2p$ is even, H_k is

$$H_k = \left(\begin{array}{c|cccc} 0 & \sigma & \alpha_2 & \cdots & \beta_{k-2} & 1 \\ \hline \bar{\sigma} & & & & & \\ \bar{\alpha}_2 & & & & & \\ \vdots & & & & & \\ \bar{\beta}_{k-2} & & & & & \\ 1 & & & & & \end{array} \right). \quad (\text{II-39})$$

The Gaussian integration over Φ leads to a new representation of the partition function

Theorem II.3 (Intermediate Field Representation of $(\bar{\phi}\phi)^k$).

$$Z_{k,c}(\lambda) = \int d\chi(\Psi) \exp[-\text{Tr} \ln(1 - g_k M(\Psi))], \quad (\text{II-40})$$

where $M_k(\Psi) = iC_k \cdot H_k(\Psi)$ and C_k , the covariance for the Φ variables, is given as in the real case by (II-21) and (II-22).

In the simplest cases $k = 3, 4$, hence for the $e^{-\lambda(\phi\bar{\phi})^6}$ and $e^{-\lambda(\phi\bar{\phi})^8}$ models, we obtain the representations :

$$Z_{3,c}(\lambda) = \int d\chi(\alpha_1, \beta_1) e^{-\text{Tr} \ln[1 - \lambda^{1/6} M_3]} \quad (\text{II-41})$$

with

$$M_3 = \begin{pmatrix} 0 & i\alpha_1 & i\beta_1 & 1 \\ i\bar{\alpha}_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \bar{\beta}_1 & 0 & 0 & 0 \end{pmatrix} \quad (\text{II-42})$$

and

$$Z_{4,c}(\lambda) = \int d\chi(\sigma, \alpha_2, \beta_2) e^{-\text{Tr} \ln[1 - \lambda^{1/8} M_4]} \quad (\text{II-43})$$

with

$$M_4 = \begin{pmatrix} 0 & i\sigma & i\alpha_2 & i\beta_2 & i \\ \bar{\alpha}_2 & 0 & 0 & 0 & 0 \\ \bar{\sigma} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \bar{\beta}_2 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{II-44})$$

III Analyticity Domains

To prove Borel-Le Roy summability of Z_k and $\log Z_k$, the key step is an upper bound on the norm of the resolvent $[1 - g_k M(\Psi)]^{-1}$ in the Nevanlinna domain for Borel-Le Roy summability of order $k - 1$. This bound must be uniform both in λ in that domain and uniform in the intermediate fields along the contours associated to $d\chi$.

Let us prove such a uniform bound in a slightly larger domain E_ρ^k consisting of all $\lambda = \rho e^{i\theta}$ with ρ small and $|\theta| \leq \frac{(k-1)\pi}{2}$ (hence in a half-disk for $\lambda^{1/k-1}$). Obviously it contains the disk $D_{\rho/2}^k$ necessary for Nevanlinna's Theorem, see Appendix below. We need to compute the eigenvalues of the matrix $1 - g_k M_k$, and to take into account the contours of integration. We do this in the real ϕ^{2k} case, the argument for the complex case $(\bar{\phi}\phi)^k$ being identical.

Lemma III.1. For $\lambda \in E_1^k$ and Ψ on the contours of integration $C_{\pm\epsilon}$ with $\epsilon = \frac{1}{4}k^{-1/2} \sin \frac{\pi}{4k}$ we have

$$\|(1 - g_k M_k)^{-1}\| \leq 2[\sin \frac{\pi}{4k}]^{-1}. \quad (\text{III-45})$$

Proof Returning to the parametrization of our contour integrals we recall that $a_j = \Re a_j - i\epsilon \tanh(\Re a_j)$ and $b_j = \Re b_j - i\epsilon \tanh(\Re b_j)$, where \Re is the real part. Hence remembering (II-12), and putting $\Psi = X + iY$, where the vectors X and Y are real, each coefficient of Y is bounded in absolute value by $\epsilon\sqrt{2}$. The matrix $M(\Psi)$ being linear in Ψ , we have $M_k(\Psi) = M_k(X) + iM_k(Y)$, and since each of the $2k$ non-zero coefficients of $M_k(Y)$ is bounded in absolute value by $\epsilon\sqrt{2}$, we can bound its Hilbert-Schmidt norm $\|M_k(Y)\|_2$ by $2\epsilon\sqrt{k}$, hence

$$\|M_k(Y)\| \leq \|M_k(Y)\|_2 = 2\epsilon\sqrt{k} \quad (\text{III-46})$$

Now let us compute the eigenvalues of the matrix $1 - g_k M_k(X)$. It has eigenvalue 1 with multiplicity $k - 1$ and two non trivial eigenvalues,

$$x_{\pm} = 1 \pm g_k \sqrt{R_k}, \quad (\text{III-47})$$

where R_k is $-(\Re\alpha_1)^2 + i(\Re\beta_1\Re\alpha_3 + \dots + \Re\beta_{k-4}\Re\alpha_{k-2} + \Re\beta_{k-2})$ if k is odd and is $i(\sigma\Re\alpha_2 + \Re\beta_2\Re\alpha_4 + \dots + \Re\beta_{k-4}\Re\alpha_{k-2} + \Re\beta_{k-2})$ if k is even.

If R_k is not zero we can state something about the argument of $\pm\sqrt{R_k}$. In the odd case, if $\Re\alpha_1 \neq 0$, we have $R_k = -a^2(1 + ib)$ with a and b real, hence $\pm\sqrt{R_k} = ia\sqrt{1 + ib}$ and the argument of $\pm\sqrt{R_k}$ lies in $I = [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [-\frac{3\pi}{4}, -\frac{\pi}{4}]$. If $\Re\alpha_1 = 0$ or in the even case the argument of $\pm\sqrt{R_k}$ belongs to $\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}\}$, hence to the boundary of I .

But in the domain E_{ρ}^k the argument of g_k is bounded by $\frac{(k-1)\pi}{4k}$ hence the argument of $\pm g_k \sqrt{R_k}$ (when $g_k R_k \neq 0$) lies in

$$\begin{aligned} I_k &= \left[\frac{\pi}{4} - \frac{(k-1)\pi}{4k}, \frac{3\pi}{4} + \frac{(k-1)\pi}{4k} \right] \cup \left[-\frac{3\pi}{4} - \frac{(k-1)\pi}{4k}, -\frac{\pi}{4} + \frac{(k-1)\pi}{4k} \right] \\ &= \left[\frac{\pi}{4k}, \pi - \frac{\pi}{4k} \right] \cup \left[-\pi + \frac{\pi}{4k}, -\frac{\pi}{4k} \right], \end{aligned} \quad (\text{III-48})$$

hence in that domain the spectrum of $1 - g_k M_k$ lies out of the disk of center 0 and radius $\sin \frac{\pi}{4k}$. Choosing $\epsilon = \frac{\sin \frac{\pi}{4k}}{4\sqrt{k}}$, and assuming $\rho \leq 1$, we have by (III-46) $\|g_k M_k(Y)\| \leq \frac{1}{2} \sin \frac{\pi}{4k}$, hence the spectrum of $(1 - g_k M_k(X) - i g_k M_k(Y))$ lies out of the disk of center 0 and radius $\frac{1}{2} \sin \frac{\pi}{4k}$, and

$$\|(1 - g_k M_k)^{-1}\| \leq 2[\sin \frac{\pi}{4k}]^{-1}. \quad (\text{III-49})$$

□

Since $(1 - g_k M_k)^{-1}$ has only two non trivial eigenvalues not equal to 1, this bound implies the same bound on the inverse square root of its determinant $\det(1 - g_k M_k)$, hence

$$|\exp[-\frac{1}{2} \text{Tr} \ln(1 - g_k M(\Psi))]| = |\det^{-1/2}(1 - g_k M_k)| \leq 2[\sin \frac{\pi}{4k}]^{-1}. \quad (\text{III-50})$$

which means a uniform upper bound on the integrand in (II-17). Analyticity of Z_k then follows by the standard theorem that a uniformly convergent integral of an analytic integrand is analytic.

In fact there is clearly some margin still in the proof of Lemma III.1 and the analyticity domain through this method always extends to a larger domain, with a wider opening angle than the minimum required for Nevanlinna's theorem. One can check that there is also analyticity in the additional domain $\theta = \frac{(k-1)\pi}{2} + \theta'$ with $\theta' \in [0, \frac{\pi}{2}[$ (and its symmetric $\theta = -\frac{(k-1)\pi}{2} - \theta'$) provided $\rho^{1/k} \leq \sin \theta'$. This gives back the usual cardioid type domain for $k = 2$.

Once analyticity of Z_k is established, the uniform estimates on the Taylor remainder at order n can be obtained simply by Taylor expanding Z_k by the Taylor formula with integral remainder. For instance in the real case we obtain, with the notations of the Appendix:

$$\begin{aligned} R^N Z_k(\lambda) &= \left[-\frac{\lambda}{2}\right]^N \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} \int d\mu(\phi) \phi^{2kN} e^{-t\lambda\phi^{2k}/2} \\ &= \left[-\frac{\lambda}{2}\right]^N \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} [2kN!!] \int d\chi(\Psi) \\ &\quad \exp\left[-\frac{1}{2} \text{Tr} \ln(1 - t^{\frac{1}{2k}} g_k M(\Psi))\right] \{[1 - t^{\frac{1}{2k}} g_k M(\Psi)]^{-1}\}_{11} \}^{kN}. \end{aligned} \quad (\text{III-51})$$

We can simply plug the previous bounds on the resolvents (which obviously also hold for $g_k \rightarrow t^{\frac{1}{2k}} g_k$, $t \in [0, 1]$) and conclude.

Since the power series for Z_k starts with 1, namely $Z_k = 1 + \lambda f(\lambda)$, with f analytic in D_k , the analyticity of $\log Z_k$ follows (for ρ small enough) from the general theorem of analyticity of composed functions. More precisely we can use, as in section IV.3 of [4] a standard Mayer expansion to compute $\log Z_k$. One simply writes

$$Z_k(\lambda) = 1 + R^1(\lambda), \quad R^1(\lambda) = -\frac{\lambda}{2} \int_0^1 dt \int d\mu(\phi) \phi^{2k} e^{-t\lambda\phi^{2k}/2}. \quad (\text{III-52})$$

Introducing many copies or ‘‘replicas’’ of the functional integral for R^1 :

$$\forall i \quad R_i^1(\lambda) = -\frac{\lambda}{2} \int_0^1 dt \int d\mu(\phi_i) \phi_i^{2k} e^{-t\lambda\phi_i^{2k}/2}, \quad (\text{III-53})$$

Defining $\epsilon_{ij} := 0 \forall i, j$ we can write

$$Z_k(\lambda) = 1 + R^1(\lambda) = \sum_{n=0}^{\infty} \prod_{i=1}^n R_i^1(\lambda) \prod_{1 \leq i < j \leq n} \epsilon_{ij}. \quad (\text{III-54})$$

Defining $\eta_{ij} := -1$, $\epsilon_{ij} = 1 + \eta_{ij} = 1 + x_{ij} \eta_{ij}|_{x_{ij}=1}$ and applying the forest formula [16, 17] as in [4] leads to

$$Z(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \prod_{i=1}^n R_i^1(\lambda) \int dw_{\mathcal{F}} \prod_{\ell \in \mathcal{F}} \eta_{\ell} \prod_{\ell \notin \mathcal{F}} [1 + \eta_{\ell} x_{\ell}^{\mathcal{F}}(\{w\})], \quad (\text{III-55})$$

- the sum over \mathcal{F} is over forests over $\{1, \dots, n\}$, including the empty forest,
- $\int dw_{\mathcal{F}}$ means integration from 0 to 1 over one parameter for each forest edge: $\int dw_{\mathcal{F}} \equiv \prod_{\ell \in \mathcal{F}} \int_0^1 dw_{\ell}$. There is no integration for the empty forest since by convention an empty product is 1. A generic integration point $w_{\mathcal{F}}$ is therefore made of $|\mathcal{F}|$ parameters $w_{\ell} \in [0, 1]$, one for each $\ell \in \mathcal{F}$,
- $x_{\ell}^{\mathcal{F}}(\{w\})$ is the infimum of the $w_{\ell'}$ parameters for ℓ' in the unique path $P_{i \rightarrow j}^{\mathcal{F}}$ from one end i of ℓ to the other end j in \mathcal{F} when this path exists, and is set to 0 if no such path exists,
- $\prod_{\ell \notin \mathcal{F}}$ runs over the edges of the complete graph K_n which do not belong to \mathcal{F} .

This formula allows easily to take the logarithm because each R_i^1 contains a different “replica” of the functional integration variables Ψ

$$\log Z_k(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T}} \prod_{i=1}^n R_i^1(\lambda) \int dw_{\mathcal{T}} \prod_{\ell \in \mathcal{T}} \eta_{\ell} \prod_{\ell \notin \mathcal{T}} [1 + \eta_{\ell} x_{\ell}^{\mathcal{T}}(\{w\})], \quad (\text{III-56})$$

where the second sum runs over *trees* spanning $\{1, \dots, n\}$. In this way analyticity and Taylor remainder estimates for $\log Z_k$ can be proved exactly as for Z_k , since the sum over n and \mathcal{T} converges for ρ small enough because $0 \leq [1 + \eta_{\ell} x_{\ell}^{\mathcal{T}}(\{w\})] \leq 1$ and because there are only n^{n-2} labeled trees at order n .

To complete the proof of Theorem II.2, one can remark first that the perturbative expansion in λ of this intermediate field representation of $\log Z_k$ is identical to the ordinary one, and second, that Borel summability of the initial Z_k can be also established in the ordinary representation (see Appendix). Therefore by unicity of the Borel sum, these two integral representations must be equal. The proof of Theorem II.3 is similar.

We would prefer to use a Loop Vertex Expansion [2] as in [1] rather than a regular Mayer expansion to compute $\log Z_k$, as it might generalize to the matrix or tensor case in a way which is uniform in the size N of the matrix or tensor, hence compatible with the so-called $1/N$ expansion. However we have not found yet how to adapt the bounds on the LVE interpolation formula to Gaussian imaginary integrals and their contour deformations. The problem comes from the many replicas introduced by the LVE (one per vertex). Each of them should have its own small contour deformation and these deformations add up in a way which we do not know how to control as the number n of loop vertices tends to infinity.

IV Appendix: Nevanlinna Theorem

We note R^N the N -th order Taylor remainder operator which acts on a smooth function $f(\lambda)$ through

$$R^N f = \lambda^N \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} f^{(N)}(t\lambda) dt. \quad (\text{IV-57})$$

Theorem IV.1. (Nevanlinna)[15]

A power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} \lambda^n$ is Borel-Le Roy summable of order k to the function $f(\lambda)$ if the following conditions are met:

- For some rational number $k > 0$, $f(\lambda)$ is analytic in the domain $D_\rho^k = \{\lambda \in \mathbb{C} : \Re \lambda^{-1/k} > \rho^{-1}\}$. C_ρ is a disk for $k = 1$.
- The function $f(\lambda)$ admits $\sum_{n=0}^{\infty} a_n \lambda^n$ as a strong asymptotic expansion to all orders as $|\lambda| \rightarrow 0$ with uniform estimate in C_ρ^k :

$$|R^N f| \leq AB^N \Gamma(kN) |\lambda|^N. \quad (\text{IV-58})$$

where A and B are some constants.

Then the Borel-Le Roy transform of order k , which is

$$B_f^{(k)}(u) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(kn+1)} u^n, \quad (\text{IV-59})$$

is holomorphic for $|u| < B^{-1}$, it admits an analytic continuation to the strip $\{u \in \mathbb{C} : |\Im u| < R, \Re u > 0\}$ and for $0 \leq R$, one has

$$f(\lambda) = \frac{1}{k\lambda} \int_0^{\infty} B_f^{(k)}(u) e^{-(\frac{u}{\lambda})^{\frac{1}{k}}} \left(\frac{u}{\lambda}\right)^{\frac{1}{k}-1} du. \quad (\text{IV-60})$$

We now give a proof of Borel-Le Roy summability of $Z_k(\lambda)$ in the standard representation.

First, $Z_k(\lambda)$ is obviously analytic in the full half complex plane $\Re(\lambda) > 0$, since in that plane $|e^{-\lambda\phi^{2k}}| \leq 1$, hence Z is a uniformly absolutely convergent integral in that domain. Putting $\lambda = \rho e^{i\theta}$, we can continue $Z_k(\lambda)$ analytically to a domain $0 < \rho < 1$ of opening angle³ $|\theta| < k\pi/2$ by rescaling ϕ . More precisely putting $\phi = e^{-i\theta/2k} \psi$ we have

$$Z_k(\lambda) = e^{-i\theta/2k} \int_{-\infty}^{+\infty} e^{-\lambda\psi^{2k}} e^{-e^{-i\theta/k}\psi^2/2} \frac{d\psi}{\sqrt{2\pi}} \quad (\text{IV-61})$$

and analyticity follows again from absolute convergence since $\cos \theta/k$ remains bounded away from 0 for $|\theta| < k\pi/2$.

³We can without too much pain improve this opening angle to $|\theta| < (k+1-\eta)\pi/2$ for any $\eta > 0$.

The uniform estimates in $AB^N\Gamma(kN)|\lambda|^N$ then easily follow from the Taylor formula with integral remainder applied to $e^{-\lambda\phi^{2k}}$, and a simple Cauchy-Schwarz bound to separate the perturbative terms from the remaining exponential of interaction in the integral remainder term.

The Borel-Le Roy summability of $Z_{k,c}(\lambda)$ is similar and the Borel-Le Roy summability of the corresponding free energies $\log Z_k(\lambda)$ and $\log Z_{k,c}(\lambda)$ follows from a Mayer expansion such as (III-56).

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