A CONSTRUCTION FOR BINARY MATROIDS*

Francisco BARAHONA
Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Michele CONFORTI

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A family of subsets of a ground set closed under the operation of taking symmetric differences is the family of cycles of a binary matroid. Its circuits are the minimal members of this collection. We use this basic property to derive binary matroids from binary matroids. In particular, we derive two matroids from graphic and cographic matroids. Cocycles of the first one are cutsets or balancing sets. Cocycles of the second one are Eulerian subgraphs or T-joins. We study the problem of finding a minimum weight circuit and cocircuit in these matroids.

1. Introduction

We assume familiarity with elementary matroid theory. For a basic reference, see [8].

Given a matroid $M$ defined on a ground set $E$, $M\setminus e$ denotes the matroid obtained from $M$ by deleting the element $e$. $M/e$ denotes the matroid obtained by contracting $e$. Circuits of $M\setminus e$ are the circuits of $M$ not containing $e$. Cocircuits of $M/e$ are cocircuits of $M$ not containing $e$. $M^{*}$ denotes the dual matroid of $M$.

Given a binary matroid $M$ a cycle (cocycle) is a disjoint union of circuits (cocircuits) of $M$.

If a matroid can be represented by a node-edge incidence matrix, we say that the matroid is graphic. The dual matroid is called cographic.

We shall derive two binary matroids associated with graphs, and study the minimum circuit problems.

2. Some matroids on graphs

In this paragraph we use standard graph theory and notation. We recall only relevant definitions.

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Given an undirected graph $G(V, E)$, a subgraph $G'(V, E')$, induced by $E' \subseteq E$, is Eulerian if the degree of each node of $G'$ is even.

A set $E' \subseteq E$ is a cut if $E' = \delta(U) = \{ e \in E, |U \cap e| = 1 \}$, for some $U \subseteq V$.

It is well known that cycles of a graphic matroid are Eulerian subgraphs and cuts are cycles of a cographic matroid.

A signed graph is a graph $G(V, E)$, where $E$ is partitioned into two sets $E_+$ and $E_-$ of positive and negative edges. A signed graph will be indicated by $G_s$. A sign of the circuit is the product of the signs of the edges in it. A signed graph is balanced if the sign of all its circuits is positive. In this case, there exists a $U \subseteq V$, such that $E_+ = \delta(U)$.

A balancing set is a set $S \subseteq E$ such that, when the signs of the elements of $S$ are changed, the resulting signed graph is balanced.

It is easy to see that $S$ is a balancing set if and only if there exists a set $U \subseteq V$ such that $S \cap E_+ = S \cap \delta(U) = E_+ \cap \delta(U)$ and $S \cap E_- = S \cap (E \setminus \delta(U)) = E_- \cap (E \setminus \delta(U))$.

A cutset has even intersection with every cycle of $G$; hence, if we switch signs to the edges in it we do not change the sign of a cycle. If all the edges have negative sign, then $E_-$ is a balancing set if and only if $E_- = E \setminus \delta(U)$, where $\delta(U)$ is a cutset.

Let $B$ be a standard representation of the graphic matroid associated with $G$, that is, $B$ is composed by an identity matrix corresponding to edges in a spanning tree $T$ of $G$, and the column corresponding to an edge $e$ not in $T$ is the representative vector of edges of $T$ in the fundamental circuit closed by $e$ with $T$.

Let $d$ be the incidence row vector of the signs of the edges of $G$, that is, $d_j = 1$ if the sign of the corresponding edge is negative, 0 otherwise; and let $\hat{B}$ be the matrix obtained from $B$ by adding the row vector $d$. The family of cutsets of $G$ together with the family of balancing sets of $G$ is the family of cocycles of a binary matroid $M(G_s)$ associated to the columns of $\hat{B}$. A set $I$ is independent in $M(G_s)$ if it corresponds to a forest or there is an edge $e$ such that $I \setminus \{e\}$ corresponds to a forest and $I$ contains a negative circuit of $G$. A circuit of $M(G_s)$ is a positive circuit of $G$ or the disjoint union of two negative circuits of $G$ that meet at most in one vertex.

When $E_- = E$ ($d$ is a vector of all ones), we obtain a construction introduced in [1], that is, the family of cutsets and the family of edgesets of the type $E \setminus \delta(U)$, $U \subseteq V$, form the family of cocycles of a binary matroid.

Incidentally, there is no loss of generality in studying balancing sets of signed graphs having all negative edges, since we can replace all positive edges with paths formed with two negative edges, and there will be correspondence between the family of negative and positive circuits in the two graphs.

Deletions ($\setminus$) and contractions ($/$) in $M(G_s)$ can be visualized on the graph itself. The circuits of $M(G_s) \setminus e$ are the circuits of $M(G_s')$, where $G_s'$ is obtained from $G_s$ by removing edge $e$. The cocircuits of $M(G_s)/e$ are minimal cutsets and
minimal balancing sets not containing $e$. These are precisely the cocircuits of $M(G'_e)$, where $G'_e$ is obtained as follows: If $e$ is an edge with positive sign, then $G'_e$ is obtained from $G_e$ by contracting $e$; if the sign of $e$ is negative, $G'_e$ is obtained from $G_e$ by first changing the sign of the edges incident at one arbitrarily selected endnode of $e$ and then contracting $e$.

We now dualize the previous construction, that is, given a cographic matroid, we augment the space of its cocycles (Eulerian subgraphs). Let $T \subseteq V$ be a set of even size of an undirected graph $G(V,E)$. A set $E' \subseteq E$ is called a $T$-join if $T$ coincides with the set of vertices of odd degree in the subgraph $G'(V,E')$ of $G$. A set $E' \subseteq E$ is an even $T$-cut (ET-cut for short) if there exists a set $V'$ of vertices with $|V' \cap T|$ even such that $E'$ is the set of edges of $G$ intersecting $V'$ in exactly one vertex. Odd $T$-cuts are defined analogously, (i.e., $|V' \cap T|$ is odd). We define $M(G,T)$ as the binary matroid whose cocycles are $T$-joins and Eulerian subgraphs of $G$. $M(G,T)$ will be called the $T$-join matroid of $G$. Cycles of $M(G,T)$ are ET-cuts.

Cocircuits of $M(G,T)$ are minimal $T$-joins and circuits of $G$ that are not $T$-Hamiltonian. A circuit of $G$ is said to be $T$-Hamiltonian if it covers all the vertices in $T$. Circuits in $M(G,T)$ are all minimal ET-cuts.

Independent sets of $M(G,T)^*$ are forests whose node set does not span $T$. Therefore the rank of $M(G,T)$ is $|E| - |V| + p + q$, where $q$ is the number of connected components of $G$, and $q$ is equal to 1 if $T \neq \emptyset$, 0 otherwise.

Matroid $M(G,T)$ can be obtained as a contradiction of a matroid introduced by Seymour [5] in the proof of the decomposition theorem for regular matroids.

Deletion and contraction in $M(G,T)$ can be easily visualized as operations in $G$ itself. In fact, the cocircuits of $M(G,T)/e$ are the cocircuits that do not contain $e$; those are exactly the cocircuits of $M(G',T)$, where $G'$ is obtained from $G$ by removing $e$. The circuits of $M(G,T)/e$ are the minimal ET-cuts that do not contain $e$; those are exactly the minimal ET'-cuts of $G''$, where $G''$ is obtained from $G$ by contracting the edge $e$, and $T'$ is obtained from $T$ as follows: If none of the extremities of $e$ belong to $T$, then $T' = T$; if both extremities of $e$ belong to $T$, then $T' = T \setminus e$; if only one of the extremities of $e$ belongs to $T$, say $u$, then $T' = T \setminus \{u\} \cup \{v\}$, where $v$ is the vertex of $G''$ that replaces the extremities of $e$. Therefore, $M(G,T)/e = M(G'',T')$. Thus, a minor of a $T$-join matroid is also a $T$-join matroid.

Tutte [7] showed that a binary matroid is regular if and only if it does not

![Fig. 1.](image-url)
contain $F_7$ (the Fano Matroid) or $F_7^*$ as minors. As we shall see, a $T$-join matroid may contain these minors.

$F_7$ is the $T$-join matroid of the graphs in Fig. 1. The vertices in $T$ are marked with squares.

$F_7^*$ is the $T$-join matroid of the graphs in Fig. 2, where the nodes in $T$ are marked with squares.

3. The minimum circuit problem

Given nonnegative weights of the elements of a binary matroid $M$, the minimum circuit (or cycle) problem is that of finding a minimum weight circuit of $M$. If $M$ has no $F_7^*$ minor the problem is solvable by decomposition methods and max flow or shortest path techniques, see [6]. Our two matroids may have $F_7^*$ as a minor.

In [1] the minimum circuit problem for the matroid $M(G_s)$ has been solved with $b$-matching techniques. The cocircuits of $M(G_s)$ are either cutsets or balancing sets. The problem of finding a minimum balancing set is NP-complete; however, the minimum cocircuit problem in $M(G_s)$ is still open. An efficient solution of this problem would yield a good characterization of “almost bipartite” graphs, that is, graphs having the cardinality of a minimum balancing set less than or equal to the cardinality of a minimum cutset, when all the edges of $E$ have negative sign.

Cocycles of the $T$-join matroid are either Eulerian subgraphs of $T$-joins. A minimum weight Eulerian subgraph is computable by shortest path techniques, and a minimum $T$-join is computable by the Chinese Postman algorithm, see [2].

Cycles of our $T$-join matroid are ET-cuts. We now study the minimum ET-cut problem. If the graph is not connected, a minimum ET-cut consists of two odd $T$-cuts in different connected components or an ET-cut in one connected component.

Padberg and Rao [4] have designed an algorithm to compute a minimum odd $T$-cut. We shall give an algorithm to find a minimum ET-cut in a connected graph. We were not able to reduce the ET-cut problem to an odd $T$-cut problem, but we can use some of the ideas of Padberg and Rao.

Given a set of weights $W_e \geq 0$, for $e \in E$, and $U \subseteq V$, let

$$W(U) = \sum \{W_e : e \in \delta(U)\}.$$
The minimum cut problem is the following: Find $U^* \subseteq V$ such that

$$W(U^*) = \min_{U \subseteq V} \{ W(U), \delta(U) \neq \emptyset \}. \quad (3.1)$$

Our problem can be stated as follows: Find $U^* \subseteq V$ such that

$$W(U^*) = \min_{U \subseteq V} \{ W(U) : |U \cap T| \text{ is even}, \delta(U) \neq \emptyset \}. \quad (3.1')$$

Obviously, if $U$ defines a minimum ET-cut, then $U$ satisfies exactly one of the following three alternatives:

1. $|U \cap T| = 0$ and $|(V \setminus U) \cap T| = |T|$, \quad (3.3)
2. $|U \cap T| = 2$ and $|(V \setminus U) \cap T| = |T| - 2$, \quad (3.4)
3. $|U \cap T| = a$ and $|(V \setminus U) \cap T| = b$, where $a$ and $b$ are even numbers greater or equal to four. \quad (3.5)

A minimum ET-cut satisfying (3.3) is computable as a min cut problem by contracting the set $T$ to a single node. The minimum ET-cut satisfying (3.4) can be found with $O(|T|^2)$ min cut calculations by picking two nodes of $T$ as a source and contracting all the remaining nodes in $T$ to a sink.

In order to study ET-cuts satisfying (3.5) we consider a minimum cut $\delta(U)$, where $U$ satisfies:

$$|U \cap T| \geq 2 \quad \text{and} \quad |(V \setminus U) \cup T| \geq 2; \quad (3.6)$$

this set $U$ can be obtained with $O(|T|^4)$ min cut calculations.

If $|U \cap T|$ is even we are done, otherwise there are three cases to study.

**Case 1.** Let $\delta(U')$ be a min cut such that $U'$ satisfies (3.5), if $|U' \cap U \cap T| \neq 1$ and $|(V \setminus (U' \cup U)) \cap T| \neq 1$, then there exists a set $U'' \subseteq U$ or $U'' \subseteq (V \setminus U)$ such that $\delta(U'')$ is an ET-cut and $W(U'') = W(U')$.

**Proof.** By submodularity of the cut function, we have

$$W(U) + W(U') \geq W(U \cup U') + W(U \cap U'). \quad (3.7)$$

**Subcase 1(a).** If $|U \cap U' \cap T|$ is odd, then $|(U' \cup U) \cap T|$ is even. Since $|U \cap U' \cap T| \neq 1$ we have $W(U \cup U') \geq W(U)$. By (3.7) we have $W(U \cup U') = W(U')$.

**Subcase 1(b).** If $|U \cap U' \cap T|$ is even, then $|(U' \cup U) \cap T|$ is odd. Since $|(V \setminus (U' \cup U)) \cap T| \neq 1$ we have $W(U) \leq W(U' \cup U)$, by (3.7) we have $W(U') = W(U' \cap U)$.

**Case 2.** Let $\delta(U')$ be a min cut with $U'$ satisfying (3.5). If $|U' \cap U \cap T| = 1$, then $W((V \setminus U) \cup U') = W(U')$. 

Proof. We have
\[ W(U) + W(U') = W(V \setminus U) + W(U') \]
\[ \geq W((V \setminus U) \cup U') + W((V \setminus U) \cap U'). \] (3.8)
Since \(|U \cap U' \cap T| = 1\) and \(|U' \cap T|\) is even, \(|(V \setminus U) \cap U' \cap T|\) is odd and \(|((V \setminus U) \cup U') \cap T|\) is even, since \(|U' \cap T| \geq 4\) we have \(|(V \setminus U) \cap U' \cap T| \geq 3\), and \(W(U) \leq W((V \setminus U) \cap U').\) This implies \(W(U') = W((V \setminus U) \cup U').\)

Case 3. Let \(\delta(U')\) be a min cut with \(U'\) satisfying (3.5). If \(|(V \setminus (U' \cup U)) \cap T| = 1\) we replace \(U'\) by \(V \setminus U'\) and \(U\) by \(V \setminus U\) and Case 2 applies.

Hence, in any of the three cases we should look for a minimum ET-cut \(\delta(U''),\) where \(U'' \subseteq U\) or \(U'' \subseteq V \setminus U.\) Hence we consider two new graphs \(G^1(V^1, E^1)\) and \(G^2(V^2, E^2),\) where \(V^1 = U \cup \{s^1\}\) and \(V^2 = (V \setminus U) \cup \{s^2\},\) where \(s^1\) and \(s^2\) are new nodes obtained by contracting \(V \setminus U\) and \(U\) respectively.

The node sets considered by this procedure form a nested family. Nested means that \(U' \cap U \neq \emptyset \Rightarrow U' \subseteq U\) or \(U \subseteq U'\) for any pair \(U, U'\) of sets in this family. A nested family of subsets of an \(n\)-element set has at most \(2n - 1\) sets.

Hence if \(|V| = n,\) a minimum cut \(\delta(U)\) with \(U\) satisfying (3.6) has to be found at most \(2n - 1\) times; each time it takes \(O(|T|^4)\) min cut computations, and a minimum cut can be found in \(O(n^4)\) operations. Thus this is an \(O(n^9)\) algorithm.

Acknowledgments

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References