High order FDTD methods for transverse magnetic modes with dispersive interfaces

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Abstract

A new finite-difference time-domain (FDTD) algorithm is introduced to solve two dimensional (2D) transverse magnetic (TM) modes with a straight dispersive interface. Driven by the consideration of simplifying interface jump conditions, the auxiliary differential equation of the Debye constitution model is rewritten to form a new Debye–Maxwell TM system. Interface auxiliary differential equations are utilized to describe the transient changes in the regularities of electromagnetic fields across a dispersive interface. The resulting time dependent jump conditions are rigorously enforced in the FDTD discretization by means of a matched interface and boundary scheme. Higher order convergences are numerically achieved for the first time in the literature in 2D FDTD simulations of dispersive inhomogeneous media.

1. Introduction

In a dispersive material, the permittivity is known to be a function of frequency so that a broadband electromagnetic wave will propagate in a frequency dependent manner. Because many natural materials including biological tissues and soils need to be modeled as dispersive media [6,14], the study of wave propagation in dispersive media is crucial to a wide range of electromagnetic applications, such as microwave imaging for early detection of breast cancer [16] and ground penetrating radar [5]. Many theoretical approaches [8,19,20,4,1] based on wave splitting, Greens functions, or homogenization techniques have been introduced in the literature to analyze the dispersive electromagnetic process in the time domain. For complex dispersive problems involving inhomogeneous media and nontrivial geometries, the electromagnetic studies are commonly conducted based on numerical simulations.

Various numerical approaches have been successfully developed in the literature to solve Maxwell’s equations in dispersive media, including finite-difference time-domain (FDTD) methods [13,9,23,3], finite element time domain methods [12,15,2,22], discontinuous Galerkin time domain methods [17,11,21], and so on. Different formulations have been utilized in these approaches to incorporate the frequency dependent material constitutive relation into the time domain Maxwell discretization. However, many of such formulations are mainly designed for a homogeneous dispersive medium, and a common difficulty exists in treating the inhomogeneous dispersive media, because the wave solution loses its regularity in a time variant manner across a dispersive interface [27]. We note that the electromagnetic wave solution also loses its regularity across a non-dispersive dielectric interface. Nevertheless, the numerical difficulty associated with a non-dispersive interface becomes less severe, in the sense that the regularity change across the dielectric interface is time independent in usual applications of the computational electromagnetics (CEM) [10,24].
To restore the accuracy reduction near a non-dispersive interface induced by the non-smoothness of wave solutions, numerous interface treatments have been established in the CEM literature [10]. Some of such interface approaches have been extended to solve the dispersive interfaces. For instance, several smoothing techniques [28,7,18], in which the permittivities in the vicinity of an interface are averaged to form a smooth effective permittivity, have been successfully constructed to accelerate the convergence of dispersive FDTD algorithms. For non-dispersive interfaces, high order interface treatments can be accomplished by rigorously enforcing the interface jump conditions in the time domain numerical discretizations [10,24,25]. However, the generalization of such rigorous interface treatments to dispersive interfaces is extremely challenging. The major obstacle is to address the time variant jump conditions. To the authors’ knowledge, the only high order method developed for dispersive interfaces is the matched interface and boundary (MIB) method [27] for solving one-dimensional (1D) transverse electromagnetic systems. The fundamental augmentation of the 1D dispersive MIB method [27], in comparing with its non-dispersive counterparts [24,25], is its capability to handle transient jump conditions. This is realized via the reformulation of jump conditions by means of an auxiliary differential equation.

The main objective of this paper is to generalize the 1D dispersive MIB method [27] to higher dimensions. In particular, we will present a new MIB approach for solving two-dimensional (2D) transverse magnetic (TM) Maxwell’s equations with straight dispersive interfaces. Based on the auxiliary differential equation (ADE) approach, the Debye dispersion model is coupled with the TM modes to derive interface auxiliary differential equations (IADEs) for describing the regularity changes in electromagnetic fields at dispersive interfaces. The resulting time dependent jump conditions can then be rigorously enforced in the FDTD discretization to produce high order convergence. This paper is organized as follows. The Maxwell formalism and the MIB algorithm are introduced in Section 2. The proposed method is illustrated through applications to several examples with different initial or boundary conditions in Section 3. Conclusions are given in Section 4.

2. Theory and algorithm

Consider isotropic, linear, and dispersive materials with both structure and the incident wave being invariant in the z direction. By dropping all z derivatives, the two-dimensional (2D) Maxwell’s equations for electromagnetic fields can be decomposed into two independent sets of equations, the transverse magnetic (TM) mode and the transverse electric (TE) mode. In this paper, we concern ourselves with the TM system in some dispersive inhomogeneous media

\[
\frac{\partial E_z}{\partial t} = \frac{1}{\mu} \frac{\partial H_y}{\partial x} - \frac{1}{\mu} \frac{\partial H_y}{\partial y}, \quad \frac{\partial H_y}{\partial t} = \frac{1}{\mu} \frac{\partial E_z}{\partial x} - \frac{1}{\mu} \frac{\partial E_z}{\partial y},
\]

where \( E_z \) is the electric field component, \( H_y \) and \( H_y \) are, respectively, the magnetic field \( x \) and \( y \) components, and \( D_z \) is the electric displacement component. Here the materials are assumed to be non-magnetic, i.e., the magnetic permeability \( \mu = \mu_0 \). Due to the frequency dependence of the dispersive media, the electric constitutive relation is better prescribed in the frequency domain, i.e., \( D_z = \epsilon(\omega)E_z \), in which the time-harmonic components are obtained via the Fourier transform of the time-varying components.

We will focus ourselves on the single-order Debye dispersion model in this work

\[
\epsilon(\omega) = \epsilon_0 \left[ \epsilon_\infty + \frac{\epsilon_s - \epsilon_\infty}{1 + j\omega \tau} \right],
\]

where \( \epsilon_0, \epsilon_s, \) and \( \epsilon_\infty \) are, respectively, the permittivities of free space, at static frequency, and at high frequency limit. Here \( \omega \) is the angular frequency and \( \tau \) is the relaxation time constant. The auxiliary differential equation (ADE) approach [13] is used to represent the constitutive Eq. (2) in the time domain

\[
\tau \frac{\partial D_z}{\partial t} + D_z = \epsilon_0 \epsilon_\infty \tau \frac{\partial E_z}{\partial t} + \epsilon_0 \epsilon_s E_z.
\]

Eqs. (1) and (3) form a 2D closed Debye–Maxwell TM system for \( D_z, E_z, H_y, \) and \( H_y \). The classical dispersive finite difference time domain (FDTD) method [13] can be formulated by discretizing such a system. It is noted that the displacement or polarization current is not included in the present Maxwell formulation, because they can be implicitly accounted for via re-defining \( \epsilon \) [2].

Consider a 2D domain of rectangle shape \((x,y) \in [a,b] \times [c,d]\). The interface \( \Gamma \) is assumed to be straight: \( x = \xi \) for \( y \in [c,d] \), which cuts the domain into two parts. A dispersive interface separating the air and a Debye medium will be studied in this paper. Nevertheless, we note that proposed approach can be easily extended to handle a more general dispersive interface separating two inhomogeneous Debye media. In the following, the left subdomain \([a, \xi] \times [c,d]\) is assumed to be vacuum with \( \epsilon = \epsilon_0 \), while the right subdomain \([\xi, b] \times [c,d]\) is a Debye dispersive medium. Define function limits and jump of function \( u(x,y) \) at \( \Gamma \) to be \( u^+ := \lim_{y \to \xi^+} \) \( u, u^- := \lim_{y \to \xi^-} u, \) and \( |u| := u^+ - u^- \). To unify the notation, we assume the material Eq. (2) in \([a, \xi] \times [c,d]\) as well with \( \epsilon_\infty = \epsilon_\infty = 1 \). Then \( \tau^{-} \) is a free parameter so that we can assume \( \tau = \tau = \tau^+ \) being a constant throughout \([a, b] \times [c,d]\).

We first establish physical jump conditions at \( x = \xi \). We have the ordinary jump conditions [24] for \( E_z \) across \( \Gamma \)

\[
[E_z] = 0, \quad \frac{\partial E_z}{\partial x} = 0.
\]
which can be simply handled by the previous interface schemes [10,24,25]. For both \( H_x \) and \( H_y \), we have continuity conditions \( |H_x| = 0 \) and \( |H_y| = 0 \). We then need to prescribe the first order jump condition for \( H_x \), but not for \( H_y \). This is because \( H_x \) will be differenti-ated only along y direction in the present study, and such a direction is actually the tangential direction of the interface \( \Gamma \). However, the first order jump condition for \( H_y \) is physically unknown. We overcome this difficulty by introducing some interface auxiliary differential equations (IADEs).

Assume \( [D_z] = \psi(t,y) \) for some unknown function \( \psi(t,y) \). By differentiating \( |H_x| = 0 \) along the tangential direction, we have \( \frac{\partial \psi}{\partial y} = 0 \). Thus, by taking jump operations to the first equation of (1), we obtain

\[
\frac{\partial H_x}{\partial y} = \frac{\partial D_z}{\partial y} + \frac{\partial H_x}{\partial t} = \frac{\partial D_z}{\partial t} = \frac{\partial \psi(t,y)}{\partial t}.
\]

(5)

In other words, the jump in the x derivative of \( H_x \) is time dependent and can be numerically estimated based on (5). For the purpose of calculating \( \psi(t,y) \), we conduct jump operations for (3). This gives rise to some IADEs

\[
\tau \frac{\partial \psi(t,y)}{\partial t} + \psi(t,y) = \epsilon_0 \tau [\epsilon_x \hat{E}_z] + \epsilon_0 \tau [\epsilon_y E_z],
\]

where \( \hat{E}_z = \frac{\partial \psi}{\partial t} \). Since both \( E_z \) and \( \hat{E}_z \) are continuous across \( \Gamma \), the jump operations on the right hand side of (6) can be rewritten as

\[
g(t,y) := \epsilon_0 \tau [\epsilon_x \hat{E}_z] + \epsilon_0 \tau [\epsilon_y E_z] = \epsilon_0 \tau (\epsilon_x^+ - \epsilon_x^-) \hat{E}_z^- + \epsilon_0 (\epsilon_y^+ - \epsilon_y^-) E_z^-.
\]

(7)

Note that \( g(t,y) \) in (7) can be evaluated from the positive side as well, i.e., based on \( \hat{E}_z^+ \) and \( E_z^+ \). The left side is used in the present study, because the incident wave propagates from the left. With \( g(t,y) \), (6) reduces to

\[
\tau \frac{\partial \psi(t,y)}{\partial t} + \psi(t,y) = g(t,y),
\]

(8)

which can be regarded as an ordinary differential equation with respect to time \( t \), along a x grid line with \( y = \text{const.} \).

Based on the above dispersive interface considerations, we propose to utilize \( \hat{E}_z \) instead of \( D_z \) to formulate a new Debye–Maxwell system for TM modes

\[
\frac{\epsilon_0 \epsilon_x}{\mu_0} \frac{\partial \hat{E}_z}{\partial t} = -\epsilon_0 \epsilon_z \hat{E}_z + \frac{\tau}{\mu_0} \left( \frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} \right) + \frac{\partial H_x}{\partial x} - \frac{\partial H_x}{\partial y},
\]

\[
\frac{\partial E_z}{\partial t} = \hat{E}_z, \quad \frac{\partial H_x}{\partial t} = \frac{1}{\mu_0} \frac{\partial E_z}{\partial x} - \frac{1}{\mu_0} \frac{\partial E_z}{\partial y}.
\]

(9)

The proposed algorithm is formulated by discretizing this closed system. In particular, new spatial discretizations will be developed only in the vicinity of the dispersive interface \( \Gamma \), while away from the interface, the standard higher order finite differences will be utilized to discretize (9) [24,25]. The classical fourth order Runge–Kutta method will be employed to update Maxwell’s Eqs. (9) in the present study, whereas other standard explicit or implicit time stepping methods may also be used for the time integration.

To facilitate the interface treatment, \( g(t,y) \) defined in (7) has to be calculated at each time step \( t_k = k\Delta t \), i.e.,

\[
g(t_k,y) = \epsilon_0 \tau (\epsilon_x^+ - \epsilon_x^-) \hat{E}_z^+(t_k,y) + \epsilon_0 (\epsilon_y^+ - \epsilon_y^-) E_z^+(t_k,y).
\]

Here \( E_z(t_k,y) \) and \( \hat{E}_z(t_k,y) \) are approximated via one-sided extrapolations based on several function values of \( E_z(t_k,y) \) and \( \hat{E}_z(t_k,y) \) from the left of the interface. With \( g(t_k,y) \), the classical fourth order Runge–Kutta method is used to integrate IADEs (8) to compute \( \psi(t_k,y) \) and \( \partial \psi(t_k,y)/\partial t \) at each time step \( t_k \).

After \( \partial \psi(t,y)/\partial t \) being accurately estimated, a matched interface and boundary (MIB) method [24,25] is utilized to impose jump conditions at the dispersive interface, i.e.,

\[
[H_y] = 0, \quad \left[ \frac{\partial H_y}{\partial x} \right] = \frac{\partial \psi(t,y)}{\partial t},
\]

(10)

for \( H_y \) and (4) for \( E_z \). In particular, a uniform staggered grid system shown in Fig. 1 is employed in the present study. As an extension of the standard Yee cell, the field components \( E_z \) and \( \hat{E}_z \) will be computed on the same collocation node, while \( H_x \) and \( H_y \) will be calculated on staggered positions. Without the loss of generality, we assume that the interface \( \Gamma \) is not located on any grid line \( x = x_i \) in the present work.

Since the jump conditions (4) and (10) are decoupled, we consider only the MIB enforcement of (4) for \( E_z \) here. The jump conditions (10) for \( H_y \) can be similarly treated. In the MIB method, the standard \((2M)\)th order central finite difference approximation [24] will be carried out away from the interface, with \( M \) being the half stencil bandwidth. The finite difference weights of nodes in the vicinity of the interface, i.e., the so-called irregular points, shall be modified in order to satisfy jump conditions. A universal rule here is that to approximate function or its derivatives on one side of interface, one never directly refers to function values from the other side. Instead, in the MIB scheme, fictitious values from the other side of the interface will be used.

For example, suppose that \( x_i < \xi < x_{i+1} \). Denote the function value and fictitious value of \( E_z \) at the node \( x_i \) as \( E_i \) and \( f_i \), respectively. The second order finite difference approximation to the double x derivative term in (9) will be modified to be
can be determined, since in terms of corresponding function values and $f_i$ and $f_{i+1}$; for $f_i$ and $f_{i+1}$ are different from those in Eqs. (12) and (13) because they involve a different set of grid points which includes two more fictitious points at the interface. These fictitious values will be resolved based on the discretized jump conditions.

At the first step, we determine two fictitious values, $f_i$ and $f_{i+1}$, by discretizing two jump conditions (4) in the same manner of Eq. (11), i.e., never referring to function values across the interface

$$
\sum_{k=1}^{L} w_{0,k} E_{i-L-k} + w_{0,L+1} f_{i+1} = w_{0,1} f_{i} + \sum_{k=2}^{L+1} w_{0,k} E_{i+k-1},
$$

(12)

$$
\sum_{k=1}^{L} w_{1,k} E_{i-L-k} + w_{1,L+1} f_{i+1} = w_{1,1} f_{i} + \sum_{k=2}^{L+1} w_{1,k} E_{i+k-1},
$$

(13)

where $w_{jk}$ and $w_{jk}^*$ for $k = 1, \ldots, L + 1$ and $j = 0, 1$ are one-sided finite difference weights, respectively, for left and right subdomains. Here the subscript $j$ represents interpolation ($j = 0$) and the first order derivative approximation ($j = 1$), and $k$ is for grid index. One-sided approximations involving $L$ grid nodes in one side are used to ensure the sufficiently high accuracy. By solving Eqs. (12) and (13), one can determine fictitious values $f_i$ and $f_{i+1}$.

We note that the solved fictitious value is actually a linear combination that represents $f_i$ or $f_{i+1}$ in terms of corresponding function values ($E_{i-L-1}, \ldots, E_{i+L}$) [24,25]. By substituting $f_i$ and $f_{i+1}$ into Eq. (11), a second order MIB method can be constructed.

To achieve the fourth order accuracy, we determine two more fictitious values by enforcing two jump conditions (4) again,

$$
\sum_{k=1}^{L} \tilde{w}_{0,k} E_{i-L-k} + \tilde{w}_{0,L+1} f_{i+1} + \tilde{w}_{0,L+2} f_{i+2} = \tilde{w}_{0,1} f_{i} + \tilde{w}_{0,2} f_{i} + \sum_{k=3}^{L+2} \tilde{w}_{0,k} E_{i+k-2},
$$

(14)

$$
\sum_{k=1}^{L} \tilde{w}_{1,k} E_{i-L-k} + \tilde{w}_{1,L+1} f_{i+1} + \tilde{w}_{1,L+2} f_{i+2} = \tilde{w}_{1,1} f_{i} + \tilde{w}_{1,2} f_{i} + \sum_{k=3}^{L+2} \tilde{w}_{1,k} E_{i+k-2},
$$

(15)

where the finite difference weights $\tilde{w}_{jk}$ and $\tilde{w}_{jk}^*$ are different from those in Eqs. (12) and (13) because they involve a different set of grid points which includes two more fictitious points at $x_{i-1}$ and $x_{i+2}$. From these two equations, two new unknowns $f_{i-1}$ and $f_{i+2}$ can be determined, since $f_i$ and $f_{i+1}$ are known. With these four fictitious values, standard fourth order central finite difference approximations can be evaluated at either $x_i$ or $x_{i+1}$, in a manner similar to Eq. (11). This gives rise to a fourth order MIB method.

By iteratively determining two more fictitious points at each step, finally we can attain 2M fictitious points for a (2M)th order central finite difference approximation across the interface. For the present straight interface problem, such a procedure can be carried out systematically, and is thus of arbitrarily high order in principle. Numerically, the order of accuracy of the MIB method is determined by 2M at regular grid points, while it is limited by the parameter $L$ at irregular points. Finally, we note that the MIB interface matching needs only be carried out once along one $x$ grid line. The solved representation coefficients can then be used in any other $x$ grid lines [24].
3. Numerical experiments

To validate the proposed MIB time-domain (MIBTD) methods, we consider three dispersive interface problems with different initial and boundary conditions. In all examples, a straight air–water interface is assumed at \( x = \zeta \) within a rectangular domain \([a, b] \times [c, d]\). Since the media are homogeneous along \( y \)-water interface, the discretization parameters in the \( y \)-direction, i.e., the number of grid nodes \( N_x \) and the half stencil width \( M_y \), will be chosen to be large enough and will be fixed in each example. Whereas, different \( N_y \) and \( M_y \) will be explored to numerically test the order of convergence for the proposed algorithm. Sufficiently small time increments \( \Delta t \) will be employed in all cases so that the approximation error is mainly due to the spatial discretization.

3.1. Example 1: planar incident wave

In our first example, the domain dimensions are chosen as \( a = c = 0 \) mm and \( b = d = 30 \) mm, with the interface location \( \zeta = 7.5 \pi \) mm. A dispersive water medium is assumed in the right subdomain with \( \varepsilon = 81 \). \( \varepsilon_s = 1.8 \), and \( \tau = 9.4 \) ps [9,27]. An incident pulse of the form \( E_x(x, y, t) = \exp(-((x + 15 - c_0 t)^2)/(2s^2)) \) is imposed as the boundary condition at the left boundary \( x = a \). Here \( s = 2.5 \) mm and \( c_0 \) is the speed of light in air. Since the incident wave is normal with respect to the interface, the electromagnetic wave solutions are actually invariant along \( y \)-direction, see Fig. 3. For simplicity, periodic conditions will be assumed on the top boundary \( y = d \) and bottom boundary \( y = c \). The computation will stop before the transmitted wave hits the right boundary \( x = b \). In this way, the electromagnetic fields remain to be negligible at \( x = b \). Consequently, this boundary can be assumed to be perfectly electric conducting (PEC) for simplicity [26]. In the following tests, the \( y \)-direction discretization parameters are fixed to be \( N_y = 10 \) and \( M_y = 4 \).

We first study the reflection coefficient of the air–water interface by considering the second order MIBTD or MIBTD2, in which we set \( M_y = 1 \) and \( L = 2 \). The number of \( x \)-grid nodes for \( E_x \) is chosen as \( N_x = 801 \) and the time increment is set to be \( \Delta t = 0.025 \) ps for a total of 10,000 steps. The time histories of \( E_x \) are tracked at two locations along one horizontal line \( y = \text{const.} \). One location is near the left boundary \( x = a \), while another is to the left of the interface \( x = \zeta \). Through a proper truncation, this yields two time domain samplings for incident and reflected pulses, respectively. The reflection coefficient can then be calculated as the ratio of the Fourier transform of the reflected wave over that of the incident wave. The reflection coefficient at an air–water interface is plotted against the analytical one [9] in Fig. 2. Obviously, our numerical results agree with the analytical one over a wide range of frequencies.

To rigorously examine the proposed higher order MIBTD methods, we next conduct a numerical convergence analysis. The parameters for the higher order MIBTD are chosen as: \( (M_y, L) = (2, 4) \) and \( (3, 6) \), respectively, for the MIBTD4 and MIBTD6. The implementation of the PEC boundary condition in the high order finite difference methods has been discussed in [26]. In the present work, a stop time \( t = 140 \) ps is chosen, which is short enough such that both reflected and transmitted pulses have not reached the horizontal boundaries, see Fig. 3. We note again that our incident wave here is constant with respect to the \( y \)-direction. The same conclusion holds for \( E_x, H_y \) and \( H_x \). In fact, all values of \( H_z \) are equal to zero throughout our computations, because the change of \( H_z \) is determined by the \( y \)-partial derivative of \( E_x \) in (9). Thus, only non-trivial \( E_x \) and \( H_x \) solutions are plotted in Fig. 3.

In the present example, a reference solution is generated by employing the MIBTD6 with a very dense grid \( N_x = 12801 \) and a small enough \( \Delta t \). Convergence analysis can then be conducted by considering mesh refinements based on nodes that are also sampled in the reference solution. Specifically, the minimal requirement for a tested mesh size \( N_x \) is that \( N_x - 1 \) is an integer factor of 12,800. For example, when \( N_x = 1601 \), the reference solution needs to be downsampled with a rate 8 along \( x \)-direction, i.e., keeping one \( E_x \) value in every eight \( E_x \) values. The downsampled reference solution can then be compared with the numerical solution to compute the maximal error for \( E_x \). We note that since the \( H_y \) nodes are staggered to the \( E_x \) nodes, the present convergence analysis is not applicable to \( H_y \).

![Fig. 2. The reflection coefficient at an air–water interface.](image-url)
The maximal errors in $E_z$ of the MIBTD methods are depicted in Fig. 4. The results of the classical dispersive FDTD [13] obtained by discretizing Eqs. (1) and (3) are also included for a comparison. In all cases, the numerical errors based on different mesh size $N = N_x$ are plotted as dashed lines. A linear least-squares fitting [25] is then conducted in the log–log scale. The fitted convergence lines are shown as solid lines in Fig. 4. Moreover, the fitted slope essentially represents the numerical convergence rate $r$ of the scheme. The traditional dispersive algorithms fail to deliver high accuracy because of the loss of regularities in wave solutions near the interface. In particular, it can be seen from Fig. 3 that $E_z$ is $C^1$ continuous cross the interface, but $H_y$ is only $C^0$ continuous. Moreover, the interface location $\xi$ is generally not located on grid during the mesh refinements, because it is an irrational number. Therefore, the conventional FDTD algorithm degrades to the first order of accuracy, i.e., $r = 1.18$ in Fig. 4. Nevertheless, after the implementing the MIB interface treatments, higher order convergence can be restored. In the present study, the numerical order $r$ of the MIBTD2, MIBTD4, and MIBTD6 methods is found to be $2.12, 4.32,$ and $5.74$, respectively, which confirms the theoretical orders of two, four, and six (see Fig. 4).

3.2. Example 2: Gaussian–Gaussian initial wave

Since all solutions in Example 1 are invariant along the $y$ direction, the two-dimensional (2D) result of the Example 1 is very close to its counterpart in one-dimension (1D), reported in our previous study [27]. We thus consider a real 2D study in the present example. The domain is chosen as $a = 0$ mm, $b = 12$ mm, $c = 0$ mm, and $d = 10$ mm. The interface is located at $\xi = 3\pi$ mm and the same Debye parameters as in Example 1 are used for the dispersive material. We introduce a Gaussian–Gaussian initial wave solution at time $t = 0$ for $E_z$, while the initial solutions for $H_x$ and $H_y$ are assumed to be zero. In particular, we set $E_z(x, y, 0) = \exp(-(x - x_c)^2 + (y - y_c)^2)/(2\sigma^2))$ with $x_c = y_c = (d - c)/2$. The Gaussian window parameter is set as $\sigma = 0.5$ mm, which ensures that the initial wave vanishes near the boundaries and the interface. This initial wave setting is applied throughout the domain at the beginning time. For the later on, all values of electric and magnetic fields are computed based on the data of the previous time. In the present study, all four boundaries of the domain are assumed to be PEC [26].

For this example, we demonstrate the second and fourth order convergences. In our computations, we choose the number of time steps and the stop time to be $N_t = 10,000$ and $t = 60$ ps, respectively. A reference solution is generated by the MIBTD4 with a dense grid ($N_x = 6401$ and $N_y = 51$). Note that one does not need to use a fine grid for the $y$ direction, since the wave solution is infinitely smooth in this direction. We thus set $N_y = 51$ and $M_y = 4$ in all computations, which are enough to resolve the Gaussian wave for our purpose and save the CPU time. The same higher order finite difference approximation with
Fig. 5. The MIBTD4 solution for the Example 2 with $N_x = 201$ and $N_y = 51$ at $t = 60$ ps. Top left: $E_z$; top right: $H_y$; bottom: $H_x$.

Fig. 6. Numerical convergence tests of $E_z$ for the Example 2.

Fig. 7. The MIBTD4 solution for Example 3 with $N_x = 201$ and $N_y = 101$ at $t = 60$ ps. Top left: $E_z$; top right: $H_y$; bottom: $H_x$. 
$N_y = 51$ and $M_y = 4$ is also used in the FDTD discretization of (1) and (3) along $y$ direction. Thus, the present FDTD is slightly different from that for Example 1. For this reason, the present finite difference discretization will be simply termed as FD2.

An example MIBTD solution is shown in Fig. 5, while the maximal errors of the MIBTD and FD2 methods by considering different $N = N_x$ values are depicted in Fig. 6. As in Example 1, a linear least-squares fitting in log–log scale is conducted in all cases. The convergence rate of the FD2 method is found to be 1.30, which is better than the previous example. This may be because of the use of higher order finite difference approximations in the $y$ direction. Based on the same stencil bandwidth, the full second order is recovered by the MIBTD2 method with $r = 2.04$. For the MIBTD4, the overall order is $r = 4.25$. Thus, a fourth order of convergence is numerically achieved in the present 2D example (see Fig. 6).

3.3. Example 3: Gaussian–Sine initial wave

We finally consider an example with a different initial condition. The same domain dimensions and Debye coefficients as in Example 2 are considered, but the initial wave is chosen as $E_z(x,y,0) = \exp\left(-\left(x - x_c\right)^2/(2a^2)\right) \sin\left(4\pi y/(d - c)\right)$, where $x_c$ and $x$ are unchanged. For the left and right boundaries of the domain, the PEC condition [26] is assumed again, because the initial solution vanishes near these boundaries. For the top and bottom boundaries, the initial solution is chosen such that it equals to zero exactly on the boundaries. So, the PEC condition can also be imposed. Because the solution is oscillatory in the $y$ direction (see Fig. 7), a larger number of grid nodes $N_y = 101$ is employed. As before, $N_x = 101$ and $M_y = 4$ are fixed in all numerical tests, including the MIBTD and FD2 methods. Through the similar convergence tests, the convergence rate for the FD2, MIBTD2, and MIBTD4 is found to be, respectively, $r = 1.36$, $r = 2.10$, and $r = 4.32$, see Fig. 8.

4. Conclusion

In summary, a new interface formulation for solving the two-dimensional (2D) transverse magnetic modes with Debye dispersive interfaces is constructed. The time dependent jump conditions are coupled with Maxwell’s equations via interface auxiliary differential equations. Higher order convergences are numerically achieved for the first time in the literature in the 2D finite difference time domain (FDTD) simulations of dispersive inhomogeneous media. Even though an air-Debye interface is considered, the proposed matched interface and boundary (MIB) approach can be easily extended to handle a more general dispersive interface separating two inhomogeneous Debye media. However, the generalization of the MIB method to curved dispersive interfaces is not trivial, and is currently under our investigation.

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References


