Internal Singularity Analysis of a Class of Lower Mobility Parallel Manipulators with Articulated Traveling Plate

Olivier Company, Sébastien Krut, and François Pierrot

Abstract—This paper deals with a particular family of lower mobility parallel kinematic manipulators. The four degrees of freedom of the end effector consist in three translations plus one rotation with a high tilting angle. Robots belonging to this family are first introduced and a common parameterization is established. Then an extended kinematic model is proposed for this family of robots using a new Jacobian matrix. Relevant information about robot kinematic singularities, internal singularities and possible end-effector motions can be obtained by resorting this matrix. The efficiency of this method is proven by applying it to several traveling plate architectures corresponding to already built robot prototypes. The results on the expected behavior are compared to the prototype’s real behavior.

The goal of this paper is to show that internal or constraint singularities can occur in lower mobility parallel kinematic manipulators and to underline the influence they have during the design stage.

Index Terms—Lower mobility parallel manipulators, singularity, internal singularity, articulated traveling plate.

I. INTRODUCTION

EVENTHough considerable research has been devoted (and is still devoted) to Gough [1] and Stewart platforms [2] (see [3] and [4] for an extensive coverage of this issue) and mathematical issues concerning their models and singularities [5], a new avenue for research has arisen concerning lower mobility parallel mechanisms. These robots have less than six degrees of freedom (dofs) and are expected to be cost-effective with respect to their design and manufacturing. Various research initiatives conducted in this domain, particularly by Clavel, have led to innovative architectures like the famous Delta robot [6]. This robot also paved the way to high speed applications, particularly pick-and-place for parallel robots, thanks to its fixed actuators that drastically reduce moving part masses.

Most research on lower mobility parallel mechanisms is devoted to three-dof manipulators that are purely translational like the Delta [6] robot or rotational like the agile eye [7].

Synthesis of lower mobility parallel manipulators has been investigated by several authors such as [8], [9], [10], [11] and [12]. Some authors use displacements group theory [13] [14] for kinematic synthesis. In the general case, irrespective of the method, synthesized parallel kinematics only take a rigid traveling plate linked to a base by several identical kinematic chains [15] into account. One of the goals of this paper is to get further insight into the possible kinematics for parallel manipulators, especially concerning the use of articulated traveling plates.

Mobility analysis of lower mobility parallel mechanisms is a crucial point for their design. Several approaches are used:

- Displacement group theory is very powerful and general but is impeded by the fact that the possible displacement of each kinematic chain must form a group, which is a major drawback: for example, there is no displacement group corresponding to a \( P(SS)_2 \) or \( R(SS)_2 \) (see Table I for the notation). For a Delta robot, three of these five-dof chains (Fig. 5) connect the base to the traveling plate (this leaves three dof for the traveling plate). As there are no five-dimension displacement groups, it is not possible to model each chain and thereby apply group properties to find the remaining possible displacements of the traveling plate.

- A kinematic condition analysis of three-dof translational parallel manipulators based on screw theory is proposed in [16], [17] and [18] also present an enumeration of four and five limb parallel manipulators using screw theory. This method has a drawback: the equations obtained are related to a particular pose and, as they apply to the neighborhood of the considered pose, they are extended. This can lead to bad results if the studied pose is singular.

<table>
<thead>
<tr>
<th>symbol</th>
<th>joint name</th>
<th>symbol</th>
<th>joint name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>prismatic</td>
<td>( R )</td>
<td>revolute</td>
</tr>
<tr>
<td>( U )</td>
<td>universal</td>
<td>( S )</td>
<td>spherical</td>
</tr>
<tr>
<td>( C )</td>
<td>cylindrical</td>
<td>( X )</td>
<td>with sensor</td>
</tr>
</tbody>
</table>

With most methods, only the kinematics model is taken into account. This model is simplified and assumes that the stiffness in the constrained directions is always infinite. The proposed method is to model the complete mechanism and then to extract and analyze the classical kinematics jacobian matrix and a newly defined constraint matrix. This paper applies this approach to a kind of lower mobility parallel mechanisms whose traveling plate is not limited to be a rigid body.

About 10 years after the creation of the Delta robot, Pierrot invented a new robot kinematic system [19], i.e. the H4 robot,
based on the same technology as Delta, and having the same
degrees of freedom (three translations and one rotation around
a given axis). At the same time, Angeles proposed another
robot with the same degrees of freedom [20] but based on the
actuation of parallelogram linkages assembled in a serial way.
The H4 robot is based on the new concept of an articulated
traveling plate for parallel mechanisms. This concept is quite
new and only a few authors have worked on it like [21] for
the Hita STT high speed machine tool. The main advantage of
the first H4 robot prototype was the same degrees of freedom
as the Delta robot, but without the central leg (transmitting
the rotation from the fixed actuator to the traveling plate)
that causes reliability problems. At the early design stage, the
authors noticed that symmetry had to be avoided in the design
of the mechanism. Regarding this, an operating condition was
derived for this particular robot.

Despite the drawbacks of the first prototype, since the
creation of H4, the articulated traveling plate concept has
been investigated in greater detail and new prototypes have
been built [22] [23] [24]. All of these robots have a total of
four degrees of freedom with the aim of achieving Schoenflies
motions (also known as SCARA motions). The rigid motion
of the traveling plate thus has three translational dofs and
internal motion is transformed into a rotation about a given
axis. Null stiffness arrangements have still been found when
designing articulated traveling plate parallel robots. The goal
of this paper is to develop a method to check if a selected
arrangement leads to null stiffness at the design stage for
robots with articulated traveling plates that use Delta-like
kinematic chains. The studied family of four-dof robots is
first described. Then a complete kinematics (i.e., not limited
to operational speeds but including internal velocities) modeling
is conducted using an approach similar to [25]. Di Gregorio
applied this approach [26] to the simpler case of the classical
Delta robot. In the third section, a method is proposed to
analyze this model regarding different kinds of singularities:
serial, parallel and internal, but this paper is mainly focused on
the internal ones. The last section is devoted to the description
of three existing prototypes. The method is then applied to
the architecture of these prototypes in order to check its efficiency.

II. FAMILY DESCRIPTION

Except for the number of actuators, H4/I4 robot family
roughly resembles the original Delta concept. The actuators
are fixed on the base and can be revolute or prismatic (see Fig.
1). However, robots belonging to this family have four pairs
of bars and an articulated traveling plate.

A. Forearm description

According to Fig. 1, a forearm is composed of two bars of
equal length. Each bar ends with ball joints but one ball joint
per bar can be replaced by a universal joint to eliminate the
internal degree of freedom (a bar can rotate about the axis
defined by ball joint centers). For each pair of bars, distances
between ball joint centers on the traveling plate side and on the
actuator side are equal. If revolute actuators are used, the axis
passing through the center of the two ball joints located on the
actuator side must be parallel to the actuator axis (see Fig. 1).
The two bars of each forearm are designed to remain parallel to
each other when the robot is running, i.e., they remain coplanar.

Some changes can be made to the robot geometry without
changing the traveling plate motion features. Spatial parallelograms with (SS)2 chains can be replaced by planar
parallelograms with RIIR chains (where II [11] [27] stands
for a (RR)2 pair), as in the orthoglide project [28] and in the
star robot [11]. Robots built with RIIR are hyperstatic and do
not require further development as parallelograms are designed
to remain planar.

It is also possible to substitute each pair of bars with a
simple bar with a universal joint at each end, like in a Tsai
mechanism [29], while respecting some design rules. The following discussion does not support both of these possible
modifications without adjustment.

B. Traveling plate description

If a rigid traveling plate is used, Grubler’s formula gives
the over-constraint rank, but rather this is not generally a
real proof. With a rigid traveling plate and four Delta-like
legs (each leg is not based on a II pair but rather on a
four bar linkage with spherical joints), the value of the over-
constraint rank is two, once the internal dof of each bar is
taken into account. There are several ways to deal with this
over-constrained problem:

- Manufacture and assemble parts in a very precise way
- Overcome constraints by adding degrees of freedom
  in joints or removing bars (each bar adds a distance
  constraint between two points)
- Add joints (and thus add parts)
- Use flexible parts to compensate for manufacturing and
  assembly errors

Fig. 1. Possible robot actuation
Of course, each of these four solutions can be combined with the others. In the following developments, the focus is on adding joints to the traveling plate to obtain an “articulated traveling plate”.

The articulated traveling plate is made of two or three bodies. The joints between these bodies can be prismatic, revolute and/or cylindrical. The four parallelograms are grouped in pairs. Each pair is connected to a different body (see Fig. 1).

After the generic modeling phase, three existing robots will be studied:

- H4 robot prototype with revolute actuators and an articulated traveling plate with two revolute joints [19]
- I4L robot prototype with prismatic actuators and an articulated traveling plate with two prismatic joints [22] plus an additional constraint
- I4R robot prototype with revolute actuators and a single prismatic joint traveling plate [30]

III. KINEMATIC MODELING

A. Systematic parameters

Systematic parameters are chosen to obtain an unique method for all of these robots. These parameters must be able to deal with different types of traveling plates equipped with revolute, cylindrical or prismatic joints. These parameters must also take traveling plates made of two or three parts into account. Rigid joints are used to solve the number of parts problem.

In Figures 2 and 3, the following notations apply:

- $i$ stands for kinematic chain number ($1 \leq i \leq 4$) and $j$ stands for bar number in a kinematic chain ($1 \leq j \leq 2$)
- $k$ stands for the traveling plate body number ($1 \leq k \leq 3$): chain numbers 1 and 2 are connected to the body with $k = 1$, chain numbers 3 and 4 are connected to body with $k = 2$. To keep the systematic aspect of this modeling, a third body ($k = 3$) is always used. It is supposed to be rigidly fixed to body 1 or 2 if the traveling plate has only two real physical parts.
- $A_{ij}$ (resp. $B_{ij}$) is the center of ball joint number $ij$ on the actuator side (resp. on the traveling plate side)
- $A_i$ (resp. $B_i$) is the middle of $A_{1i}A_{2i}$ (resp. $B_{1i}B_{2i}$)
- $C_k$ is the center of the joint between traveling plate parts 3 and $k$
- $D$ is the Tool Controlled Point (TCP)
- $v_k$ is the vector giving the direction of the joint axis between traveling plate bodies 3 and $k$
- $\delta_k$ and $\delta'_k$ are coefficients depending on the joint type. Values are given in Table II. $\delta = 1$ allows a translation motion along $v_k$ and $\delta' = 1$ allows a rotation motion around $v_k$
- $\epsilon_k$ is joint $k$ parameter (relative linear or angular displacement between body $k$ and body 3 along axis $k$)

Some vectors are also defined (see Figures 2 and 3):

- $\vec{d}_{k}$ is the vector linking point $C_k$ to point $B_i$
- $\vec{r}_{i}$ is the vector tangent to point $A_{ij}$ trajectory corresponding to the current pose (if a linear actuator is used, this vector will be unitary, and if a revolute actuator is used, the norm of this vector will be equal to the distance from point $A_{ij}$ to the revolute joint axis)
- $\vec{l}_{i} = \vec{l}_{1i} = \vec{l}_{2i}$ is the vector linking point $A_{ij}$ to point $B_{ij}$
- $\vec{e}_{ij} = \vec{c}_k + \vec{d}_{ij}$
- $\vec{\epsilon}_{ij} = \vec{c}_k + \vec{d}_{i}$
- $\vec{f}_{i}$ is the vector linking point $B_{1i}$ to point $B_{2i}$

Reference frame axes are noted $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$.

B. Complete modeling

Classical kinematic jacobian matrices linking operational velocities $\dot{x}$ and articualr velocities $\dot{q}$ is obtained by the equiprojectivity property of velocities of two points belonging to a rigid body [19]:

\begin{table}[h]
\centering
\caption{Coefficients depending on the joint type}
\begin{tabular}{|c|c|c|}
\hline
joint name & $\delta$ & $\delta'$ \\
\hline
rigid & 0 & 0 \\
prismatic & 1 & 0 \\
revolute & 0 & 1 \\
cylindrical & 1 & 1 \\
\hline
\end{tabular}
\end{table}
\[ J_x \dot{x} = J_q \dot{q} \] (1)

If \( J_x \) is a full rank matrix, i.e. the current pose is not singular:

\[ \dot{x} = J_x^{-1} J_q \dot{q} = J \dot{q} \] (2)

The goal of this complete modeling is to find a model that generates information about kinematic chain relative positions when the mechanism behavior is considered. The jacobian kinematic matrix does not give any relevant information about the mechanism structural stiffness and internal motions because this matrix is only focused on transformation between the joint space and operational space velocities.

Let \( \dot{s}_D \) be the vector composed of the point \( D \) (TCP) velocity:

\[ \dot{s}_D = \begin{bmatrix} s \\ \omega \end{bmatrix} \] (3)

where \( s \) is the cartesian velocity of point \( D \) and \( \omega \) is the angular velocity of body number 3.

The velocity of point \( C_k \) belonging to body 3 is:

\[ \dot{s}_{C_k} = \dot{s}_D + \begin{bmatrix} \omega \times c_k \\ 0_3 \end{bmatrix} \] (4)

where \( 0_n \) is a n-dimensional zero vector.

The velocity of point \( C_k \) belonging to body \( k (k \in \{1, 2\}) \) is:

\[ \dot{s}_{C_{k \in 3}} = \dot{s}_{C_{k \in 3}} + \dot{s}_k \] (5)

with:

\[ \dot{s}_k = \begin{bmatrix} \delta_k \dot{q}_k v_k \\ \delta_k' \dot{q}_k' v_k \end{bmatrix} \] (6)

The velocity of point \( B_{ij} \) belonging to body \( k (k \in \{1, 2\}) \) is (see Table II):

\[ \dot{s}_{B_{ij \in k}} = \dot{s}_{C_{k \in 3}} + \begin{bmatrix} (\delta_k' \dot{q}_k' v_k + \omega) \times d_{ij} \\ 0_3 \end{bmatrix} \] (7)

As points \( B_{ij} \) are centers of ball joints that connect body \( k (k \in \{1, 2\}) \) to bars, the linear velocity of point \( B_{ij} \) belonging to body \( k \) is equal to the linear velocity of point \( B_{ij} \) belonging to the considered bar. According to the geometry presented in Fig. 1, the velocities of points \( A_{i1} \) and \( A_{i2} \) are equal:

\[ \dot{s}_{A_{i1}} = \dot{s}_{A_{i2}} = \dot{s}_{A_{i}} \] (8)

Furthermore:

- there are only pure translations for linear actuators,
- the distances between the rotation axis and points \( A_{i1} \) and \( A_{i2} \) are equal for revolute actuators.

\[ \dot{s}_{A_{i}} = \dot{q}_i r_i \] (9)

Using the velocity properties of rigid bodies, the following linear system is obtained:

\[ J_{act1} \dot{q} = J_{tp1} \dot{x}_1 \] (10)

with:

\[ J_{act1} = \begin{bmatrix} r_1^T l_{11} & 0 & 0 & 0 \\ r_1^T l_{12} & 0 & 0 & 0 \\ 0 & r_2^T l_{21} & 0 & 0 \\ 0 & r_2^T l_{22} & 0 & 0 \\ 0 & 0 & r_3^T l_{31} & 0 \\ 0 & 0 & r_3^T l_{32} & 0 \\ 0 & 0 & 0 & r_4^T l_{41} \\ 0 & 0 & 0 & r_4^T l_{42} \end{bmatrix} \] (11)

and:

\[ J_{tp1} = \begin{bmatrix} I_{11} & [e_{11} \times l_{11}]^T & h_{11} & 0 \\ I_{12} & [e_{12} \times l_{12}]^T & h_{12} & 0 \\ I_{21} & [e_{21} \times l_{21}]^T & h_{21} & 0 \\ I_{22} & [e_{22} \times l_{22}]^T & h_{22} & 0 \\ I_{31} & [e_{31} \times l_{31}]^T & 0 & h_{31} \\ I_{32} & [e_{32} \times l_{32}]^T & 0 & h_{32} \\ I_{41} & [e_{41} \times l_{41}]^T & 0 & h_{41} \\ I_{42} & [e_{42} \times l_{42}]^T & 0 & h_{42} \end{bmatrix} \] (12)

with:

\[ \begin{align*}
& h_{ij} = I_{ij}^T (\delta_k v_k + \delta_k' v_k + d_{ij}) \\
& \dot{q}_i = [\dot{q}_i] \\
& \dot{x}_1 = [s_{D}^T \dot{\epsilon}_1 \dot{\epsilon}_2]^T
\end{align*} \]

Eq 10 describes all the distance constraints imposed by the bars, and not the classical input/output relation given by the jacobian matrix.

IV. KINEMATIC ANALYSIS

A. Introduction

Kinematic jacobian matrix analysis is usually enough to find singular positions. This is particularly true for hexapods because this matrix contains all relevant information. This is not true, for example, for a Delta robot because some parameters such as the distance between points \( A_{11} \) and \( A_{12} \) (the same for points \( B \) ) have no influence in the kinematic jacobian matrix, even though they are essential for evaluating mechanism stiffness. In the same way, the orientation of \( \vec{f}_i \) is not considered in the kinematic jacobian matrix. This is particularly problematic because, if all vectors \( \vec{f}_i \) \( (i \in \{1, 2, 3\}) \) are parallel (see Fig. 4), the mechanism will have a null stiffness in one direction, even if the jacobian kinematic matrix is well conditioned. For a complete study of mechanism degrees of freedom, the following points should be considered, especially for architectures composed of Delta kinematic chains:

- a geometry analysis using pose transformation laws (assuming that parallelograms are planar)
- a kinematic analysis based on jacobian kinematic matrices \( J_x, J_q \) and \( J \) defined in equations 1 and 2, by checking that the mechanism has no under-mobilities (\( det (J_q) = 0 \) and no over-mobilities (\( det (J_x) = 0 \))
- a more complete kinematic analysis to focus on internal singularities. This last point is the purpose of the proposed method presented below.

Singularities of robots belonging to the H4/I4 family occur when a bad arrangement is selected at the design stage.
for parallelograms. Fig. 4 shows good and bad in-the-plane arrangements but off-the-plane arrangements are also valid [31]. This arrangement plays an important role in robot stiffness. Classical kinematic jacobian analysis does not generate complete information, e.g. no information is taken into account for all non-controlled motions. This point is crucial for robots with less than six degrees of freedom. A complete generic kinematic analysis is hard to derive because a huge number of kinematic equations have to be considered. The proposed method is based on a kinematic approach but only requires eight linear equations. This method, designed for the H4/I4 family, can be easily extended to other families of parallel kinematic robots.

![Fig. 4. Different arrangements of the Delta robot traveling plate](image)

**B. Development**

Hervé’s work on displacement group theory gives an answer to the geometry analysis of such mechanisms [32]. If \( \{T\} \) denotes the subgroup of spatial translation displacements and \( \{X(u)\} \) denotes the subgroup of Schoenflies displacements (three translations plus one rotation around an axis of given direction), where \( u \) stands for the unitary vector giving the direction of the rotation axis. If a closed loop composed of two chains producing Schoenflies displacements with \( u \neq v \), then:

\[
\{X(u)\} \cap \{X(v)\} = \{T\} \tag{13}
\]

That is to say, that this mechanism will have only three remaining translations.

For the studied family, the weak point of this reasoning concerns the fact that \( P(SS)_2 \) and \( R(SS)_2 \) chains have three translational and two rotational displacement capabilities. These five displacements cannot form a group because the union property is not valid. The only way to deal with this problem is to add the hypothesis that parallelograms remain planar during displacement and then to replace \( P(SS)_2 \) chains by chains providing Schoenflies displacements such as PRIIR chains (this also applies to \( R(SS)_2 \) chains), but important information is lost due to this strong hypothesis.

The proposed analysis method does not use group theory because \( P(SS)_2 \) chains are considered (see Fig. 5). The goal is to find the linear system of the smallest rank describing the complete kinematic mechanism, where “complete” means that all possible singularities can occur, especially those relative to internal mobilities.

Except for the rotation of bars about their own axis, which is well known, (10) contains all relevant information about machine kinematics. By multiplying (10) by the invertible matrix \( M \) defined by:

\[
M = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\end{bmatrix} \tag{14}
\]

where:

\[
det(M) = 1 \tag{15}
\]

Matrix \( M \) adds consecutive lines two by two and subtracts consecutive lines two by two without changing system (10) rank. The goal of this operation is to have a four by four zero block at the end of \( J_{act1} \).

The following linear system is obtained:

\[
J_{act2} \dot{q} = J_{tp2} \dot{x}_1 \tag{16}
\]

with:

- \( J_{act2} = MJ_{act1} = \begin{bmatrix} J_q \\ 0 \end{bmatrix} \)
- \( J_q = \text{diag}([l^T_i, r_i]) \)
- \( J_{tp2} = MJ_{tp1} \)

Matrix \( J_{tp2} \) elements are:

\[
J_{tp2} = \begin{bmatrix}
l_1^T & p_1^T & g_1 & 0 \\
l_2^T & p_2^T & g_2 & 0 \\
l_3^T & p_3^T & 0 & g_3 \\
l_4^T & p_4^T & 0 & g_4 \\
0 & w_1 & m_1 & 0 \\
0 & w_2 & m_2 & 0 \\
0 & w_3 & 0 & m_3 \\
0 & w_4 & 0 & m_4
\end{bmatrix} \tag{17}
\]
where:

- \( g_i = I_T^T (\delta_i \mathbf{v}_i + \delta'_i \mathbf{v}_i \times \mathbf{d}_i) \)
- \( m_i = I_T^T (\delta'_i \mathbf{v}_i \times f_i) \)
- \( w_i = f_i \times l_i \)
- \( p_i = e_i \times l_i \)

This linear system can be seen as a linear system relative to unknown \( \mathbf{x}_1 \), this means that mechanism under-mobilities will not be considered (analysis of \( \mathbf{J}_4 \) matrix). The linear system 16 turns into:

\[
\mathbf{J}_{tp2} \mathbf{x}_1 = \begin{bmatrix}
I_T^T \mathbf{r}_i \dot{q}_i \\
0
\end{bmatrix}
\]

This system has eight equations and eight unknowns.

If the rank of \( \mathbf{J}_{tp2} \) is not equal to eight, the system has an infinite number of solutions. For the mechanism to run properly, all velocities must be correctly determined (single solution), i.e. the system is a Cramer system (\( \det(\mathbf{J}_{tp2}) \neq 0 \)). Determinant analysis of \( \mathbf{J}_{tp2} \) is necessary for searching geometrical running conditions.

C. Linear system decoupling

To avoid having to perform a complete analysis of the linear system, this system will be turned into a block triangular system:

\[
\mathbf{J}_{tp2} \mathbf{x}_2 = \begin{bmatrix}
I_T^T \mathbf{r}_i \dot{q}_i \\
0
\end{bmatrix}
\]

with:

- \( \mathbf{J}_{tp3} = \begin{bmatrix}
\mathbf{J}_x & \mathbf{J}_{int} \\
0 & \mathbf{J}_{int}
\end{bmatrix} \)
- \( \mathbf{x}_2 = \begin{bmatrix}
\mathbf{x} \\
\dot{\mathbf{x}}
\end{bmatrix} \)

\( \mathbf{J}_x \) is the kinematic jacobian matrix and \( \dot{\mathbf{x}} \) is the vector of operational velocities. This new system is obtained from (19). The method to obtain linear system 19 is as follows:

The first step is to sort \( \mathbf{J}_{tp3} \) (and consequently vector \( \mathbf{x}_1 \)) to certify that the four first columns correspond to the four operational speeds of the mechanism. In a practical way, only the fourth column is concerned by this operation because the three first columns are associated to operational speeds \( \dot{x}, \dot{y} \) and \( \dot{z} \) in the correct order. For some cases, when a gear system (e.g. amplification device in Fig. 8), a rack-and-pinion device (Fig. 17) or a cable and pulley system (Fig. 12) is used, the fourth operational velocity is not explicitly visible in vector \( \mathbf{x}_1 \). In this case, a transformation ratio \( k \) must be used to convert the internal dof into the operational dof. Let’s note \( \mathbf{J}_{tp2} \) the matrix used in the new sorted system:

\[
\mathbf{J}_{tp2} \mathbf{x}_1 = \mathbf{J}_{tp2} \mathbf{R}^{-1} \mathbf{R} \mathbf{x}_1 = \mathbf{J}_{tp2}' \mathbf{x}_1'
\]

with, for example, an expression of the resorting matrix \( \mathbf{R} \) which is invertible:

\[
\mathbf{R} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

The second step is to combine columns to create a four-by-four zero block in the lower left corner. Only the fourth column is concerned in this step because a four-by-three zero block is already located at the lower left corner. Let’s note \( \mathbf{C}_i \) the \( i^{th} \) column of \( \mathbf{J}_{tp2}' \):

\[
\mathbf{J}_{tp2}' \mathbf{x}_1' = \mathbf{C}_1 \dot{x} + \mathbf{C}_2 \dot{y} + \mathbf{C}_3 \dot{z} + \mathbf{C}_4 \dot{x}_4 + ... + \mathbf{C}_8 \dot{x}_8
\]

Eq. (22) can be resorted like this:

\[
\mathbf{J}_{tp2}' \mathbf{x}_1' = \begin{bmatrix}
\dot{x} \mathbf{C}_1 \\
\dot{y} \mathbf{C}_2 \\
\dot{z} \mathbf{C}_3 \\
\dot{x}_4 (\mathbf{C}_4 + k_5 \mathbf{C}_5 + k_6 \mathbf{C}_6 + k_7 \mathbf{C}_7 + k_8 \mathbf{C}_8) \\
\dot{x}_5 - k_5 \dot{x}_4 \mathbf{C}_5 \\
\dot{x}_6 - k_6 \dot{x}_4 \mathbf{C}_6 \\
\dot{x}_7 - k_7 \dot{x}_4 \mathbf{C}_7 \\
\dot{x}_8 - k_8 \dot{x}_4 \mathbf{C}_8
\end{bmatrix}
\]

where \( k_5, k_6, k_7, k_8 \) are four scalar variables that do not depend on time. These variables are used to set the four last lines of the fourth column at zero, i.e. corresponding to operational variable \( \dot{x}_4 \).

In a matrix formulation, this can be written as follows:

\[
\mathbf{J}_{tp2}' \mathbf{x}_1' = \mathbf{J}_{tp2} \mathbf{P}^{-1} \mathbf{P} \dot{\mathbf{x}}_1 = \mathbf{J}_{tp3} \mathbf{x}_2
\]

with \( \mathbf{P} \) being a change of variable matrix:

\[
\mathbf{P} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -k_5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -k_6 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -k_7 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -k_8 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

As \( \det(\mathbf{P}) = 1 \neq 0 \) this is a full rank change of variable.

D. Internal singularities

System 19 is an upper triangular system, so:

\[
\det(\mathbf{J}_{tp3}) = \det(\mathbf{J}_x) \det(\mathbf{J}_{int})
\]

As \( \mathbf{J}_x \) analysis gives information about over-mobility singularities, and as this system is representative of all mechanism degrees of freedom, internal singularities will come only from \( \mathbf{J}_{int} \) analysis. If a pose has no internal singularity, then:

\[
\det(\mathbf{J}_{int}) \neq 0
\]
To solve system 19, it is possible to proceed by matrix block; firstly, we can note that:

$$\mathbf{J}_{int} \alpha = 0 \quad (28)$$

If the mechanism has no internal singularity ($det(\mathbf{J}_{int}) \neq 0$), the only solution for system 28 is:

$$\dot{\alpha} = 0 \quad (29)$$

This implies:

$$\dot{x}_5 = k_5 \dot{x}_4, \quad \dot{x}_6 = k_6 \dot{x}_4, \quad \dot{x}_7 = k_7 \dot{x}_4, \quad \dot{x}_8 = k_8 \dot{x}_4 \quad (30)$$

All of these equations represent the relationship between internal velocities. In the second stage of the analysis, as $\dot{\alpha} = 0$, system 19 will lead to the classical (31) which expresses the relationship between joint and operational velocities.

$$\mathbf{J}_x \ddot{x} = \mathbf{J}_q \dot{q} \quad (31)$$

In next section, this method is applied to three existing prototypes of existing parallel robots belonging to H4/I4 family.

V. ANALYSIS OF EXISTING ROBOT PROTOTYPES

A. First H4 robot prototype

The H4 robot prototype was designed in 1999 [19] and is the first member of the H4/I4 family. Various arrangements of the H4 architecture have been studied [33] (see Figures 6 and 7). The main innovation of this concept concerns the use of an "H-shaped" articulated traveling plate (see Fig. 8). The traveling plate was composed of three bodies and two revolute joints ($\delta_1 = \delta_2 = 0$ and $\delta'_1 = \delta'_2 = 1$). On the first traveling plate prototype, the gripper was fixed on the central part (body number three according to previous notations). Rotation of this body about the vertical axis was exactly the fourth operational speed. The joint-and-loop graph of this robot is presented in Fig. 9.

For this robot, the $\mathbf{J}_{tp2}$ matrix can be written:

$$\mathbf{J}_{tp2} = \begin{bmatrix}
I^T_1 & p_1^T & (v_1 \times d_1).l_1 & 0 \\
I^T_2 & p_2^T & (v_1 \times d_2).l_2 & 0 \\
I^T_3 & p_3^T & 0 & (v_2 \times d_3).l_3 \\
I^T_4 & p_4^T & 0 & (v_2 \times d_4).l_4 \\
0 & w_1^T & (v_1 \times f_1).l_1 & 0 \\
0 & w_2^T & (v_1 \times f_2).l_2 & 0 \\
0 & w_3^T & 0 & (v_2 \times f_3).l_3 \\
0 & w_4^T & 0 & (v_2 \times f_4).l_4 
\end{bmatrix} \quad (32)$$

As the fourth operational speed is $\omega_z$, column numbers four and six have to be exchanged:

$$\mathbf{J}_{tp2} = \begin{bmatrix}
I^T_1 & p_1.e_z & p_1.e_x & p_1.e_y \\
I^T_2 & p_2.e_z & p_2.e_x & p_2.e_y \\
I^T_3 & p_3.e_z & p_3.e_x & p_3.e_y \\
I^T_4 & p_4.e_z & p_4.e_x & p_4.e_y \\
0 & w_1.e_z & w_1.e_x & w_1.e_y \\
0 & w_2.e_z & w_2.e_x & w_2.e_y \\
0 & w_3.e_z & w_3.e_x & w_3.e_y \\
0 & w_4.e_z & w_4.e_x & w_4.e_y \\
(v_1 \times d_1).l_1 & 0 \\
(v_1 \times d_2).l_2 & 0 \\
0 & (v_2 \times d_3).l_3 \\
0 & (v_2 \times d_4).l_4 \\
(v_1 \times f_1).l_1 & 0 \\
(v_1 \times f_2).l_2 & 0 \\
0 & (v_2 \times f_3).l_3 \\
0 & (v_2 \times f_4).l_4 
\end{bmatrix} \quad (33)
Fig. 9. H4 robot joint and loop graph

consequently with:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\omega_x \\
\omega_y \\
\dot{\epsilon}_1 \\
\dot{\epsilon}_2
\end{bmatrix} = T \begin{bmatrix}
1^T \\
1^T \\
2^T \\
3^T \\
4^T \\
1 \\
0
\end{bmatrix} \begin{bmatrix}
p_1.e_z \\
p_2.e_z \\
p_3.e_z \\
p_4.e_z \\
w_1.e_x \\
w_2.e_x \\
w_3.e_x \\
w_4.e_x
\end{bmatrix} \begin{bmatrix}
p_1.e_y \\
p_2.e_y \\
p_3.e_y \\
p_4.e_y \\
w_1.e_y \\
w_2.e_y \\
w_3.e_y \\
w_4.e_y
\end{bmatrix}
\]

(34)

If the mechanism behavior is as expected, the revolute joint axes remain vertical \((v_1 = v_2 = e_z)\). Then, by subtracting columns seven and eight from column four:

\[
J_{tp2} = \begin{bmatrix}
1^T \\
1^T \\
2^T \\
3^T \\
4^T \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
p_1.e_z \\
p_2.e_z \\
p_3.e_z \\
p_4.e_z \\
w_1.e_x \\
w_2.e_x \\
w_3.e_x \\
w_4.e_x
\end{bmatrix} \begin{bmatrix}
p_1.e_y \\
p_2.e_y \\
p_3.e_y \\
p_4.e_y \\
w_1.e_y \\
w_2.e_y \\
w_3.e_y \\
w_4.e_y
\end{bmatrix}
\]

(35)

\[
J_{int} = \begin{bmatrix}
w_1.e_x & w_1.e_y & (v_1 \times f_1).l_1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(36)

The next step is to develop determinant of the \(J_{int}\) matrix, which leads to:

\[
det(J_{int}) = -\left( (w_1 \times w_2) \times (w_3 \times w_4) \right).e_z
\]

(37)

So the condition to be verified to have no internal singularity is:

\[
\left( ([f_1 \times l_1] \times [f_2 \times l_2]) \times ([f_3 \times l_3] \times [f_4 \times l_4]) \right).e_z \neq 0
\]

(38)

The condition obtained by this new method is the same that one presented in [19] and [34]. In other words, this condition shows that it is impossible to have a perfectly symmetrical arrangement for the robot architecture, as verified in the first prototype design (see Fig. 10). The consequence of this last point is crucial because the lack of symmetry in this mechanism induces bad characteristics in terms of isotropy and stiffness (verified on the robot prototype [35]).

B. Rotative I4 robot

The I4R robot is partly made of ABB Flexpicker parts (arms and forearms) and aims to have the same usable workspace, without having to cope with the passive leg at the center for rotation transmission (see Fig. 11). This robot has been designed for high speed pick and place, so its workspace is a 1 meter radius and 0.2 height cylinder. Its traveling plate is made of two bodies linked by a prismatic joint (see Fig. 12). The rotation motion of the tool is obtained by transforming the translation motion into rotation using a cable and pulley system.

Fig. 10. Different arrangements of the H4 robot traveling plate

(a) Bad arrangement (b) Good arrangement

Fig. 11. I4R robot prototype

The I4R robot can be considered as an isostatic mechanism assuming that the behavior of the prismatic joint of the traveling plate has minor rotational stiffness around the translation axis, so the prismatic joint can be modeled as a cylindrical joint (see Fig. 13). This implies:

\[
v_1 = v_2 = e_z, \delta_1 = 1, \delta'_1 = 0, \delta_2 = 0 \text{ and } \delta'_2 = 1
\]

(39)

The matrix to be studied to find internal singularities (according to the Fig. 13 joint and loop graph) is:
Fig. 12. I4R robot traveling plate

Fig. 13. I4R robot isostatic joint and loop graph

Fig. 14. I4R robot hyperstatic joint and loop graph

Fig. 15. I4L robot joint and loop graph

Fig. 16. I4L robot prototype

second one, as shown in the Fig. 14 joint and loop graph. It is obvious that this mechanism becomes hyperstatic. In that case, the method remains the same, but one equation has to be eliminated from the system: the one relative to the joint variable of the complete joint. So $J_{\text{int}}$ loses one line and becomes a $3 \times 4$ matrix and the condition to verify is that the rank of this matrix is 3. The result is that this rank is always three, so there is no special condition to obtain a mechanism with the expected behavior.

C. Linear I4 robot

The particularities of the I4L robot (see Fig. 15), regarding the two previously introduced robots, are:

- to use linear actuators having the same direction (see Fig. 16)
- to have two parallel prismatic joints on the traveling plate (see Fig. 17)
- due to the previous particularity, to have an additional
constraint generating a coupling relation between displacements of the two prismatic joints (see Fig. 15).

Indeed, the matrix to be analyzed in the case of the I4L robot with no such coupling is:

\[
J_{tp2} = \begin{bmatrix}
\mathbf{l}_1^T & \mathbf{p}_1^T & \mathbf{v}_1 & \mathbf{l}_1 & 0 \\
\mathbf{l}_2^T & \mathbf{p}_2^T & \mathbf{v}_1 & \mathbf{l}_2 & 0 \\
\mathbf{l}_3^T & \mathbf{p}_3^T & \mathbf{v}_2 & \mathbf{l}_3 & 0 \\
\mathbf{l}_4^T & \mathbf{p}_4^T & \mathbf{v}_2 & \mathbf{l}_4 & 0 \\
0 & \mathbf{w}_1^T & 0 & 0 \\
0 & \mathbf{w}_2^T & 0 & 0 \\
0 & \mathbf{w}_3^T & 0 & 0 \\
0 & \mathbf{w}_4^T & 0 & 0 
\end{bmatrix}
\] (43)

It is obvious that \( \det(J_{tp2}) = 0 \). Considering (18), this means that there are uncontrolled velocities (internal degrees of freedom) in the mechanism. To solve this problem, a coupling between two speeds must be added (see joint and loop graph on Fig. 15). Let’s introduce the operational rotation speed \( \omega_c \):

\[
\omega_c = k \epsilon_1 = -k \epsilon_2
\] (44)

with \( k \neq 0 \). Taking into account these additional equations, system 18) can be rewritten, while resorting the new operational speed \( \omega_c \):

\[
J_{tp2} \begin{bmatrix} \dot{x} \\ \omega_c \\ \omega \end{bmatrix} = \begin{bmatrix} \mathbf{r}_f \cdot \mathbf{l}_i \dot{q}_i \\ 0 \end{bmatrix}
\] (45)

with:

\[
J_{tp2} = \begin{bmatrix}
\mathbf{l}_1^T & k \cdot \mathbf{v}_1 & \mathbf{l}_1 & 0 \\
\mathbf{l}_2^T & k \cdot \mathbf{v}_1 & \mathbf{l}_2 & 0 \\
\mathbf{l}_3^T & -k \cdot \mathbf{v}_2 & \mathbf{l}_3 & 0 \\
\mathbf{l}_4^T & -k \cdot \mathbf{v}_2 & \mathbf{l}_4 & 0 \\
0 & 0 & \mathbf{w}_1^T & 0 \\
0 & 0 & \mathbf{w}_2^T & 0 \\
0 & 0 & \mathbf{w}_3^T & 0 \\
0 & 0 & \mathbf{w}_4^T & 0 
\end{bmatrix}
\] (46)

This matrix is no longer square (8 lines, 7 columns). The left upper \( 4 \times 4 \) block is the usual jacobian matrix \( J_x \). \( J_{int} \) (4 lines, 3 columns) is expressed as:

\[
J_{int} = \begin{bmatrix}
\mathbf{w}_1^T \\
\mathbf{w}_2^T \\
\mathbf{w}_3^T \\
\mathbf{w}_4^T
\end{bmatrix}
\] (47)

The fact that this matrix is not square shows that we turned the initial isostatic mechanism into an hyperstatic (rank 1) one by adding the coupling of two speeds. To check the lack of internal singularities, we have to verify that \( \text{rank}(J_{int}) = 3 \), that is to say:

\[
\exists (i, j, k), i \neq j \neq k \in \{1, 2, 3, 4\} / D_{ijk} \neq 0
\] (48)

Assuming that:

\[
D_{ijk} = (\mathbf{w}_i \times \mathbf{w}_j) \cdot \mathbf{w}_k
\] (49)

With the arrangement presented in Figures 16 and 17, this condition is respected in the robot workspace. We recall that such an arrangement of parallelograms was forbidden for a H4 robot due to internal singularities, as shown in Fig. 10.

VI. CONCLUSION

In this paper, we underlined that a complete modeling and analysis of lower mobility parallel manipulators is needed to take into account all kind of singularities. We focused on internal singularities of a family of lower mobility parallel kinematic robots. These robots have an articulated traveling plate and four degrees of freedom. We developed a common method to analyze the kinematics of all of these robots. A jacobian matrix including the kinematics and the constraints is derived. Then the kinematics jacobian matrix and the constraint matrices are extracted and analyzed for various traveling plate designs. The working conditions of three existing robot prototypes have been verified using this method. The next step in this work will be to derive from the analysis of internal singularities all possible arrangements and stiffness behavior of robots with articulated traveling plates.

REFERENCES


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