

## Existence of Confinement Phase in Quantum Electrodynamics

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### Abstract

We show that the four-dimensional U(1) gauge theory in the continuum formulation has a confining phase (exhibiting area law of the Wilson loop) in the strong coupling region above a critical coupling  $g_c$ . This result is obtained by taking into account topological non-trivial sectors in gauge theory. The derivation is based on the reformulation of gauge theory as a deformation of a topological field theory and subsequent dimensional reduction of the D-dimensional topological field theory to the (D-2)-dimensional nonlinear sigma model. The topological field theory part of the four-dimensional U(1) gauge theory is exactly equivalent to the two-dimensional O(2) nonlinear sigma model. The confining (resp. Coulomb) phase of U(1) gauge theory corresponds to the high (resp. low) temperature phase of O(2) nonlinear sigma model and the critical point  $g_c$  is determined by the Berezinskii-Kosterlitz-Thouless phase transition temperature. The quark (charge) confinement is caused by the vortex condensation. Thus the continuum gauge theory has the direct correspondence to the compact formulation of lattice gauge theory.

Key words: quark confinement, abelian gauge theory, topological field theory, dimensional reduction, nonlinear sigma model, instanton, vortex, monopole

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# 1 Introduction

In this paper we study the phase structure of the *continuum* Abelian U(1) gauge theory by including the effect due to compactness of U(1) group. The reason of taking into compactness is as follows. From the viewpoint of unified field theory, the Abelian group should be embedded as a subgroup in larger non-Abelian gauge group. In view of this, the Abelian group should be compact. Another importance of compactness of Abelian gauge group stems from a possibility of explaining quantization of charge [1]. In non-compact QED there are no reasons for charge quantization.

In this paper we show that the four-dimensional U(1) gauge theory has a confinement phase in the strong coupling region  $g > g_c$  due to the compactness (periodicity) leading to the non-trivial topological configuration. If we neglect the periodicity, we have a free U(1) gauge theory which has only one phase, the Coulomb phase, as expected. This work confirms the claim made by Polyakov [2, 3]. However, the claim that the abelian gauge theory has a confinement phase sounds strange from the conventional wisdom based on continuum abelian gauge theory. We clarify the meaning of this statement in what follows.

More than twenty years ago, it has been pointed out by many authors that the four-dimensional SU(2) non-Abelian gauge theory bears many similarities with a two-dimensional O(3) nonlinear sigma model (NLSM). Both theories possess asymptotic freedom, multi-instanton (and anti-instanton) solution, dynamical mass generation and scale invariance (i.e., no intrinsic scale parameter), see [4] for references.

These similarities can be seen also in the lattice regularized versions of these models, between spin models and lattice gauge theories [5]. Naively the scaling limit of the classical O(3) Heisenberg model is the O(3) NLSM, whereas that of the SU(2) lattice gauge theory is the SU(2) gauge theory. One can take the scaling limit of a lattice theory at a second order phase transition point. Hence the scaling limit is taken by approaching the critical point  $T \rightarrow T_c$  (or  $g \rightarrow g_c$ ) as the lattice spacing  $a$  goes to zero,  $a \rightarrow 0$ , in such a way that the physical quantities remain finite.

In two-dimensional classical O(3) Heisenberg model, two-point correlation function decays exponentially at any finite temperature. This corresponds to a claim in four-dimensional lattice SU(2) gauge theory that confinement phase survives as long as the coupling constant  $g$  is positive, even if  $g \ll 1$ . Both models have a phase transition at  $T = 0$  ( $g = 0$ ), i.e.,  $T_c = 0$  ( $g_c = 0$ ) which is believed to be the second order.

In a previous paper [4], it has been shown that these similarities between two models are not merely an accident; actually we have proved the exact equivalence between (D-2)-dimensional O(3) NLSM and the  $D$ -dimensional topological field theory obtained by removing the perturbative deformation (topological trivial sector) from the SU(2) non-Abelian gauge theory in  $D$ -dimensions ( $D \geq 3$ ). This proof is based on an idea of the dimensional reduction of Parisi and Sourlas [6]. The case of  $D = 4$  is the most interesting case of physical reality.

What can we say in the Abelian case? For this, recall a fact that the two-dimensional O(2) NLSM or XY model undergoes a phase transition without the appearance of a spontaneous magnetization. This absence of order parameter in two

dimensions is consistent with the Coleman-Mermin-Wagner (CMW) theorem [7]. The low temperature phase ( $T < T_c$ ) contains massless spin waves. On the other hand, the high temperature phase ( $T > T_c$ ) is completely disordered. For this phase transition, the periodicity of the angular variable  $\varphi$  is very important. The model has topological singularities, called the vortices. These vortices condense in high temperature and disorder the correlation function [8]. This phase transition is called the Berezinskii-Kosterlitz-Thouless (BKT) transition [8]. The vortex part is equivalent to the neutral Coulomb gas and sine-Gordon theory, see [9, 10, 11]. Although the existence of BKT transition is rather subtle, the existence of BKT transition was rigorously proved by Fröhlich and Spencer [12].

In lattice formulation, it is well known that all of these properties in two-dimensional abelian spin models have correspondences in the Abelian gauge theory in four dimensions. The vortices in two dimensions are closely related to magnetic monopoles in four dimensions, see [13, 14, 15]. The condensation of closed loops of magnetic monopole currents leads to quark (charge) confinement in the strongly coupled phase of Abelian gauge theory, since electric flux cannot easily penetrate such a medium (which we call the dual Meissner effect). It will be worth remarking that it has been shown that the dual superconductor picture of quantum chromodynamics (QCD) vacuum can be derived without any assumption [16] from QCD in the continuum formulation.

In lattice gauge theory, charge confinement in the sense of area decay of the Wilson loop is derived in the strong coupling region as shown by the strong coupling expansion [5, 17]. Quite remarkably, the quark (charge) confinement in lattice gauge theory occurs irrespective of the details of the gauge group, as long as it is compact (discrete [18, 19] or continuous), even for abelian gauge group. However, one expects that the U(1) lattice gauge theory in four dimensions ( $U(1)_4$ ) has a Coulomb phase in the weak coupling region, which was proved rigorously by Guth [20] and Fröhlich and Spencer [21]. Therefore the U(1) lattice gauge theory undergoes a phase transition at a finite non-zero coupling  $g_c$ . In continuum gauge theory, such non-trivial phase structure was suggested to occur due to topological non-trivial configurations by Polyakov [2, 1]. Actually he has shown the confinement phase in three-dimensional U(1) gauge theory for arbitrary gauge coupling, in agreement with the lattice analysis. In four dimensions, he claimed that the weak coupling U(1) gauge theory does not confine. Accordingly, it is expected that the U(1) gauge theory in four dimensions have two phases, confinement and deconfinement (Coulomb) phases, whereas only one phase i.e., confinement phase exists in three dimensions.

In this paper, we show that the continuum four-dimensional U(1) gauge theory has two phases, (strong coupling) confinement phase and (weak coupling) Coulomb phase, which have direct correspondence with the high-temperature and low-temperature phases in O(2) NLSM respectively. The phase transition point corresponds to the BKT transition in XY model. Therefore, the phase transition point  $g_c$  is determined by the BKT transition temperature  $T_c$ . This is one of the main results of this paper. This result is obtained as a specific case of the previous paper [4].

Therefore, in the strong coupling phase ( $g > g_c$ ), the continuum U(1) gauge theory confines quarks and the gauge field becomes massive, in agreement with the result

of lattice gauge theory. In the weak coupling phase, on the other hand, quarks are liberated and gauge field remains massless. In the weak coupling phase ( $g < g_c$ ), the beta function of the renormalization group [22] is identically zero and  $0 < g < g_c$  is the line of fixed points, if we neglect the perturbative deformations. For a review of lattice gauge theory, see e.g. [23, 24, 25].

The plan of this paper is as follows. In section 2, we give the reformulation of Abelian gauge theory as a deformation of a topological field theory. In section 3, using the reformulation of section 2, we evaluate the Wilson loop expectation value in four-dimensional Abelian gauge theory. In the final section we give some remarks on the renormalization group properties and extension of our scheme to other dimensional cases. More details about the interplay between the Abelian and non-Abelian cases are given in a forthcoming paper.

## 2 Abelian gauge theory as a deformation of TFT and dimensional reduction

Now we reformulate Quantum electrodynamics (QED) as a deformation of a topological field theory. It is obtained as a special case of non-Abelian gauge theory given in the previous paper [4].

### 2.1 Decomposition into perturbative and topological non-trivial sectors

QED on the  $D$ -dimensional space-time is defined by the action,

$$S_{QED}^{tot} = \int d^D x (\mathcal{L}_{QED}[a, \psi] + \mathcal{L}_{GF}), \quad (2.1)$$

$$\mathcal{L}_{QED}[a, \psi] := -\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \bar{\psi} (i\gamma^\mu D_\mu[a] - m)\psi, \quad (2.2)$$

where

$$f_{\mu\nu}(x) := \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x), \quad (2.3)$$

$$D_\mu[a] := \partial_\mu - i g a_\mu. \quad (2.4)$$

The gauge transformation of the  $U(1)$  gauge field  $a_\mu(x)$  and the fermion field  $\psi$  is defined by

$$a_\mu(x) \rightarrow a_\mu^U(x) := a_\mu(x) + \frac{i}{g} U(x) \partial_\mu U^\dagger(x), \quad U(x) \in U(1), \quad (2.5)$$

$$\psi(x) \rightarrow \psi^U(x) := U(x) \psi(x). \quad (2.6)$$

The gauge fixing term  $\mathcal{L}_{GF}$  is given by

$$\mathcal{L}_{GF} := -i \delta_B G_{gf}[a_\mu, C, \bar{C}, \phi], \quad (2.7)$$

using the nilpotent BRST transformation  $\delta_B$ ,

$$\begin{aligned}
\delta_B a_\mu &= \partial_\mu C, \\
\delta_B C &= 0, \\
\delta_B \bar{C} &= i\phi, \\
\delta_B \phi &= 0, \\
\delta_B \psi &= C\psi,
\end{aligned} \tag{2.8}$$

where  $\phi$  is the Lagrange multiplier field.

The partition function of QED with the source term

$$S_J[a_\mu, C, \bar{C}, \phi, \psi, \bar{\psi}] := \int d^D x (J^\mu a_\mu + J_c C + J_{\bar{c}} \bar{C} + J_\phi \phi + \bar{\eta} \psi + \eta \bar{\psi}) \tag{2.9}$$

is given by

$$Z_{QED}[J] := \int [da_\mu][dC][d\bar{C}][d\phi][d\psi][d\bar{\psi}] \exp \left\{ iS_{QED}^{tot} + iS_J \right\}. \tag{2.10}$$

To reformulate QED as a deformation of a topological field theory according to [4], we first regard the U(1) gauge field  $a_\mu$  and the fermion field  $\psi$  as the gauge transformation of the U(1) gauge fields  $v_\mu$  and  $\Psi$ ,

$$a_\mu(x) := v_\mu(x) + \omega_\mu(x), \quad \omega_\mu(x) := \frac{i}{g} U(x) \partial_\mu U^\dagger(x). \tag{2.11}$$

$$\psi(x) := U(x) \Psi(x), \tag{2.12}$$

Here <sup>1</sup>  $v_\mu$  and  $\Psi$  are identified with the perturbative (topological trivial) sector ( $Q = 0$ ) and  $\omega_\mu$  belongs to the topological nontrivial sector ( $Q \neq 0$ ).

Furthermore we introduce new ghost field  $\gamma$ , anti-ghost field  $\bar{\gamma}$  and the multiplier field  $\beta$  in the perturbative sector. They are subject to a new BRST transformation  $\tilde{\delta}_B$ ,

$$\begin{aligned}
\tilde{\delta}_B v_\mu &= \partial_\mu \gamma, \\
\tilde{\delta}_B \gamma &= 0, \\
\tilde{\delta}_B \bar{\gamma} &= i\beta, \\
\tilde{\delta}_B \beta &= 0, \\
\tilde{\delta}_B \Psi &= \gamma \Psi.
\end{aligned} \tag{2.13}$$

Then the partition function of QED is rewritten as

$$\begin{aligned}
Z_{QED}[J] &= \int [dU][dC][d\bar{C}][d\phi] \int [dv_\mu][d\gamma][d\bar{\gamma}][d\beta][d\Psi][d\bar{\Psi}] \\
&\times \exp \left\{ i \int d^D x \left[ -i\delta_B G_{gf}[\omega_\mu + v_\mu, C, \bar{C}, \phi] \right. \right. \\
&\quad \left. \left. + \mathcal{L}_{QED}[v, \Psi] - i\tilde{\delta}_B \tilde{G}_{gf}(v_\mu, \gamma, \bar{\gamma}, \beta) \right] \right. \\
&\quad \left. + iS_J[\omega_\mu + v_\mu, C, \bar{C}, \phi, U\Psi, \bar{\Psi}U^\dagger] \right\},
\end{aligned} \tag{2.14}$$

where  $\tilde{G}_{gf}$  is a gauge fixing functional for the perturbative (topological trivial) sector.

<sup>1</sup> This decomposition  $a_\mu = v_\mu + \omega_\mu$  of  $a_\mu$  corresponds to the superposition of two independent configuration,  $\varphi = \varphi_{SW} + \varphi_V$  (spin waves and vortex parts) in XY model.

## 2.2 Gauge fixing

The Lorentz gauge is given by

$$F[a] := \partial_\mu a^\mu = 0. \quad (2.15)$$

The most familiar choice of  $G_{gf}$

$$G_{gf} = \bar{C}(\partial_\mu a^\mu + \frac{\alpha}{2}\phi) \quad (2.16)$$

yields the familiar form of the gauge fixing term,

$$\mathcal{L}_{GF} := -i\delta_B G_{gf}[a_\mu, C, \bar{C}, \phi] = \phi\partial_\mu a^\mu + i\bar{C}\partial^\mu\partial_\mu C + \frac{\alpha}{2}\phi^2. \quad (2.17)$$

In this paper we propose to use a choice,

$$G_{gf}^{U(1)} = -\bar{\delta}_B \left( \frac{1}{2}a_\mu^2 + iC\bar{C} \right), \quad (2.18)$$

where  $\bar{\delta}_B$  is the anti-BRST transformation [4],

$$\begin{aligned} \bar{\delta}_B a_\mu &= \partial_\mu \bar{C}, \\ \bar{\delta}_B C &= i\bar{\phi}, \\ \bar{\delta}_B \bar{C} &= 0, \\ \bar{\delta}_B \bar{\phi} &= 0, \\ \bar{\delta}_B \psi &= \bar{C}\psi, \\ \phi + \bar{\phi} &= 0. \end{aligned} \quad (2.19)$$

Apart from a total derivative term, this choice yields

$$\mathcal{L}_{GF} = i\delta_B \bar{\delta}_B \left( \frac{1}{2}a_\mu^2 + iC\bar{C} \right) = -i\delta_B [\bar{C}(\partial_\mu a^\mu - \phi)]. \quad (2.20)$$

Therefore, the choice (2.18) corresponds in (2.17) to the choice

$$\alpha = -2, \quad (2.21)$$

which has appeared also in the non-Abelian case [4]. The above choice for  $G_{gf}^{U(1)}$  yields the decomposition,

$$\mathcal{L}_{GF} = -i\delta_B G_{gf}^{U(1)}[\omega_\mu + v_\mu, C, \bar{C}, \phi] \quad (2.22)$$

$$\begin{aligned} &= i\delta_B \bar{\delta}_B \left( \frac{1}{2}(\omega_\mu + v_\mu)^2 + iC\bar{C} \right) \\ &= \mathcal{L}_{TFT} + i v_\mu \delta_B \bar{\delta}_B \omega_\mu, \end{aligned} \quad (2.23)$$

where we have defined

$$\mathcal{L}_{TFT} := i\delta_B \bar{\delta}_B \left( \frac{1}{2}\omega_\mu^2 + iC\bar{C} \right). \quad (2.24)$$

Here we have used that the action of  $\delta_B$  is trivial in the perturbative sector,

$$\delta_B v_\mu = 0 = \bar{\delta}_B v_\mu, \quad (2.25)$$

while

$$\delta_B \omega_\mu = \partial_\mu C, \quad \bar{\delta}_B \omega_\mu = \partial_\mu \bar{C}. \quad (2.26)$$

## 2.3 Deformation of topological field theory

Finally, the partition function of QED is cast into the form,

$$Z_{QED}[J] := \int [dU][dC][d\bar{C}][d\phi] \exp \left\{ iS_{TFT}[\omega_\mu, C, \bar{C}, \phi] \right. \\ \left. + i \int d^D x [J^\mu \omega_\mu + J_c C + J_{\bar{c}} \bar{C} + J_\phi \phi] + iW[U; J^\mu, \bar{\eta}, \eta] \right\}, \quad (2.27)$$

where  $W[U; J^\mu, \bar{\eta}, \eta]$  is the generating functional of QED in the perturbative sector (pQED) given by

$$e^{iW[U; J^\mu, \bar{\eta}, \eta]} := \int [dv_\mu][d\gamma][d\bar{\gamma}][d\beta][d\Psi][d\bar{\Psi}] \exp \left\{ iS_{pQED}[v, \Psi, \gamma, \bar{\gamma}, \beta] \right. \\ \left. + i \int d^D x [v_\mu \mathcal{J}_\mu + \bar{\eta} U \Psi + \eta \bar{\Psi} U^\dagger] \right\}, \quad (2.28)$$

$$S_{pQED}[v, \Psi, \gamma, \bar{\gamma}, \beta] := \int d^D x \left[ \mathcal{L}_{QED}[v, \Psi] - i\tilde{\delta}_B \tilde{G}_{gf}(v_\mu, \gamma, \bar{\gamma}, \beta) \right], \quad (2.29)$$

$$\mathcal{J}_\mu := J_\mu + i\delta_B \bar{\delta}_B \omega_\mu. \quad (2.30)$$

The correlation functions of the original (fundamental) field  $a_\mu, \psi, \bar{\psi}$  is obtained by differentiating  $Z_{QED}[J]$  with respect to the corresponding source  $J_\mu, \bar{\eta}, \eta$ .

All the field configurations are classified according to the integer-valued topological charge  $Q$  which is specified later. The above reformulation of gauge theory is the decomposition of the original theory into topological trivial sector with  $Q = 0$  and topological non-trivial sector with  $Q \neq 0$ . This corresponds to the decomposition of XY model into spin wave part ( $Q = 0$ ) and the vortex part ( $Q \neq 0$ ) where  $Q$  is given by the winding number of the vortex solution. However, the XY model does not have any local gauge invariance.

The integration over the fields  $(U, C, \bar{C}, \phi)$  in TFT should be treated non-perturbatively by taking into account the topological non-trivial configurations. The deformation  $W[U; J^\mu, \bar{\eta}, \eta]$  of TFT should be calculated according to the ordinary perturbation theory in the coupling constant  $e$ . The perturbative expansion around the TFT means the integration over the new fields  $(v_\mu, \gamma, \bar{\gamma}, \beta)$  based on the perturbative expansion in powers of the coupling constant  $e$ .

## 2.4 Dimensional reduction to O(2) NLSM

Following the argument given in [4] based on the Parisi-Sourlas dimensional reduction, it turns out that the D-dimensional TFT (as the topological non-trivial sector of D-dimensional U(1) abelian gauge theory) with an action

$$S_{TFT}[\omega_\mu, C, \bar{C}, \phi] = \int d^D x i\delta_B \bar{\delta}_B \left( \frac{1}{2} \omega_\mu(x) \omega_\mu(x) + iC(x) \bar{C}(x) \right) \quad (2.31)$$

is equivalent to the (D-2)-dimensional O(2) NLSM with an action,

$$S_{O(2)NLSM}[U] := \int d^{D-2} z \pi \omega_\mu(z) \omega_\mu(z), \quad \omega_\mu(z) := \frac{i}{g} U(z) \partial_\mu U^\dagger(z), \quad (2.32)$$

$$= \int d^{D-2} z \frac{\pi}{g^2} \partial_\mu U(z) \partial_\mu U^\dagger(z). \quad (2.33)$$

Dimensional reduction is a simple consequence of a fact that the action (2.31) has hidden supersymmetry and can be rewritten in the  $\text{OSp}(D/2)$  symmetric form in the superspace formulation, see [4].

### 3 Quark confinement in Abelian gauge theory

Now we calculate the Wilson loop expectation in the  $U(1)$  gauge theory based on the reformulation given in the previous section. In what follows we move to the Euclidean formulation.

#### 3.1 Dimensional reduction of Wilson loop

We define the Wilson loop operator for the closed loop  $C$  by

$$W_C[a] = \exp\left(iq \oint_C a_\mu(x) dx^\mu\right), \quad (3.1)$$

where  $q$  is a test charge. In the abelian gauge theory, the Wilson loop factorizes,

$$W_C[a] = \exp\left(iq \oint_C \omega_\mu(x) dx^\mu\right) \exp\left(iq \oint_C v_\mu(x) dx^\mu\right) =: W_C[\omega] W_C[v]. \quad (3.2)$$

For  $Q = 0$  sector, we choose the gauge-fixing function,

$$\tilde{G}_{gf} = \bar{\gamma}(\partial_\mu v^\mu + \frac{\xi}{2}\beta), \quad (3.3)$$

with a gauge-fixing parameter  $\xi$ . For the  $U(1)$  gauge theory with an action (omitting matter fields),

$$S_{pU(1)}[v, \gamma, \bar{\gamma}, \beta] := \frac{1}{4g^2}(\partial_\mu v_\nu - \partial_\nu v_\mu)^2 - i\tilde{\delta}_B \tilde{G}_{gf}, \quad (3.4)$$

the perturbative part is given by

$$e^{iW[\omega, J, 0, 0]} = \int [dv_\mu][d\gamma][d\bar{\gamma}][d\beta] \exp\left\{-S_{pU(1)}[v, \gamma, \bar{\gamma}, \beta] + i \int d^D x v_\mu \mathcal{J}_\mu\right\}. \quad (3.5)$$

Integrating out the fields  $\gamma, \bar{\gamma}, \beta$  yields

$$e^{iW[\omega, J, 0, 0]} = \int [dv_\mu] \exp\left\{-S[v] - \int d^D x i v_\mu(x) [J_\mu(x) + i\delta_B \bar{\delta}_B \omega_\mu(x)]\right\}, \quad (3.6)$$

where

$$S[v] := \int d^D x \left[ \frac{1}{4g^2}(\partial_\mu v_\nu - \partial_\nu v_\mu)^2 + \frac{1}{2\xi}(\partial_\mu v^\mu)^2 \right]. \quad (3.7)$$

For the calculation of the Wilson loop expectation,  $J^\mu$  is taken to be the current along the closed loop  $C$  such that

$$\int d^D x v_\mu(x) J^\mu(x) = q \oint_C v_\mu(x) dx^\mu. \quad (3.8)$$



It is easy to see that

$$e^{iW[\omega, J, 0, 0]} = \langle W_C[v] e^{(v_\mu, \delta_B \bar{\delta}_B \omega_\mu)} \rangle_{pU(1)} \int [dv_\mu] e^{-S[v] + (v_\mu, \delta_B \bar{\delta}_B \omega_\mu)}, \quad (3.9)$$

where

$$(v_\mu, \delta_B \bar{\delta}_B \omega_\mu) := \int d^D x v_\mu(x) \delta_B \bar{\delta}_B \omega_\mu(x). \quad (3.10)$$

The Wilson loop expectation is rewritten as

$$\langle W_C[a] \rangle_{U(1)} = \frac{\langle W_C[\omega] e^{iW[\omega, J, 0, 0]} \rangle_{TFT}}{\langle e^{iW[\omega, 0, 0, 0]} \rangle_{TFT}}, \quad (3.11)$$

where

$$e^{iW[\omega, 0, 0, 0]} = \langle e^{(v_\mu, \delta_B \bar{\delta}_B \omega_\mu)} \rangle_{pU(1)} \int [dv_\mu] e^{-S[v] + (v_\mu, \delta_B \bar{\delta}_B \omega_\mu)}. \quad (3.12)$$

Expanding the exponential

$$e^{(v_\mu, \delta_B \bar{\delta}_B \omega_\mu)} = e^{\int d^D x \delta_B \bar{\delta}_B (v_\mu \omega_\mu)(x)} = e^{\int d^D x \{Q_B, \bar{\delta}_B (v_\mu \omega_\mu)(x)\}} \quad (3.13)$$

into power series and using a fact that the vacuum of TFT obeys

$$Q_B |0\rangle_{TFT} = 0, \quad (3.14)$$

we find that this term does not contribute to the expectation value (3.11). Therefore, the Wilson loop expectation is completely separated into the topological part ( $Q \neq 0$ ) and the perturbative part ( $Q = 0$ ),

$$\langle W_C[a] \rangle_{U(1)} = \langle W_C[\omega] \langle W_C[v] \rangle_{pU(1)} \rangle_{TFT} = \langle W_C[\omega] \rangle_{TFT} \langle W_C[v] \rangle_{pU(1)}. \quad (3.15)$$

This corresponds to the result eq.(10) of Polyakov [2]. This property does not hold in the non-Abelian case, which makes the systematic calculation rather difficult.

In order to use the dimensional reduction for calculating the topological part, we choose the loop  $C$  so that  $C$  is contained in the  $D - 2$  dimensional space. For  $D = 4$ , the loop  $C$  must be planar. Then the dimensional reduction of the topological part leads to the equivalence of the Wilson loop expectation value between the  $D$ -dimensional  $U(1)$  TFT and  $(D-2)$ -dimensional  $O(2)$  NLSM,

$$\langle W_C[\omega(z)] \rangle_{TFT_D} = \langle W_C[\omega(z)] \rangle_{O(2)NLSM_{D-2}}, \quad z \in R^{D-2}. \quad (3.16)$$

if the Wilson loop has its support on the  $(D-2)$ -dimensional space on which the NLSM is defined. Hence, the calculation of the Wilson loop in the four-dimensional  $U(1)$  gauge theory is reduced to those in the two-dimensional  $O(2)$  NLSM and the four-dimensional perturbative  $U(1)$  gauge theory,

$$\langle W_C[a] \rangle_{U(1)_D} = \langle W_C[\omega] \rangle_{O(2)NLSM_{D-2}} \langle W_C[v] \rangle_{pU(1)_D}. \quad (3.17)$$

For large rectangular loop with sides  $R, T$ , the static potential is obtained by

$$V(R) = \lim_{T \rightarrow \infty} \frac{-1}{T} \ln \langle W_C[a] \rangle. \quad (3.18)$$

In four dimensions ( $D = 4$ ), it is well known [23] that for large Wilson loop  $\langle W_C[v] \rangle_{pU(1)_4}$  gives the Coulomb potential,

$$V(R) = -\frac{g^2}{4\pi} \frac{1}{R}. \quad (3.19)$$

In the following, we show that  $\langle W_C[\omega] \rangle_{O(2)NLSM_2}$  exhibits area law for strong coupling  $g > g_c$  with a finite and non-zero value of a critical point  $g_c$ . This confinement-deconfinement transition corresponds exactly to the BKT transition.

### 3.2 O(2) NLSM and Wilson loop

Defining the angle variable  $\varphi(z)$  for  $U(z) \in U(1)$ ,

$$U(z) = e^{i\varphi(z)}, \quad (3.20)$$

we obtain

$$\omega_\mu(z) = \frac{i}{g} U(z) \partial_\mu U^\dagger(z) = \frac{1}{g} \partial_\mu \varphi(z). \quad (3.21)$$

Then the action of O(2) NLSM reads

$$S_{O(2)}[U] = \int d^{D-2}z \pi \omega_\mu(z) \omega_\mu(z) = \frac{\beta}{2} \int d^{D-2}z \partial_\mu \varphi(z) \partial_\mu \varphi(z), \quad \beta := \frac{2\pi}{g^2}. \quad (3.22)$$

The partition function is defined by

$$Z_{O(2)}[U] = \int [dU] \exp(-S_{O(2)}[U]), \quad [dU] = \prod_x \frac{d\varphi(x)}{2\pi}, \quad (3.23)$$

using the Haar measure  $dU$  on  $U(1)$ . For the notation of the two-dimensional vector,

$$\mathbf{S}(x) = (\cos \varphi(x), \sin \varphi(x)), \quad (3.24)$$

the action reads

$$S_{O(2)}[U] = \frac{\beta}{2} \int d^{D-2}z \partial_\mu \mathbf{S}(x) \cdot \partial_\mu \mathbf{S}(x). \quad (3.25)$$

Hence the O(2) NLSM is regarded as a continuum version of classical planar spin model. In the following, we identify  $\beta$  with the inverse temperature  $1/T$ . Hence the high (resp. low) temperature of the spin model corresponds to strong (resp. weak) coupling of the gauge theory.

It should be remarked that O(2) NLSM with the action (3.22) is not a free scalar field theory, since this theory is periodic in the angle variable  $\varphi$  (modulo  $2\pi$ ). Of

course, if we neglect this periodicity and treat the variable  $\varphi$  as a non-compact variable  $\varphi(x) \in (-\infty, +\infty)$ , we have a trivial theory, i.e., free massless scalar field theory. In this case, the Contour integral is zero,

$$\oint_C \omega_\mu(z) dz^\mu = \frac{1}{g} \oint_C \partial_\mu \varphi(z) dz^\mu = 0. \quad (3.26)$$

Hence, the Wilson loop  $W_c[\omega]$  is trivial,  $W_c[\omega] \equiv 1$  and hence the total static quark potential comes from the the Wilson loop expectation  $W_c[v]$  of perturbative U(1) gauge theory and is equal to the Coulomb potential. Thus we obtain a trivial result that the four-dimensional non-compact Abelian gauge theory fails to confine quarks (charges).

However, the periodicity (or compactness) leads to topological non-trivial solutions which are seeds for confinement, as shown in what follows.

In the following, we restrict our consideration to  $d := D - 2 = 2$  case. The extremum of the classical action (3.22) is obtained as a solution of the classical field equation,

$$\nabla^2 \varphi = 0 \pmod{2\pi}. \quad (3.27)$$

The harmonic function  $\varphi$  is constant or has singularities. We require  $\varphi$  be constant at infinity and assume only isolated singularities around which  $\varphi$  varies by  $\pm 2\pi$  as one turns anticlockwise. Then the solution <sup>2</sup> is written as

$$\varphi(z) = \sum_i Q_i \arctan \frac{(z - z_i)_2}{(z - z_i)_1} = \sum_i Q_i \text{Im} \ln(z - z_i), \quad z := x_1 + ix_2. \quad (3.28)$$

This denotes a sum of vortex excitations located at points  $x_i \in R^2$  and of vorticity  $Q_i$  (integers). The solution has an alternative form,

$$\varphi(z) = \ln \prod_i \frac{(z - z_i^+)/|z - z_i^+|}{(z - z_i^-)/|z - z_i^-|} = \sum_i \left[ \ln \frac{(z - z_i^+)}{|z - z_i^+|} - \ln \frac{(z - z_i^-)}{|z - z_i^-|} \right]. \quad (3.29)$$

This means that vortices of intensity  $\pm 1$  are centered at the points  $z_i^\pm$  and that intensities of higher magnitude are obtained when several  $z_i^+$  (or  $z_i^-$ ) coincide. Note that the angle  $\varphi$  is a multi-valued function, but  $e^{i\varphi}$  is well-defined everywhere, except at the singular points.

The contribution of the solution (3.28) to  $\omega_\mu$  is

$$\omega_\mu(z) = \frac{1}{g} \partial_\mu \varphi(z) = \frac{1}{g} \sum_i Q_i \epsilon_{\mu\nu} \frac{(z - z_i)_\nu}{(z - z_i)^2} = \frac{1}{g} \sum_i Q_i \epsilon_{\mu\nu} \partial_\nu \ln |z - z_i|. \quad (3.30)$$

Therefore, the integral of one-form  $\omega := \omega_\mu dx^\mu$  along the closed loop  $C$  is

$$\oint_C \omega_\mu dx^\mu = \oint_C \omega = \frac{1}{g} \sum_i \int_0^{2\pi} d\Theta_i = \sum_i \frac{2\pi}{g} Q_i, \quad (3.31)$$

---

<sup>2</sup> The two-dimensional Laplace equation is equivalent to the Cauchy-Riemann equation. Hence the solution is given by the holomorphic function. If one avoids branch cuts, it is a meromorphic function.

where the sum runs over all the vortices inside the closed loop  $C$  and  $\Theta_i$  is an angle around  $z = z_i$ ,

$$\Theta_i(z) := \arctan \frac{(z - z_i)_2}{(z - z_i)_1}. \quad (3.32)$$

Note that  $\omega$  is a closed form,  $d\omega = 0$ , but it is not an exact form, that is, there does not exist a function (zero-form) such that  $\omega = df$  with  $f$  being defined everywhere in  $R^2 - \{0\}$ . Domain of  $f(z) = \Theta(z)$  is restricted to  $R^2 - R_+$ , in other words, for one unit vortex at the origin,

$$\omega_\mu = \epsilon_{\mu\nu} \frac{x_\nu}{x^2} - 2\pi\theta(x_1)\delta(x_2)\delta_{\mu 2}. \quad (3.33)$$

This is analogous to the case of the magnetic monopole in three dimensions where the magnetic field is given by

$$H_\mu = \frac{1}{2} \frac{x_\mu}{|x|^3} - 2\pi\delta_{3\mu}\delta(x_1)\delta(x_2)\theta(x_3). \quad (3.34)$$

The singular line of (3.33) in two dimensions does not contribute to the action, so does the Dirac string (on the positive  $Z$  axis) in three dimensions.

In order to calculate the classical action for the singular configuration (3.28), we consider a disk of radius  $R_0$ ,  $D_i := \{|z - z_i| < R_0; z \in R^2\}$  (centered on each singular point  $z_i$ ) which is small with respect to the distances between vortices. Let  $\mathcal{R}$  be the remaining domain of integration outside the vortices. The classical action consists of two parts, the self-energy (action) part of vortices,

$$S^{(1)} = \frac{\beta}{2} \sum_i \int_{|z - z_i| < R_0} d^2z (\nabla\varphi(z))^2, \quad (3.35)$$

and the remaining part,

$$\begin{aligned} S^{(2)} &:= \frac{\beta}{2} \int_{\mathcal{R}} d^2z (\nabla\varphi(z))^2 \\ &= -2\pi\beta \sum_{i \neq j} Q_i Q_j \ln |z_i - z_j| + \sum_i Q_i^2 \pi\beta \ln 1/R_0. \end{aligned} \quad (3.36)$$

Summing over all vortex sectors leads to the partition function of the form,

$$Z_C = \sum_{n=0}^{\infty} \frac{\zeta^n}{(n!)^2} \int \prod_{j=1}^n d^2z_j \exp \left[ (2\pi)^2 \beta \sum_{i,j} Q_i Q_j \Delta(z_i, z_j) \right], \quad \zeta := e^{-S^{(1)}}, \quad (3.37)$$

where  $\zeta$  comes from the self-energy (action) part of vortices and  $\Delta$  expresses the two-dimensional inverse Laplacian given by

$$\Delta(x, 0) = \frac{1}{2\pi} \ln \frac{R}{|x|}. \quad (3.38)$$

Therefore the partition function just agrees with the two-dimensional neutral Coulomb gas (i.e., a gas of classical charged particles with a Coulomb interaction and globally neutral,  $\sum_i Q_i = 0$ ).

The transition temperature is estimated as follows. The contribution to the free energy from one vortex pair at distance  $r_{12}$  in a box of linear dimension  $L$  is

$$\begin{aligned} F &\sim \ln \int_{|z_1 - z_2| > R_0} d^2 z_1 d^2 z_2 \exp(-2\pi\beta \ln |z_1 - z_2|/R_0) \\ &\sim \ln[L^4 \exp(-2\pi\beta \ln L/R_0)] \\ &\sim (4 - 2\pi\beta) \ln L. \end{aligned} \tag{3.39}$$

The vortices always arise in pairs of opposite Coulomb charges to yield finite energy configurations and each pair forms an elementary dipole (Coulomb dipole gas). If  $2\pi\beta > 4$  (the low temperature or weak coupling phase), the contribution from the vortex pair is negligible in the limit  $L \rightarrow \infty$ . In this phase, charges are bound and one has a dielectric medium. As  $\beta \rightarrow \infty$ , few vortices are present and their correlation decreases rapidly with the relative distance. This describes a dielectric medium of neutral bound states. In this regime, the correlation  $\langle U(z)U(z') \rangle$  decays polynomially,

$$\langle U(z)U(z') \rangle = |z - z'|^{-\frac{1}{4\pi\beta}}. \tag{3.40}$$

On the other hand, if  $2\pi\beta < 4$  (the high temperature or strong coupling phase), an instability occurs and the creation of well-separated vortices is favored and disorder increases. In high temperature phase, one has a plasma of free charges. The vortex expectation decays exponentially yielding exponential decay of  $\langle U(z)U(z') \rangle$ ,

$$\langle U(z)U(z') \rangle = |z - z'|^{-\frac{1}{4\pi\beta}} e^{-m(\beta)|z - z'|}. \tag{3.41}$$

Hence a naive estimate of the critical temperature is obtained

$$\beta_c = 2/\pi, \quad g_c^2 = \pi^2. \tag{3.42}$$

This is the phase transition without the appearance of a spontaneous magnetization. The phase transition can be interpreted as a dipole condensation. The critical point separates the dissociated dipole phase from the condensed phase.

In order to calculate the Wilson loop, we use the equivalence of the Coulomb gas to the sine-Gordon model (see Appendix A for proof). The sine-Gordon model is defined by the action and the partition function,

$$S_{sG}(\phi) := \alpha \int d^d x \left[ \frac{1}{2} (\partial_\mu \phi(x))^2 - h \cos \phi(x) \right], \tag{3.43}$$

$$Z_{sG}(h) := \int [d\phi] \exp[-S_{sG}(\phi)]. \tag{3.44}$$

This is equivalent to the partition function of a globally neutral gas of particles of charges  $Q_i = \pm 1$  through a Coulomb potential in  $d$  dimensions,

$$\begin{aligned} Z_{C,\pm 1}(h) &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2} \prod_{i=1}^n \int d^d x_i d^d y_j \exp \left\{ -\frac{1}{T} \left[ \sum_{i < j} [V(|x_i - x_j|) + V(|y_i - y_j|)] \right. \right. \\ &\quad \left. \left. - \sum_{i,j} V(|x_i - y_j|) \right] \right\} \end{aligned} \tag{3.45}$$

at temperature  $T$  with the fugacity  $z$ ,

$$\alpha = T = 1/(4\pi^2\beta) = \frac{g^2}{8\pi^3}, \quad \alpha h = 2z = 2\zeta. \quad (3.46)$$

Note that  $\alpha\beta = 1/(4\pi^2)$ . It is known that the transition point of sine-Gordon model is

$$\alpha_c = 1/(8\pi), \quad (3.47)$$

which is in agreement with (3.42). The relation of  $\varphi$  and  $\phi$  is given by

$$\phi(x, t) = \int_x^\infty dy \dot{\varphi}(y, t), \quad (3.48)$$

or

$$\partial_\mu \varphi = \epsilon_{\mu\nu} \partial^\nu \phi. \quad (3.49)$$

Therefore the fields  $\varphi$  and  $\phi$  are dual variables. Therefore, O(2) NLSM is equivalent to the Coulomb gas and moreover it is equivalent to sine-Gordon model when charge of the Coulomb gas (or vorticity of O(2) NLSM) is restricted to  $Q_i = \pm 1$ . Taking into account  $Q_i > 1$  will lead to the  $\cos(Q\phi)$  term. The above consideration can be transferred into the lattice formulation, see [23].

### 3.3 Wilson loop and area decay

The Wilson loop expectation is calculated using the equivalent sine-Gordon model. The generating functional for the charge density  $\rho$

$$\rho(x) := \sum_i Q_i \delta^{(2)}(x - x_i), \quad (3.50)$$

is obtained as

$$\frac{Z_{sG}[\eta]}{Z_{sG}(0)} = \langle e^{i \int d^2x \rho(x) \eta(x)} \rangle_{sG} = \langle e^{i \sum_i Q_i \eta(x_i)} \rangle_{sG}, \quad (3.51)$$

$$Z_{sG}[\eta] = \int [d\phi] \exp \left\{ -\alpha \int d^2x \left[ \frac{1}{2} (\partial_\mu \phi(x))^2 - h \cos[\phi(x) + \eta(x)] \right] \right\}. \quad (3.52)$$

In two dimensions, we can introduce the dual vector field,

$$H_\mu = \epsilon_{\mu\nu} \omega_\nu. \quad (3.53)$$

Then the dual field is connected with the charge density  $\rho$  as follows,

$$\oint_C \omega_\mu(z) dz^\mu = \int_S \epsilon_{\mu\nu} \partial_\mu \omega_\nu(z) d^2z = \int \partial_\mu H_\mu(z) d^2z = \frac{2\pi}{g} \int \rho(z) d^2z. \quad (3.54)$$

Note that the rotation of  $\omega_\mu$  or the divergence of  $H_\mu$

$$\epsilon_{\mu\nu} \partial_\mu \omega_\nu = \partial_\mu H_\mu = \frac{2\pi}{g} \rho \quad (3.55)$$

measures the density of topological charge. If we identify the right-hand-side with the magnetic charge, this implies Dirac quantization condition,

$$g_m = \frac{2\pi}{g} Q \quad (Q : \text{integer}). \quad (3.56)$$

The Wilson loop is calculated from the  $\eta$  given by

$$\eta(x) = \frac{q}{g} \oint_C dz^\mu \epsilon_{\mu\nu} \frac{(z-x)_\nu}{(z-x)^2}, \quad (3.57)$$

since

$$q \oint_C \omega_\mu dz^\mu = \int d^d x \rho(x) \eta(x) = \sum_i Q_i \eta(x_i). \quad (3.58)$$

If the argument  $x$  of  $\eta(x)$  is outside the loop,  $\eta(x) = 0$ , while  $\eta(x) = 2\pi q/g$  if  $x$  is inside the loop.

In high temperature phase, photon is massive, whereas photon is massless in the low temperature phase. This is because in the high temperature phase, random distribution of free vortices with long range interaction spoil the correlation. Therefore, in the high temperature phase, the Wilson loop expectation is estimated by the steepest descent as

$$\langle W_C[\omega] \rangle_{sG} \cong \exp \left\{ -\alpha \int d^2 x \left[ \frac{1}{2} (\partial_\mu [\phi_{cl}(x) - \eta(x)])^2 - h \cos \phi_{cl}(x) \right] \right\}, \quad (3.59)$$

where  $\phi_{cl}$  is determined by the Debye equation,

$$\nabla^2 [\phi_{cl}(x) - \eta(x)] = h \sin \phi_{cl}(x). \quad (3.60)$$

Corrections to this field due to fluctuation are exponentially small in high temperature phase. The loop  $C$  is placed in two-dimensional plane. We can perform the calculation in the same way as done by Polyakov [3].

Instead of repeating similar calculation to [3], we use the Villain form [26],

$$e^{J \cos \phi} \rightarrow e^J \sum_{m \in Z} e^{-\frac{J}{2} (\phi - 2\pi m)^2}, \quad (3.61)$$

to estimate the Wilson loop expectation. Then the partition function is replaced with (apart from field-independent constants)

$$\begin{aligned} Z_{sG}[\eta] &= \int [d\phi] e^{-\alpha \int d^d x [\frac{1}{2} (\partial_\mu \phi(x))^2]} \prod_{x \in R^d} e^{\alpha h \cos[\phi(x) + \eta(x)]} \\ &= \int [d\phi] e^{-\alpha \int d^d x [\frac{1}{2} (\partial_\mu \phi(x))^2]} \prod_{x \in R^d} \sum_{m(x) \in Z} e^{-\frac{\alpha h}{2} [\phi(x) + \eta(x) - 2\pi m(x)]^2} \\ &= \sum_{\{m(x) \in Z; x \in R^d\}} \int [d\phi] e^{-\alpha \int d^d x \frac{1}{2} [-\phi(x) \partial^2 \phi(x)]} e^{-\frac{\alpha h}{2} \int d^d x [\phi(x) + \eta(x) - 2\pi m(x)]^2} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\{m(x) \in Z; x \in R^d\}} e^{-\frac{\alpha h}{2} \int d^d x [\eta(x) - 2\pi m(x)]^2} \\
&\quad \times \int [d\phi] e^{-\int d^d x \{ \frac{\alpha}{2} \phi(x) (-\partial^2 + h) \phi(x) + \alpha h \phi(x) [\eta(x) - 2\pi m(x)] \}} \\
&= \sum_{\{m(x) \in Z; x \in R^d\}} \exp \left[ -\frac{\alpha h}{2} \int d^d x \{ [\eta(x) - 2\pi m(x)]^2 \right. \\
&\quad \left. - h [\eta(x) - 2\pi m(x)] (-\partial^2 + h)^{-1} [\eta(x) - 2\pi m(x)] \right]. \tag{3.62}
\end{aligned}$$

When  $\eta = 0$ , the denominator is obtained,

$$Z_{sG}[0] = \sum_{\{m(x) \in Z; x \in R^d\}} \exp \left[ -(2\pi)^2 \frac{\alpha h}{2} \int d^d x \{ m(x)^2 - h m(x) (-\partial^2 + h)^{-1} m(x) \} \right] \tag{3.63}$$

Note that the field  $\eta(x)$  has its support on  $S(\partial S = C)$  and has the value  $2\pi(q/g)$ . If  $q$  is an integral multiple of  $g$  (the elementary charge), we have  $\eta \in 2\pi Z$ . This is absorbed by the shift of  $m$ . Therefore, in this case, charge confinement does not occur. This is interpreted as the charge screening.

A naive estimate of the ratio  $Z_{sG}[\eta]/Z_{sG}[0]$  is given when  $\eta \notin 2\pi Z$  in Appendix B. Finally we obtain the area decay of the Wilson loop expectation,

$$\langle W_C[\omega] \rangle_{sG} \cong e^{-\sigma A(C)}, \tag{3.64}$$

$$\sigma = \left( 2\pi \frac{q}{g} \right)^2 \frac{\alpha h}{2} = \left( 2\pi \frac{q}{g} \right)^2 \zeta \sim e^{-S^{(1)}}. \tag{3.65}$$

This implies the linear static potential

$$V(R) = \sigma R \tag{3.66}$$

between two fixed electric charges and electric string with uniform energy density  $\sigma$  which is called the string tension. Therefore condensation of topological non-trivial configuration leads to quark confinement. The total static potential is given by

$$V(R) = -\frac{g^2}{4\pi} \frac{1}{R} + \sigma R. \tag{3.67}$$

Even the continuum Abelian gauge theory has a confinement phase, exhibiting rich phase structure as in lattice U(1) gauge theory. This in turn proves the quark confinement for non-Abelian gauge group for all the coupling, as given in the previous paper [4].

## 4 Remarks

### 4.1 Renormalization group result

The O(N) NLSM has the renormalization group beta function,

$$\beta(g) := \mu \frac{dg(\mu)}{d\mu} = -\frac{N-2}{8\pi^2} g^3 + O(g^5). \tag{4.68}$$



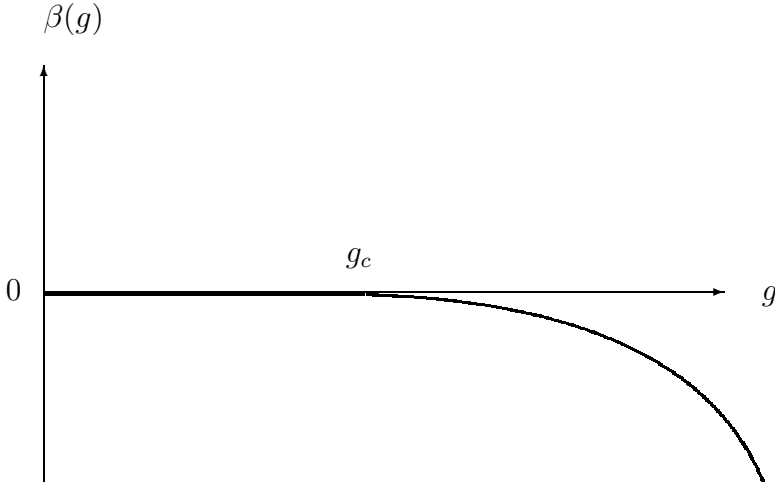


Figure 1: Renormalization group beta function of U(1) gauge theory

For O(2) NLSM,  $\beta(g) \equiv 0$  ( $g \ll 1$ ). This is consistent with the fact that at low temperature, the inverse correlation length or mass  $m = \xi^{-1}$  of O(2) NLSM or XY model vanishes for all  $T < T_c$ . The theory is scale invariant. For high temperature, the renormalization group study of XY model shows that the mass behaves as

$$m(T) \sim \exp\left(-\frac{c}{\sqrt{T - T_c}}\right), \quad g \downarrow g_c. \quad (4.69)$$

These results are translated into the four-dimensional Abelian gauge theory. In the weak coupling phase ( $g < g_c$ ), the beta function of the renormalization group [22] is identically zero

$$\beta(g) \equiv 0 \quad (0 < g < g_c). \quad (4.70)$$

Therefore,  $0 < g < g_c$  is the line of fixed points, if we neglect the perturbative deformations. However, in strong coupling phase ( $g > g_c$ ), we have the beta function

$$\beta(g) \sim -(g^2 - g_c^2)^{3/2} < 0. \quad (g > g_c). \quad (4.71)$$

If the mass scale is generated by dimensional transmutation, these results are quite analogous to the dynamical fermion mass and beta function in quenched massless QED [27, 28]. The relationship with the fermionic model is more suggestive by the following correspondence.

The two-dimensional sine-Gordon model is equivalent to the massive Thirring model [29] with an action,

$$S[\psi, \bar{\psi}] = \int d^2x [\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi - \frac{G}{2}(\bar{\psi}\gamma_\mu\psi)^2]. \quad (4.72)$$

The correspondence between two theories is given by

$$1 + \frac{G}{\pi} = 4\pi\alpha, \quad (4.73)$$

$$\bar{\psi}\gamma_\mu\psi = -\frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\nu\phi, \quad (4.74)$$

$$m\bar{\psi}\psi \rightarrow -\alpha h \cos\phi. \quad (4.75)$$

The transition point  $\alpha_c = 1/(8\pi)$  corresponds to  $G_c = -\pi/2$ . More details will be given elsewhere.

## 4.2 Lower and higher dimensional cases

Using the equivalence between the TFT part of  $U(1)_D$  gauge theory and  $O(2)_{D-2}$  NLSM, we can study other dimensional cases.

For  $D = 3$ , the equivalent  $O(2)$  NLSM is one dimensional,  $O(2)_1$ . This is not a field theory model, but a quantum mechanical model of the plane rotor. There is no phase transition in this model. This implies that the three-dimensional  $U(1)$  gauge theory has only the confinement phase. This will be understood as the tunneling effect among classical vacua in the sense that the double-well anharmonic oscillator is related to the one-dimensional Ising model. It will be interesting to see agreement (or disagreement) on confinement mechanism between our approach and the Polyakov approach [2, 3].

For  $D = 5$ , the  $O(2)$  NLSM is three dimensional,  $O(2)_3$ . The three-dimensional  $O(2)$  NLSM has two phases on the lattice [30]. The phase transition is first order. This result is consistent with the mean field study of five-dimensional lattice  $U(1)$  gauge theory [24]. This theory has a finite non-zero critical coupling  $g_c$ . In the strong coupling phase, quark confinement is expected to occur. However, the phase transition is first order. Therefore, on the lattice, it is impossible to take the continuum limit at this point. In view of this, the construction of continuum  $U(1)$  gauge theory from the lattice regularized theory is problematic in five dimensions and higher dimensions, unless the action is modified.

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# A Equivalence between Coulomb gas and sine-Gordon model

The partition function can be rewritten as

$$\begin{aligned}
Z_{sG}(h) &= \int [d\phi] e^{-\alpha \int d^d x \frac{1}{2} (\partial_\mu \phi(x))^2} \sum_{n=0}^{\infty} \frac{(\alpha h)^n}{n!} \left[ \int d^d x \cos \phi(x) \right]^n \\
&= \int [d\phi] e^{-\alpha \int d^d x \frac{1}{2} (\partial_\mu \phi(x))^2} \sum_{n=0}^{\infty} \frac{(\alpha h)^{2n}}{(2n)!} \frac{1}{2^{2n}} \left[ \int d^d x (e^{i\phi(x)} + e^{-i\phi(x)}) \right]^{2n} \\
&= \int [d\phi] e^{-\alpha \int d^d x \frac{1}{2} (\partial_\mu \phi(x))^2} \sum_{n=0}^{\infty} \frac{(\alpha h)^{2n}}{(2n)!} \frac{1}{2^{2n}} \binom{2n}{n} \prod_{i=1}^n \int d^d x_i d^d y_i e^{i\phi(x_i) - i\phi(y_i)} \\
&= \int [d\phi] e^{-\alpha \int d^d x \frac{1}{2} (\partial_\mu \phi(x))^2} \sum_{n=0}^{\infty} \frac{(\alpha h/2)^{2n}}{(n!)^2} \prod_{i=1}^n \int d^d x_i d^d y_i e^{i\phi(x_i) - i\phi(y_i)}. \quad (\text{A.1})
\end{aligned}$$

Note that

$$\begin{aligned}
[[e^{\int d^d x J(x)\phi(x)}]] &:= \int [d\phi] \exp \left[ -\frac{\alpha}{2} \int d^d x (\partial_\mu \phi(x))^2 + \int d^d x J(x)\phi(x) \right] \\
&= \exp \left[ \frac{1}{2\alpha} \int d^d x \int d^d y J(x)\Delta(x, y)J(y) \right], \quad (\text{A.2})
\end{aligned}$$

where  $\Delta(x, y)$  is the massless scalar field propagator,

$$\Delta(x, y) := \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y)}}{p^2}. \quad (\text{A.3})$$

In particular, for  $J(x) = i \sum_j q_j \delta(x - x_j)$ ,

$$[[\prod_{j=1}^n e^{iq_j \phi(x_j)}]] = \begin{cases} \exp \left[ -\frac{1}{2\alpha} \sum_{j,k} q_j q_k \Delta(x_j, x_k) \right] & \text{for } \sum_j q_j = 0 \\ 0 & \text{for } \sum_j q_j \neq 0 \end{cases}, \quad (\text{A.4})$$

where the latter case is a result of invariance under constant translation of the field  $\varphi(z) \rightarrow \varphi(x) + c$ . Using this result for  $q_j = \pm 1$ ,

$$\begin{aligned}
Z_{sG}(h) &= \sum_{n=0}^{\infty} \frac{(\alpha h/2)^{2n}}{(n!)^2} \prod_{i=1}^n \int d^d x_i d^d y_i \int [d\phi] e^{-\alpha \int d^d x \frac{1}{2} (\partial_\mu \phi(x))^2} \prod_{i=1}^n e^{i\phi(x_i) - i\phi(y_i)} \\
&= \sum_{n=0}^{\infty} \frac{(\alpha h/2)^{2n}}{(n!)^2} \prod_{i=1}^n \int d^d x_i d^d y_i \exp \left\{ -\frac{1}{2\alpha} \left[ \sum_{i<j} [\Delta(x_i - x_j) + \Delta(y_i - y_j)] \right. \right. \\
&\quad \left. \left. - \sum_{i,j} \Delta(x_i - y_j) \right] \right\}. \quad (\text{A.5})
\end{aligned}$$

## B Estimate of sine-Gordon partition function

Note that the quantity

$$\varrho(x)^2 - h\varrho(x)(-\partial^2 + h)^{-1}\varrho(x) = \varrho(x)\frac{-\partial^2}{-\partial^2 + h}\varrho(x) \quad (\text{B.1})$$

is positive, since  $(-\partial^2)/(-\partial^2 + h)$  is a positive operator. Therefore,

$$e^{-\frac{\alpha h}{2}(2\pi)^2 \int_S d^d x \{\varrho(x)^2 - h\varrho(x)(-\partial^2 + h)^{-1}\varrho(x)\}}, \quad \varrho(x) = \left| \frac{\eta(x)}{2\pi} - m(x) \right| \quad (\text{B.2})$$

is monotonically (rapidly) decreasing in  $\{|\varrho(x)|; x \in S\}$ . Therefore, in the partition function,

$$Z_{sG}[\eta] = \sum_{\{m(x) \in \mathbb{Z}; x \in R^d\}} \exp \left[ -\frac{\alpha h}{2}(2\pi)^2 \int d^d x \{\varrho(x)^2 - h\varrho(x)(-\partial^2 + h)^{-1}\varrho(x)\} \right], \quad (\text{B.3})$$

the most dominant contribution comes from a set of configurations  $\{|\varrho(x)|; x \in S\}$  which gives the smallest value for  $\int d^d x \{\varrho(x)^2 - h\varrho(x)(-\partial^2 + h)^{-1}\varrho(x)\}$ .

If the argument  $x$  of  $\eta(x)$  is outside the loop  $C$  ( $C = \partial S$ ),  $\eta(x) = 0$ , while  $\eta(x) = 2\pi q/g$  if  $x$  is inside the loop. For  $Z_{sG}[0]$ ,  $\{m(x) \equiv 0\}$  gives the most dominant contribution. Hence, we see

$$Z_{sG}[0] = 1 + \dots \quad (\text{B.4})$$

For  $Z_{sG}[\eta]$ , the most dominant contribution comes from a set of integers  $\{m(x)\}$  whose value is the nearest to  $q/g$  where  $\varrho(x) = 2\pi|q/g - m(x)|$ . Since the integral part of  $q/g$  is absorbed in the shift of  $m(x)$ , it is sufficient to consider the case  $0 < q/g < 1$  without loss of generality. For a half-integer  $q/g$ , i.e.,  $q/g = \pm 1/2$ , we see that  $\{m(x) \equiv 0, \pm 1\}$  give the smallest value of  $2\pi|q/g - m(x)|$  for  $x \in S$ . For  $0 < q/g < 1/2$  (resp.  $1/2 < q/g < 1$ ), the most dominant contribution is given by  $\{m(x) \equiv 0\}$  (resp.  $\{m(x) \equiv 1\}$ ). Thus, the rough estimate, for example, in the case of  $0 < q/g < 1/2$  leads to

$$Z_{sG}[\eta] = e^{-\frac{\alpha h}{2} \int_S d^d x \{\eta(x)^2 - h\eta(x)(-\partial^2 + h)^{-1}\eta(x)\}} + \dots \quad (\text{B.5})$$

A naive estimate of the ratio  $Z_{sG}[\eta]/Z_{sG}[0]$  is given for  $\eta \notin 2\pi\mathbb{Z}$  by

$$\langle W_C[\omega] \rangle_{sG} = \frac{Z_{sG}[\eta]}{Z_{sG}[0]} = e^{-\frac{\alpha h}{2} \int_S d^d x \eta(x)[1 - h(-\partial^2 + h)^{-1}]\eta(x)} + \dots \quad (\text{B.6})$$

Here  $(-\partial^2 + h)^{-1}(x, y)$  is the massive scalar propagator with mass  $m \sim \sqrt{h}$  and hence has an exponential damping factor  $e^{-m|x-y|}$ . Therefore, the integral  $\int_S d^d x \eta(x)(-\partial^2 + h)^{-1}\eta(x)$  converges to a finite value even for large  $S$ . The term  $\int_S d^d x \eta(x)^2$  is proportional to the area of  $S$ . This term leads to the area decay of the Wilson loop expectation.

## References

- [1] A.M. Polyakov, Gauge fields and strings (Harwood Academic Publishers, London, 1987).
- [2] A.M. Polyakov, Compact gauge fields and the infrared catastrophe, Phys. Lett. B 59, 82-84 (1975).
- [3] A.M. Polyakov, Quark confinement and topology of gauge theories, Nucl. Phys. B 120, 429-458 (1977).
- [4] K.-I. Kondo, Yang-Mills Theory as a Deformation of Topological Field Theory, Dimensional Reduction and Quark Confinement, hep-th/9801024.
- [5] K. Wilson, Confinement of quarks, Phys. Rev. D 10, 2445-2459 (1974).
- [6] G. Parisi and N. Surlas, Random magnetic fields, supersymmetry, and negative dimensions, Phys. Rev. Lett. 43, 744-745 (1979).
- [7] N.D. Mermin and H. Wagner, Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic Heisenberg models, Phys. Rev. Lett. 17, 1133-1136 (1966).  
N.D. Mermin, Absence of ordering in certain classical systems, J. Math. Phys. 8, 1061-1064 (1967).  
P.C. Hohenberg, Existence of long-range order in one and two dimensions, Phys. Rev. 158, 383-386 (1967).  
S. Coleman, There are no Goldstone bosons in two dimensions, Commun. Math. Phys. 31, 259-264 (1973).
- [8] V.L. Berezinskii, Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group I. classical systems, Soviet Physics JETP 32, 493-500 (1971).  
J.M. Kosterlitz, The critical properties of the two dimensional XY model, J. Phys. C 7, 1046-1060 (1974).  
J.M. Kosterlitz and D.V. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, J. Phys. C 6, 1181-1203 (1973).
- [9] J. Fröhlich, Classical and quantum statistical mechanics in one and two dimensions: two-component Yukawa- and Coulomb systems, Commun. Math. Phys. 47, 233-268 (1976).
- [10] J.V. Jose, L.P. Kadanoff, S. Kirkpatrick and D.R. Nelson, Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model, Phys. Rev. B 16, 1217-1241 (1977).
- [11] D.J. Amit, Y.Y. Goldschmidt and G. Grinstein, Renormalization group analysis of the phase transition in the 2D Coulomb gas, sine-Gordon theory and XY-model, J. Phys. A 13, 585-620 (1980).
- [12] J. Fröhlich and T. Spencer, The Kosterlitz-Thouless transition in two dimensional abelian spin systems and the Coulomb gas, Commun. Math. Phys. 81, 527-602 (1981).
- [13] T. Banks, R. Myerson and J. Kogut, Phase transitions in abelian lattice gauge theories, Nucl. Phys. B 129, 493-510 (1977).

- [14] J. Glimm and A. Jaffe, Instantons in a  $U(1)$  lattice gauge theory, *Commun. Math. Phys.* 56, 195-212 (1977).
- [15] M.E. Peskin, Mandelstam-'t Hooft duality in abelian lattice models, *Ann. Phys.* 113, 122-152 (1978).
- [16] K.-I. Kondo, Abelian-projected effective gauge theory of QCD with asymptotic freedom and quark confinement, hep-th/9709109 (revised), *Phys. Rev. D*, to be published.
- [17] K. Osterwalder and E. Seiler, Gauge field theories on the lattice, *Ann. Phys.* 110, 440-471 (1978).
- [18] S. Elitzur, R.B. Pearson and J. Shigemitsu, Phase structure of discrete Abelian spin and gauge systems, *Phys. Rev. D* 19, 3698-3714 (1979).
- [19] A. Ukawa, P. Windey and A. Guth, Dual variables for lattice gauge theories and phase structure of  $Z(N)$  systems, *Phys. Rev. D* 21, 1013-1036 (1980).
- [20] A. Guth, Existence proof of a nonconfining phase in four-dimensional  $U(1)$  lattice gauge theory, *Phys. Rev. D* 21, 2291-2307 (1980).
- [21] J. Fröhlich and T. Spencer, Massless phases and symmetry restoration in Abelian gauge theories and spin systems, *Commun. Math. Phys.* 83, 411-454 (1982).
- [22] K.G. Wilson and J. Kogut, The renormalization group and the  $\epsilon$  expansion, *Phys. Rep.* 12 C, 75-200 (1974).
- [23] J.B. Kogut, Introduction to lattice gauge theory and spin systems, *Rev. Mod. Phys.* 51, 659-713 (1979).
- [24] J.-M. Drouffe and J.-B. Zuber, Strong coupling and mean field methods in lattice gauge theories, *Phys. Rep.* 102, 1-119 (1983).
- [25] E. Seiler, Gauge theories as a problem of constructive field theory and statistical mechanics, *Lecture Note in Physics*, Vol. 159 (Springer, 1982).
- [26] J. Villain, Theory of one- and two-dimensional magnets with an easy magnetization plane. II. The planar, classical, two-dimensional magnet, *J. Physique* 36, 581-590 (1975).
- [27] V.A. Miransky, Dynamics of spontaneous chiral symmetry breaking and the continuum limit in quantum electrodynamics, *Nuovo Cimento* 90A, 149-170 (1985).
- [28] K.-I. Kondo, T. Mino and K. Yamawaki, Critical line and dilaton in scale-invariant QED, *Phys. Rev. D* 39, 2430-2433 (1989).
- [29] S. Coleman, Quantum sine-Gordon equation as the massive Thirring model, *Phys. Rev. D* 11, 2088-2097 (1975).  
K.-I. Kondo, Bosonization and duality of massive Thirring model, *Prog. Theor. Phys.* 94, 899-914 (1995).
- [30] P.R. Thomas and M. Stone, Nature of the phase transition in a non-linear  $O(2)_3$  model, *Nucl. Phys. B* 144, 513-524 (1978).