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Revisiting Dirichlet's Solution over an $\mathbb{R}^2$ Ball Using Poisson's Kernel

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Abstract

The present work is aimed at revisiting the continuity of the solution to Dirichlet's homogeneous problem applied to a unit radius ball, $\overline{B}$ in $\mathbb{R}^2$ by using Poisson's Kernel. Continuity in the solution is rigorously justified over ∂B. This reasoning is supported by the theorem of existence of solutions to Dirichlet's problem for balls [4] as well as by the uniform convergence of Poisson's kernel $P_r(\theta)$ in $0 \leq r < 1$ (with constant r). Therefore, the coefficients of the corresponding Fourier series for $n = 1, 2, \cdots$ are as follows

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1 \quad \text{and} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} P_r(\theta)\cos n\theta d\theta = 2r^n.$$

From this series representation of $P_r(\theta)$ it can be readily observed that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi - \theta)d\theta = \frac{1}{2\pi}\int_{\partial B} K(x, y, v, w)ds = 1.$$

This result, coupled with the continuity of function f defined over ∂B, which indicates the values of u on the boundary of Ω, implies the continuity of u as a solution over $\overline{B}$.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ and $f : \partial\Omega \to \mathbb{R}$ be a continuous function over $\partial\Omega$. The maximum principle [5], [4], [2] implies that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $\Delta u = 0$ in $\mathring{\Omega}$, then $u(x, y)$ is uniquely determined for its values over $\partial\Omega$. That is to say, the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u \equiv f & \text{in } \partial\Omega \end{cases} \tag{1}$$

has a single solution u in $C^2(\Omega) \cap C(\overline{\Omega})$. If u_1 and u_2 are solutions to problem (1) and $v = u_1 - u_2$ is considered, then $\Delta v = 0$ in Ω and $v \equiv 0$ over $\partial\Omega$, that is v, is harmonic in Ω, taking the value of zero on the boundary. By using the maximum principle [5], it can be concluded that $v \equiv 0$ in $\overline{\Omega}$, which implies that $u_1 = u_2$ in $\overline{\Omega}$.

Dirichlet's problem (1) consists in finding a function $u(x, y)$ once its corresponding values on $\partial\Omega$ are given, considering that $\Omega = B_R(0)$ is a ball with radius R centered at the origin.

An important complex-variable-derived result that is used in the present work is that both the real and the imaginary parts of an analytic function defined in Ω are harmonic functions in Ω [3], [1].

The construction of the solution to the problem in (1) uses Poisson's formula [4]. Here, the result in [4] is used for the particular case where $n = 2$ and $\Omega = B_1(0)$.

In [4], it is shown that function $u(x, y)$, which satisfies (1), belongs to the space defined by $C^2(\Omega) \cap C(\overline{\Omega})$ and also that u is harmonic in Ω since the Green function $G = G(x, y)$, for Ω and $\partial G/\partial\vec{n}$, is harmonic in (x, y). The existence of G suggests the representation of function u in terms of its corresponding values over $\partial\Omega$.

In [1], a functional linear operator called Poisson's Integral is defined for any function $U(\theta)$ that is piecewise continuous in $-\pi \le \theta \le \pi$. The operator on U is defined as follows:

$$P(U)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re}\left[\frac{e^{i\theta} + z}{e^{i\theta} - z}\right] U(\theta) d\theta$$

H. A. Schwarz [1] shows that $P(U)(z)$ represents a harmonic function for $|z| < 1$ and $P(U)(z) \to U(\theta_0)$ when $z \to e^{i\theta_0}$, assuming that U is continuous at θ_0. Schwarz also presents an interesting geometric interpretation of Poisson's formula.

2 Leading result from revisiting

Let $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ have boundary $\partial\Omega = \{(x, y) : x^2 + y^2 = 1\}$ and let $f : \partial\Omega \to \mathbb{R}$ be a continuous function on $\partial\Omega$, then there is a unique function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfies (1). Function u is defined by Poisson's Integral formula as [4]

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi - \theta) f(e^{i\theta}) d\theta, \tag{2}$$

for all $0 \leq r < 1, -\pi \leq \theta \leq \pi, -\pi \leq \varphi \leq \pi$ [4], [3]. When constructing the solution to problem (1), some properties of complex analytic functions are used, [3], [1], together with Poisson's Integral formula [4].

3 Preliminary

This section presents and proves some of the basic properties of Poisson's kernel [3], [6]

1. In [3], [1] it is shown that a complex function $g(z) = u(z) + iv(z)$ is analytic over a region Ω, if and only if $\mathrm{Re}[g(z)] = u(x, y)$ and $\mathrm{Im}[g(z)] = v(x, y)$ are harmonic in Ω, that is, $\Delta u = 0$ and $\Delta v = 0$ in Ω.

2. For all $-\infty < \theta < \infty, 0 \leq r < 1$, the following function is defined

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta} \tag{3}$$

$P_r(\theta)$ is named Poisson's kernel. Let $z = re^{i\theta}$, then $|z| < 1$ and therefore

$$\frac{1 + re^{i\theta}}{1 - re^{i\theta}} = \frac{1 + z}{1 - z} = (1 + z)(1 + z + z^2 + \cdots) \tag{4}$$

$$= 1 + 2\sum_{n=1}^{\infty} z^n = 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta} \tag{5}$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n (\cos n\theta + i \sin n\theta) \tag{6}$$

$$\mathrm{Re}\left[\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right] = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta \tag{7}$$

$$= 1 + \sum_{n=1}^{\infty} r^n (e^{in\theta} + e^{-in\theta}) = P_r(\theta) \tag{8}$$

also

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1-re^{-i\theta}+re^{i\theta}-r^2}{|1-re^{i\theta}|^2} \tag{9}$$

then

$$P_r(\theta) = \text{Re}\left[\frac{1+re^{i\theta}}{1-re^{i\theta}}\right] = \frac{1-r^2}{1-2r\cos\theta+r^2} \tag{10}$$

3. The following is proof of the uniform convergence of $P_r(\theta)$ for all $-\infty < \theta < \infty$. Let $0 \leq r < 1$ be a fixed value r, then the series $2\sum_{n=1}^{\infty} r^n$ converges and also

$$|P_r(\theta)| = \left|\sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta}\right| \tag{11}$$

$$\leq \sum_{n=0}^{\infty} r^n + \sum_{n=1}^{\infty} r^n \tag{12}$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n \tag{13}$$

Based on criterion M, by Weierstrass, $P_r(\theta)$ converges uniformly for all $0 \leq r < 1$, with fixed r and $-\infty < \theta < \infty$.

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta &= \sum_{n=-\infty}^{\infty} r^{|n|}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}d\theta \\ &= \frac{1}{2\pi}\left(\sum_{n=0}^{\infty} r^n \int_{\pi}^{\pi} e^{in\theta}d\theta + \sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} e^{-in\theta}d\theta\right) \\ &= 1 + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} \cos n\theta d\theta = 1 \qquad (14)\end{aligned}$$

Then, it is possible to evaluate (14) by substituting $t = \tan(\theta/2)$. Similarly, it can be verified that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi-\theta)d\theta = 1 + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} \cos n(\varphi-\theta)d\theta = 1 \tag{15}$$

4. Trivially, $P_r(\theta)$ is a positive, even and periodic function for all θ since (9) suggests that

$$P_r(\theta) > 0,\ P_r(\theta) = P_r(-\theta)\ \text{ y } P_r(\theta+2\pi) = P_r(\theta) \tag{16}$$

4 Proof of (1)

According to [4], function $u : \overline{\Omega} \to \mathbb{R}$ is defined as

$$u(x,y) = \begin{cases} \dfrac{1-|(x,y)|^2}{2\pi} \displaystyle\int_{\partial\Omega} \dfrac{f(v,w)}{|(x,y)-(v,w)|^2} ds & \text{if } (x,y) \in \Omega \\ f(x,y) & \text{if } (x,y) \in \partial\Omega \end{cases} \tag{17}$$

If $(x,y) \in \Omega$ and $(v,w) \in \partial\Omega$, there is $0 \le r < 1$, $-\pi \le \theta \le \pi$, $-\pi \le \varphi \le \pi$, such that $v = \cos\theta$, $w = \sin\theta$, $x = r\cos\varphi$, $y = r\sin\varphi$. Using (17), then

$$u(x,y) = u(re^{i\varphi}) = \frac{1-r^2}{2\pi} \int_{\partial\Omega} \frac{f(v,w)}{(x-v)^2 + (y-w)^2} ds \tag{18}$$

$$= \frac{1-r^2}{2\pi} \int_{-\pi}^{\pi} \frac{f(\cos\theta, \sin\theta)}{1 - 2r\cos(\varphi-\theta) + r^2} d\theta \tag{19}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\varphi-\theta) f(e^{i\theta}) d\theta \tag{20}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Re}\left[\frac{1 + re^{(\varphi-\theta)}}{1 - re^{(\varphi-\theta)}}\right] f(e^{i\theta}) d\theta \tag{21}$$

$$= \mathrm{Re}\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) f(e^{i\theta}) d\theta\right] \tag{22}$$

,where $z = re^{i\varphi}$. In [3], it is shown that

$$g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) f(e^{i\theta}) d\theta \tag{23}$$

is an analytic function in Ω; therefore, $\mathrm{Re}\,[g(z)]$ is a harmonic function in Ω, that is $\Delta u \equiv 0$ in Ω.

Let

$$K(x,y,v,w) = \frac{1-|(x,y)|^2}{|(x,y)-(v,w)|^2}, \text{ for all } (x,y) \in \Omega \text{ and } (v,w) \in \partial\Omega \tag{24}$$

If $x = r\cos\varphi$, $y = r\sin\varphi$, $v = \cos\theta$, $w = \sin\theta$, then $K(x,y,v,w) = P_r(\varphi-\theta)$ and therefore

$$\frac{1}{2\pi} \int_{\partial\Omega} K(x,y,v,w) ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\varphi-\theta) d\theta = 1 \tag{25}$$

Since u is harmonic in Ω, then u is continuous in Ω [3]. To complete the proof, continuity of u at every point of $\partial\Omega$ must be shown.

Let $(x_0, y_0) \in \partial\Omega$ and let $\epsilon > 0$ be an arbitrary value. Then $\delta > 0$ can be chosen such that, if $|(x,y) - (x_0,y_0)| < \delta$ and $(x,y) \in \partial\Omega$, then, due to the continuity of f in $\partial\Omega$, $|f(x,y) - f(x_0, y_0)| < \epsilon$ is obtained, and since $f(\partial\Omega)$ is closed and bounded in $\mathbb{R}$, There is a value of $M > 0$ such that $||f||_\infty \leq M$ for all $(x,y) \in \partial\Omega$.

Let $(x,y) \in \Omega$, y $|(x,y) - (x_0, y_0)| < \delta/2$;

$$|u(x,y) - u(x_0,y_0)| = \frac{1 - |(x,y)|^2}{2\pi} \left| \int_{\partial\Omega} \frac{f(v,w)}{|(x,y) - (v,w)|^2} ds - u(x_0,y_0) \right| \quad (26)$$

Then by (25)

$$|u(x,y) - u(x_0,y_0)| = \left| \frac{1}{2\pi} \int_{\partial\Omega} K(x,y,v,w) \left[f(v,w) - f(x_0,y_0) \right] ds \right| \quad (27)$$

Let $\tilde{B}$ be the set $\partial\Omega \cap B_\delta(x_0,y_0) = \{(v,w) : |(v,w) - (x_0,y_0)| < \delta\}$, and let $\partial\Omega - \tilde{B} = \{(v,w) : |(v,w) - (x_0,y_0)| \geq \delta\}$ from (27), therefore

$$\begin{aligned} |u(x,y) - u(x_0,y_0)| \leq & \left| \frac{1}{2\pi} \int_{\tilde{B}} K(x,y,v,w) \left[f(v,w) - f(x_0,y_0) \right] ds \right| \\ & + \left| \frac{1}{2\pi} \int_{\partial\Omega - \tilde{B}} K(x,y,v,w) \left[f(v,w) - f(x_0,y_0) \right] ds \right| \end{aligned}$$

If $(v,w) \in \tilde{B}$, then

$$\left| \frac{1}{2\pi} \int_{\tilde{B}} K(x,y,v,w) \left[f(v,w) - f(x_0,y_0) \right] ds \right| < \epsilon \quad (28)$$

Now, since

$$||(x_0,y_0) - (v,w))| - |(x,y) - (x_0,y_0)|| \leq |(x,y) - (v,w)| \quad (29)$$

from (29), and also if $(v,w) \in \partial\Omega - \tilde{B}$, then the following holds

$$K(x,y,v,w) = \frac{1 - |(x,y)|^2}{|(x,y) - (v,w)|^2} \leq \frac{1 - |(x,y)|^2}{(\delta/2)^2} \quad (30)$$

From (28) and (30) it is observed that

$$|u(x,y) - u(x_0,y_0)| \leq \epsilon + \frac{2M(1 - |(x,y)|^2)}{(\delta/2)^2} \quad (31)$$

Therefore, if

$$|(x,y) - (x_0, y_0| \leq \frac{\epsilon(\delta/2)^2}{2M} \quad \text{and} \quad 1 - |(x,y)| \leq |(x,y) - (x_0,y_0)| \leq \frac{\epsilon(\delta/2)^2}{2M}$$

then $|u(x,y) - u(x_0,y_0)| < \epsilon + \epsilon$. Therefore u is a continuous function at (x_0,y_0).

5 Conclusions

The present work has shown that the solution $u(x, y)$ to problem (1) is the real part of an analytic function $g(z) = g(x + iy)$ defined in (23) for a unit-radius ball B centered at the origin; therefore, $\Delta u = 0$ in B and u is continuous within B.

Regarding the continuity analysis of u on the boundary of B, a local argument was considered together with two cases for $(x_0, y_0) \in \partial B$, namely

1. For $(x, y) \in \partial B$, it was considered that f is continuous and bounded on ∂B, and therefore if $|(x, y) - (x_0, y_0)|$ is small enough, then $|u(x, y) - u(x_0, y_0)| < \epsilon$.

2. if $(x, y) \in B$, then $u(x, y) \to f(x_0, y_0)$ when $(x, y) \to (x_0, y_0)$.

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