Abstract: This paper addresses the problem of static output-feedback controller design for nonlinear interval time-delay systems based on the Takagi-Sugeno fuzzy model. Sufficient conditions to guarantee the robust stability for considered systems are derived in terms of the matrix measures of the system matrices of the linear subsystems in the consequent parts of fuzzy rules. By using linear matrix inequalities, we provide a method of obtaining a static output-feedback controller that can stabilize the system. Although the results obtained in this paper are believed to be more restrictive, it can provide an effective solution to stabilize more complex systems that are uncertain, time-delay, and ill defined. Finally, an example is provided to illustrate the effectiveness of our approach.

Index Terms — Fuzzy control; Matrix measure; Robust control; Linear Matrix Inequality; time-delay.

I. INTRODUCTION

Recently, based on the Takagi-Sugeno (T-S) fuzzy model [1,22], researchers have proposed many methods of designing controllers for nonlinear systems with uncertainties. The main feature of a T-S fuzzy model is to characterize the local dynamics of each fuzzy rule by a linear system model. The appeal of the T-S model is that the stability and performance characteristics of the system can be verified by using a Lyapunov function approach [2]-[6]. The basic idea of these methods is to design feedback controllers for individual local models respectively and then to construct a global controller from these local controllers so that the stability of the overall fuzzy system is guaranteed. State feedback controllers were addressed in most research works. However, in practical applications, state variables are often not measurable. Therefore, in the literatures [24]-[26], fuzzy observers are proposed to estimate inaccessible states of the controlled fuzzy systems. Although dynamic output feedback controllers can overcome the problem that states are not accessible, the design of state estimators will increase the difficulty of stabilizing the overall fuzzy system. If possible, we may like to consider the controller synthesis problem via static output feedback. This is a more challenging problem. In this paper, we shall propose an LMI-based fuzzy static output feedback design method for nonlinear interval time-delay systems.

Time delays occur in many dynamical systems such as biological systems, chemical systems, electrical networks, hydraulic, and rolling mill systems. Time delays are often a source of instability and poor performance. The study of stability and stabilization for a class of nonlinear time-delay systems based on the T-S fuzzy model has attracted considerable attentions [14]-[16]. As an example, Cao et al. [14] dealt with the issue of stability analysis and control design for both continuous and discrete time nonlinear systems with time delays in T-S fuzzy model-based control. Generally, to consider the stability and stabilization issue for nonlinear time-delay systems with uncertainties is desired. However, to our best knowledge, the research found in the literature is only for nonlinear systems either with uncertainties [7]-[13] or with time delays [14]-[16]. Kiriakids [7] treated the issue of stability robustness against modeling uncertainty in T-S fuzzy model-based control. Parametric uncertainties regarding those approaches are also dealt with in [8]-[13]. In this paper, we propose a way of conducting stability analysis and control design for nonlinear interval time-delay systems under the representation of the T-S fuzzy model. It is shown that the static output-feedback controller design problem for nonlinear interval time delay systems is solvable if a matrix measure assignment problem is solvable. Afterward, the matrix measure assignment problem is shown to be equivalent to a LMI feasibility problem. Thus, the solutions can be found through LMI tools.

II. TAKAGI-SUGENO FUZZY MODELS AND MATRIX MEASURE

A. Takagi-Sugeno Fuzzy Model

Takagi and Sugeno [1] proposed an elegant modeling method, which is often referred to as the T-S fuzzy model, to represent or approximate a nonlinear dynamical system. In each fuzzy rule, a linear model is given to describe the nonlinear system locally. The overall system is described by fuzzy “blending” those local models. A typical IF-THEN rule of T-S fuzzy models is represented as

Rule $i$: IF $z_i(t)$ is $M_{ij}$ and ... $z_p(t)$ is $M_{pj}$

THEN $\dot{x}(t) = A_i x(t) + B_i u(t)$, for $i = 1,2,...,r$ \quad (1)

$y(t) = C_i x(t)$

where $z_i(t)$ ... $z_p(t)$ are the premise variables, and $M_{ij}$ is the corresponding fuzzy set for $j = 1, ... p$.

$x(t) = [x_i(t), x_j(t), ... , x_q(t)]^T$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $A_i \in \mathbb{R}^{m \times m}$, $B_i \in \mathbb{R}^{m \times n}$, $C_i \in \mathbb{R}^{r \times m}$. $y(t) \in \mathbb{R}^r$ is the controlled output and $r$ is the number of IF-THEN rules. In our implementation, we simply set $p=q$ and $z_i(t) = y_i(t)$, ..., $z_p(t) = y_p(t)$.

By using the center of gravity method for defuzzification, the final output of the T-S fuzzy model is inferred as:
\[
    \dot{x}(t) = \sum_{i=1}^{r} w_i(z(t))(A_i x(t) + B_i u(t)) \\
    = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) \\
    y(t) = \sum_{i=1}^{r} w_i(z(t))(C_i x(t)) = \sum_{i=1}^{r} h_i(z(t))(C_i x(t))
\]

where \( z(t) = [z_1(t), z_2(t), \ldots, z_r(t)]^T \).

\( w_i(z(t)) = \prod_{j=1}^{p} M'_j(z_{j}(t)) \) and \( M_j(z_{j}(t)) \) is the fuzzy membership grade of \( z_j(t) \) in \( M'_j \). It is assumed in this paper that \( w_i(z(t)) \geq 0 \), \( i = 1, 2, \ldots, r \) and \( \sum_{i=1}^{r} w_i(z(t)) > 0 \) for all \( t \).

Therefore, \( h_i(z(t)) \geq 0 \), \( i = 1, 2, \ldots, r \) and \( \sum_{i=1}^{r} h_i(z(t)) = 1 \), for all \( t \).

**B. The Properties of Matrix Measure**

We now introduce several properties about matrix measure as follows. The matrix measure of a constant matrix \( M \) is defined as

\[
    \mu_\gamma(M) = \lim_{\theta \to 0^+} \frac{\| I + \theta M \|_\gamma - 1 }{\theta}
\]

where \( \| \cdot \|_\gamma \) is a suitable matrix norm (see [17]).

**Lemma 1** [17]: The matrix measure has the following properties.

(a) \( \mu_\gamma(\cdot) \) is convex; i.e.,

\[
    \mu_\gamma\left( \sum_{j=1}^{K} \alpha_j M_j \right) \leq \sum_{j=1}^{K} \alpha_j \mu_\gamma(M_j) \text{ for all } \alpha_j \geq 0.
\]

(b) For any norm and any constant matrix \( M \)

\[
    -\| M \| \leq \mu_\gamma(-M) \leq \mu_\gamma(M) \leq \| M \|.\]

(c) Suppose \( m_{ij} \) is the \( \gamma \)-th element of \( M \), then

\[
    \mu_1(M) = \max_{i} \left[ \text{Re}(m_{ij}) + \sum_{j=1}^{K} m_{ij} \right],
\]

\[
    \mu_2(M) = \max_{i} \left[ \lambda_i(M + M^*) / 2 \right].
\]

**III. NEW STABILITY CONDITIONS FOR INTERVAL TIME-DELAY DYNAMIC NONLINEAR SYSTEMS**

Consider a nonlinear interval time-delay system described by the following T-S fuzzy model:

**Rule**: IF \( y_j(t) \) is \( M'_j \) and \( y_q(t) \) is \( M'_q \) THEN

\[
    \dot{x}(t) = \hat{A}_j x(t) + \hat{D}_j x(t-\tau) + B_j u(t) \\
    y(t) = C_j x(t)
\]

for \( i = 1, 2, \ldots, r \)

\[
    y(t) = \sum_{i=1}^{r} h_i(y_i(t))(\hat{A}_j x(t) + \hat{D}_j x(t-\tau) + B_j u(t))
\]

\[
    y(t) = \sum_{i=1}^{r} h_i(y_i(t))(C_j x(t))
\]

with \( h_i(y_i(t)) \geq 0 \), \( i = 1, 2, \ldots, r \) and \( \sum_{i=1}^{r} h_i(y_i(t)) = 1 \) for all \( t \), where \( \tau \) is the time-delay of the system. Assume that the matrices \( \hat{A}_j \) and \( \hat{D}_j \) are

\[
    \hat{A}_j = [a_{ij}] \quad a_{ij} \leq a_{ij} \leq \overline{a}_{ij} \quad i = 1, 2, \ldots, r
\]

\[
    \hat{D}_j = [d_{ij}] \quad d_{ij} \leq d_{ij} \leq \overline{d}_{ij} \quad i = 1, 2, \ldots, r
\]

where \( a_{ij} \) is the \( \gamma \)-th element of the matrix \( \hat{A}_j \), \( d_{ij} \) and \( \overline{d}_{ij} \) denote \( a_{ij} \)'s lower bound and upper bound, respectively, \( a_{ij} \) is the \( \gamma \)-th element of the matrix \( \hat{D}_j \), and \( d_{ij} \) and \( \overline{d}_{ij} \) denote \( d_{ij} \)'s lower bound and upper bound, respectively. We further assume that those bounds \( a_{ij} \), \( a_{ij} \), \( a_{ij} \), \( a_{ij} \), \( a_{ij} \), \( a_{ij} \) are known real values. Denote

\[
    \overline{A}_j = [\overline{a}_{ij}], \quad \overline{D}_j = [\overline{d}_{ij}], \quad i = 1, 2, \ldots, r
\]

Since the consequent parts of T-S fuzzy models are described by linear state equations, the linear control theory can be used to design the consequent parts of a fuzzy controller. Fuzzy controllers for stabilizing the fuzzy system (10) can be designed via parallel-distributed control (PDC) [2] [4] [18]. In PDC, fuzzy controllers share the same premise parts with (10); that is, the controller for Rule \( i \) is
if \( y_i(t) \) is \( M_i' \) and ... \( y_q(t) \) is \( M_q' \) THEN \( \mathbf{u}(t) = \mathbf{F}_i \mathbf{y}(t) \), \( i = 1, 2, \ldots, r \). (16)

Then, the overall output of this fuzzy controller is

\[
\mathbf{u}(t) = \sum_{i=1}^{r} \hat{h}_i(y(t))\mathbf{F}_i \mathbf{y}(t),
\]

where \( \hat{h}_i(y(t)) \) is the same as that of the \( \mathbf{h} \) rule of the fuzzy system (10). By substituting (17) into (10), we get

\[
\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \hat{h}_j(y(t))\hat{h}_j(y(t))\hat{h}_j(y(t))x(t) + \mathbf{D}_i x(t - \tau) \quad (18)
\]

Let \( \hat{A}_i = A_i + \Delta A_i \) and \( \hat{D}_i = D_i + \Delta D_i \). It is known that \( \| \Delta A \| < \| M \| \) and \( \| \Delta D \| < \| N \| \) (see [23]), for \( i = 1, 2, \ldots, r \).

We first derive a new sufficient condition to ensure the stability of the following T-S fuzzy model (without uncertainty)

\[
\dot{x}(t) = \sum_{i=1}^{r} \hat{h}_i(y(t))(A_i x(t) + D_i x(t - \tau))
\]

\[
y(t) = \sum_{i=1}^{r} \hat{h}_i(y(t))(C_i x(t))
\]

**Theorem 1:** For any \( \varepsilon > 0 \), if

\[
\mu_2(A_i) < -\frac{1}{2\varepsilon} - \frac{1}{\varepsilon} \| D_i \|^2 - \frac{1}{2\varepsilon} \| F_i \|^2 i = 1, 2, \ldots, r, \quad (20)
\]

then the equilibrium of the unforced system (19) is asymptotically stable for all \( \tau \).

**Proof:** This proof is omitted.

Theorem 1 provides a simple method to verify the stability of the time-delay fuzzy system (19). In what follows, we shall consider the stability conditions of time-delay fuzzy systems with uncertainty.

**Theorem 2:** Consider the interval time-delay fuzzy system

\[
\dot{x}(t) = \sum_{i=1}^{r} \hat{h}_i(y(t)) (\hat{A}_i x(t) + \hat{D}_i x(t - \tau))
\]

\[
y(t) = \sum_{i=1}^{r} \hat{h}_i(y(t))(C_i x(t))
\]

where \( \hat{A}_i \) and \( \hat{D}_i \) are defined in (11). If

\[
\mu_2(A_i) < -\frac{1}{2\varepsilon} - \frac{1}{\varepsilon} \| D_i \|^2 - \frac{1}{2\varepsilon} \| F_i \|^2 \quad (21)
\]

then (21) is robustly asymptotically stable for all \( \tau \).

**Proof:** This proof is omitted.

With the above theorems, we have the following corollary.

**Corollary 1:** Suppose that static output feedback gains \( \mathbf{F}_j \), \( j = 1, 2, \ldots, r \), satisfy the following conditions

\[
\mu_2(A_i + B_j F_j C_i) < -\frac{1}{2\varepsilon} - \frac{1}{\varepsilon} \| D_i \|^2 -\frac{1}{2\varepsilon} \| N \|^2 \quad (23)
\]

for \( i, j, k = 1, 2, \ldots, r \), then the equilibrium of the closed-loop nonlinear time-delay system with norm-bounded uncertainties represented as

\[
\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \hat{h}_j(y(t))\hat{h}_j(y(t))h_j(y(t))x(t) + \hat{D}_i x(t - \tau) \quad (24)
\]

\[
y(t) = \sum_{i=1}^{r} \hat{h}_i(y(t))(C_i x(t))
\]

is robustly asymptotically stable for all \( \tau \).

Corollary 1 reveals that if each constant control gain in each local model satisfies (23), then the overall nonlinear interval time-delay system is robustly asymptotically stable.

For simplicity of notation, define

\[
\gamma_i = -\frac{1}{2\varepsilon} - \frac{1}{\varepsilon} \| D_i \|^2 - \frac{1}{2\varepsilon} \| N \|^2 \quad (25)
\]

Now, we turn our attention to reduce the matrix measure assignment problem (25) to an LMI feasibility problem. For a matrix \( \mathbf{U} \), define \( \mathbf{U}^{-\perp} \) as a matrix whose columns form bases of the null bases of \( \mathbf{U} \). Then, we can have the following theorem.

**Theorem 3:**

1. The static output feedback gains \( \mathbf{F}_j \), \( j = 1, 2, \ldots, r \), satisfy the following conditions

\[
\mu_2(A_i + B_j F_j C_i) < \gamma_j \quad i, j, k = 1, 2, \ldots, r \quad (26)
\]

if and only if \( \mathbf{F}_j \) satisfy the following LMIs

\[
(A_i + A_i^T - 2 \gamma I) + B_j F_j C_i + C_j^T F_j B_j^T < 0 \quad (27)
\]

2. There exists an \( \mathbf{F}_j \) that satisfies (27) if and only if

\[
(B_j^T A_i + A_i^T - 2 \gamma I) B_j < 0 \quad i, j, k = 1, 2, \ldots, r \quad (28)
\]

and

\[
(C_j^T A_i + A_i^T - 2 \gamma I) C_j < 0 \quad i, j, k = 1, 2, \ldots, r \quad (29)
\]

**Proof:** This proof is omitted.

Theorem 3 tells us that if (28) and (29) hold, then there exist static output feedback gains \( \mathbf{F}_j \), \( j = 1, 2, \ldots, r \), that satisfy LMIs (27). In fact, such static output feedback gains \( \mathbf{F}_j \) also satisfy (26). Note that static output feedback gains \( \mathbf{F}_j \) satisfying LMIs (27) can easily be obtained by using Matlab’s LMI Control Toolbox. The obtained static output feedback gains \( \mathbf{F}_j \) then can also solve the considered problem.
IV. ILLUSTRATIVE EXAMPLE

In this section, an example is used to verify the performance of the proposed controller. Assume that an unknown nonlinear interval time-delay system can be modeled as the following simple T-S fuzzy model system:

Rule 1: IF \( y_1(t) \) is \( M_1 \) THEN
\[
\dot{x}(t) = \hat{A}_1 x(t) + \hat{D}_1 x(t - \tau) + B_1 u(t)
\]
\[
y(t) = C_1 x(t)
\]
Rule 2: IF \( y_2(t) \) is \( M_2 \) THEN
\[
\dot{x}(t) = \hat{A}_2 x(t) + \hat{D}_2 x(t - \tau) + B_2 u(t)
\]
\[
y(t) = C_2 x(t)
\]

where \( x(t) = [x_1(t), x_2(t), x_3(t)]^T \).

\[
\psi(t) = 0, \quad t \in [-\tau, 0], \quad \tau = 1, \text{ and}
\]
\[
\Delta = \begin{bmatrix} 3.8 & -5.6 & 3.5 \\ -4.3 & 19.1 & -19.2 \\ 29.4 & 3.6 & -36.5 \\ -0.4 & 0.7 & -0.8 \end{bmatrix}, \quad \hat{\Delta} = \begin{bmatrix} 4.6 & -4.2 & 4.5 \\ -3.7 & 22.9 & -16.8 \\ 34.6 & 4.4 & -30.5 \\ -0.2 & 1 & -0.3 \end{bmatrix}
\]
\[
\tilde{\alpha} = \begin{bmatrix} 0.3 & -2.2 & -1.1 \\ -1.5 & -0.5 & -2.3 \end{bmatrix}, \quad \hat{\alpha} = \begin{bmatrix} 0.8 & -1.8 & 0.1 \\ -0.7 & 0.5 & -1.2 \end{bmatrix}
\]
\[
\Delta_1 = \begin{bmatrix} 5.2 & -6.7 & 7.1 \\ -9.3 & 11.9 & -15.9 \\ 21.9 & 8.6 & 24.6 \end{bmatrix}, \quad \hat{\Delta}_1 = \begin{bmatrix} 5.6 & -6.2 & 9.3 \\ -8.2 & 15.1 & -13.7 \\ 25.3 & 9.9 & -27.3 \end{bmatrix}
\]
\[
\Delta_2 = \begin{bmatrix} -0.7 & 0.2 & -1.1 \\ 0.9 & 2.8 & 3 \end{bmatrix}, \quad \hat{\Delta}_2 = \begin{bmatrix} -0.3 & 0.8 & -0.7 \\ 1.3 & 1.5 & 1 \end{bmatrix}
\]
\[
B_1 = \begin{bmatrix} -7 \\ 5 \\ 3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}
\]
\[
C_1 = \begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -6 & 9 & 2 \\ 1 & -4 & -2 \end{bmatrix}
\]

Figure 1 shows the membership functions of \( M_1 \) and \( M_2 \) for \( y_1(t) \). The following simulated systems are chosen in random from the above low-bounded and upper-bounded data. Figure 2 shows the output responses of the unforced system for the initial condition \( x(0) = [10 \ 15 \ 5]^T \) for \( t \in [0, 1] \). Now we want to design a fuzzy controller via PDC to stabilize the system. By using Theorem 3 and setting \( \gamma = 1 \), we obtain the static output feedback gains of the fuzzy controller:

\[
E = \begin{bmatrix} -6.4121 & -8.2238 \\ -3.8267 & -7.4992 \end{bmatrix}, \quad F = \begin{bmatrix} -5.5867 & -5.5340 \\ -4.0118 & -7.2285 \end{bmatrix}
\]

It is easy to check the stability of the overall fuzzy system by (23). Figure 3 shows the output responses of \( y_1 \) in the closed-loop system for various initial conditions \( x(0) \)’s. Figure 4 shows the output responses of \( y_2 \) for various initial conditions \( x(0) \)’s. Figure 5 shows the responses of \( x_1 \) for various initial conditions \( x_i(0) \)’s. Figure 6 shows the responses of \( x_2 \) for various initial conditions \( x_i(0) \)’s. Figure 7 shows the responses of \( x_3 \) for various initial conditions \( x_i(0) \)’s. From these simulation results, it is evident that the designed T-S fuzzy model based static output feedback controller not only can stabilize the nonlinear system, but has strong robustness against norm-bounded uncertainties and time delays.

V. CONCLUSIONS

In this paper, we have proposed a new robust fuzzy static output feedback controller design methodology for nonlinear interval time-delay systems with norm-bounded uncertainties and time delays. Finding an admissible solution to the matrix measure assignment problem can solve the problem of stabilizing controller design via static output-feedback for nonlinear interval time-delay systems represented by a T-S fuzzy model. The designed controller can indeed tolerate norm-bounded parametric uncertainties and against the effect of time delays. In the paper, we have shown that the matrix measure assignment problem is equivalent to an LMI feasibility problem. A sufficient condition for the existence of the fuzzy static output-feedback gain is obtained. Simulation results have verified and confirmed the effectiveness of the new approach in controlling a nonlinear interval time-delay system.

REFERENCES


Fig. 1. Membership function of the example.

Fig. 2. Output response of the unforced system for $x(0)=[10 15 5]^T$. 
Fig. 3. Output $y_1(t)$ response of the closed-loop system for various $x(0)$ and $\tau = 1$.

Fig. 4. Output $y_2(t)$ response of the closed-loop system for various $x(0)$ and $\tau = 1$.

Fig. 5. State $x_1(t)$ response of the closed-loop system for various $x_1(0)$ and $\tau = 1$.

Fig. 6. State $x_2(t)$ response of the closed-loop system for various $x_2(0)$ and $\tau = 1$.

Fig. 7. State $x_3(t)$ response of the closed-loop system for various $x_3(0)$ and $\tau = 1$. 