SPECTRAL ELEMENT METHOD IN STRUCTURAL DYNAMICS

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Preface

Owing to the rapid developments in computer technology, impressive progress in the computational methods used in engineering and science has been made over recent decades. The classical finite element method (FEM) has probably been the most popular in many areas of engineering and science, being one of the most convenient and easy-to-use computational methods. Though the FEM is applicable to most geometries, boundary conditions and material variations, it can be extremely expensive and it is often impossible to work out solutions to the large scale finite element models using a desktop computer. Thus, an alternative method that can provide accurate solutions while reducing the computational burden, but retaining the key advantage features of FEM, is mandatory, even today.

The FEM is a time-domain solution method in which the finite element equation is formulated in the time domain and solved by using a numerical integration method. On the other hand, the spectral element method (SEM) is a frequency-domain solution method in which the spectral element equation is formulated in the frequency domain and solved by using the fast Fourier transform (FFT) based spectral analysis method. In SEM, the exact dynamic stiffness matrix, known as the spectral element matrix, is formulated in the frequency domain by using exact wave solutions for the governing differential equations. Accordingly, in theory, the SEM will provide exact frequency-domain solutions while using only a minimum number of degrees-of-freedom.

Although there have been a huge number of journal publications since the basic concept of SEM was initially introduced more than two decades ago, very few books exist on the subject. Thus, the present book presents a concise introduction to the theory of SEM and its applications to various problems in structural dynamics and other related areas. It is intended as a valuable reference book for graduate students, professors, and professional researchers in the areas of mechanical engineering, civil engineering, aerospace engineering, naval architecture, structural engineering, applied mechanics, biomechanical engineering, and other related areas including computational methods. This book could be also used as a textbook for graduate students.

There are three parts to the book. The first part (Chapters 1 and 2) addresses the background and history of the SEM and the fundamentals of the spectral analysis of signals. In the second part (Chapters 3 and 4), the methods of spectral element formulation and the spectral element analysis method are addressed. The last part, from Chapter 5 to Chapter 16, presents the applications of SEM to various engineering problems in structural dynamics and related areas. The following overview summarizes the key features of each chapter.

Chapter 1 addresses key features of the SEM, together with a brief historical perspective on the development of SEM and its applications.
Chapter 2 introduces the fundamentals of the DFT (discrete Fourier transform) and FFT theories necessary for the spectral analysis of signals.

Chapter 3 is devoted to various methods of spectral element formulation: the force-displacement relation method, the variational method, and the state-vector equation method.

Chapter 4 addresses the general procedure of spectral element analysis: formulation and assembly of spectral elements; imposition of boundary conditions; and the computation of eigensolutions and time responses.

Chapter 5 deals with the dynamics of beams and Levy-type plates. For distributed dynamic forces, the two-element method is introduced as an approximation approach.

Chapter 6 is devoted to the spectral element modeling and analysis of the flow-induced vibrations of pipelines conveying two types of internal fluids: steady fluid and unsteady fluid.

Chapter 7 addresses the spectral element modeling and analysis of the dynamics and stability of axially moving structures, such as the string, the Bernoulli–Euler beam, the Timoshenko beam, and the thin plates.

Chapter 8 is devoted to the spectral element modeling and analysis of the dynamics of rotor systems, which consist of the flexible shafts, thin and thick rigid disks and bearing supports.

Chapter 9 discusses the spectral element modeling and analysis of the dynamics of multi-layered beam structures, such as the elastic–elastic two-layer beams and the elastic–viscoelastic–elastic passive constrained layer damping (PCLD) beams.

Chapter 10 discusses the spectral element modeling and analysis of the dynamics of adaptive structures, such as the elastic–piezoelectric two-layer beams and the active constrained layered damping (ACLD) beams with and without active control.

Chapter 11 is devoted to the spectral element modeling and analysis of the dynamics of composite laminate beams subjected to axial-bending-shear coupled vibration and to bending-torsion-shear coupled vibration.

Chapter 12 addresses the SEM-based continuum modeling method and the spectral transfer matrix method for the efficient dynamic analysis of various types of periodic lattice structures.

Chapter 13 extends the SEM application to a biomechanics problem: blood flow analysis. The pseudo-force based non-linear spectral element modeling is introduced.

Chapter 14 discusses the SEM-based methods to identify non-ideal structural boundaries and the joints in a beam structure from experimentally measured frequency response function (FRF) data.

Chapter 15 addresses the SEM-based linear and non-linear structural damage identification methods to identify multiple local damage generated in a structure from experimentally measured FRF data.

Chapter 16 discusses other promising SEM applications, such as the SEM–FEM hybrid method and the impact force identification, while providing a brief list of applications from the existing literature.

In most chapters the derivations of the associated governing equations are provided. Furthermore, the conventional finite element models are also provided in the appendices of each chapter for the use in validating or comparing with the corresponding spectral element models. Though no source codes are included in this book, some code examples (M files) are available for download from the book’s companion website http://www.wiley.com/go/ulee.

This book is the product not only of my knowledge, research and teaching experience, but of numerous discussions with my past and present graduate students over a period of more than 15 years. I wish to acknowledge the contributions of my graduate students during the various
stages of the manuscript, without which this book would not have been completed. All support and cooperation from the staff of John Wiley & Sons, senior commissioning editor Mr. James Murphy, project editor Mr. Roger Bullen, and production editor Ms. Sarah Karim are accordingly acknowledged. The author also acknowledges the financial support from Inha University (Inha University Research Grant) during the course of manuscript preparation. Finally, I dedicate this book to my mother Chungkyung Koh and to the memory of my father Seokbong Lee.

The author will be pleased to hear from readers who find misprints and errors, or who can provide hints to other ways of improving the book in any future editions.

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Part One

Introduction to the Spectral Element Method and Spectral Analysis of Signals
Introduction

1.1 Theoretical Background

1.1.1 Finite Element Method

As the dynamic behavior and characteristics of a structure are of great importance in engineering, it is necessary to predict them accurately in an efficient and economic manner. The finite element method (FEM) is probably one of the powerful and popular computation methods to have been used in many areas of engineering and science.

As an illustrative example, the vibration pattern of a structure certainly varies depending on the vibration frequency, and its wavelength at high frequency is very low. Because a sufficiently accurate dynamic response can only be obtained by capturing all necessary high frequency wave modes, the mesh (finite element) size used in the finite element modeling must be sufficiently small, comparable to the lowest wavelength of the vibrating structure.

However, as the conventional finite element models are formulated by using frequency-independent (static or fixed) polynomial shape (interpolation) functions, the FEM cannot capture all necessary high frequency wave modes of interest. Thus the FEM solutions become significantly inaccurate, especially at high frequencies, where associated wavelengths are very short. The so-called $h$-method is one of well-known approaches to improving the FEM accuracy by refining the meshes. Unfortunately this approach will make the size of the system extremely large, and hence from the computational aspect, the conventional FEM often becomes prohibitive for most complex, large flexible structures. It is known that, as a rough guide, the mesh size must be 10–20 times smaller than the wavelength of the highest frequency wave mode of interest [6].

An alternative approach to improve the solution accuracy is to use the shape functions, which can vary depending on vibration frequency. Accordingly, the candidate shape functions will be frequency dependent and they are known as dynamic shape functions in the literature. As the dynamic shape functions can readily capture all necessary high frequency wave modes of interest, extremely highly accurate solutions can be obtained, and the need to refine the meshes is no longer necessary. This elegant concept has led to the so-called dynamic stiffness method (DSM) [11, 108].
1.1.2 Dynamic Stiffness Method

The exact dynamic stiffness matrix is used in the DSM. The exact dynamic stiffness matrix is formulated in the frequency domain by using exact dynamic shape functions that are derived from exact wave solutions to the governing differential equations. To obtain the exact wave solutions in the frequency domain, the time-domain governing differential equations are transformed into the frequency domain by assuming harmonic solutions of a single frequency. Accordingly the exact dynamic stiffness matrix is also frequency dependent and it can be considered as a mixture of the inertia, stiffness and damping properties of a structure element. As the exact dynamic stiffness matrix is formulated by using exact frequency dependent dynamic shape functions derived from the exact wave solutions, it automatically deals with the continuous mass distribution in a structure element exactly. Thus the structure element represented by an exact dynamic stiffness matrix is often called a continuum element. Consequently, the DSM guarantees exact frequency-domain solutions to the governing differential equations (or the mathematical model) adopted for the problem under consideration: this is why the DSM is referred to as an exact solution method in the literature. Of course the absolute accuracy of the DSM will be limited to the accuracy level of the governing differential equations adopted to formulate the exact dynamic stiffness matrix. For instance, the DSM based on the Timoshenko-beam model will provide more accurate frequency-domain solutions when compared with that based on the Bernoulli–Euler beam model. However, as the assumptions made for the DSM will be less severe than those made for the conventional FEM and other approximate solution methods, the DSM will still provide better solutions.

As the exact dynamic stiffness matrix is formulated by using exact dynamic shape functions, it treats the mass distribution in a structure member exactly. Thus only one single element is sufficient for a regular part of a structure (without any structural or material discontinuities inside), regardless of its length between any two successive structural or material discontinuities, to acquire exact solutions. That is, we no longer need to refine a regular part of a structure into multiple fine meshes. This will significantly reduce the size of the problem, in other words, the total number of meshes and degrees-of-freedom (DOFs). In due course, this will significantly reduce the computation cost and time, together with improving the solution accuracy by reducing the computer round-off errors or numerical errors that are inevitable for most large size problems. In addition, the DSM provides an infinite number of eigensolutions from the exact dynamic stiffness matrix represented in terms of a minimum number of DOFs.

Because the exact dynamic stiffness matrices are stiffness formulated (as the conventional finite element stiffness matrices are), they can be assembled in a completely analogous way to that used in the FEM. Thus, the meshing and assembly features of the FEM can be applied equally to the DSM. The only difference from FEM is that the assembly must be repeated at all discrete frequencies by using the Do-Loop routine. By combining the advantage features of the DSM with those of the spectral analysis method (SAM), Beskos [124] introduced the fundamental concept of the spectral element method (SEM) for the first time in his 1978 work.

1.1.3 Spectral Analysis Method

The solution methods for the governing differential equations formulated in the time domain can be categorized into two major groups. The first group consists of the time-domain methods, such as the numerical integration methods and the modal analysis method, which is commonly
used for the vibration analysis. The second group consists of the frequency-domain methods. The spectral analysis method (SAM) is one of the frequency-domain methods most popularly reported in the literature. It is worth remembering that throughout this book, the word “frequency” is used to mean the Fourier transform of the “time” (that is, “time frequency”) rather than that of the “spatial” coordinate (that is, “space frequency”).

In SAM, the solutions to the governing differential equations are represented by the superposition of an infinite number of wave modes of different frequencies (or periods). This corresponds to the continuous Fourier transform of the solutions. This approach involves determining an infinite set of spectral components (or Fourier coefficients) in the frequency domain and performing the inverse Fourier transform to reconstruct the time histories of the solutions. The continuous Fourier transform is feasible only when the function to be transformed is mathematically simple and the inverse transform is the biggest impediment to most practical cases, especially when dealing with digitized experimental data measured through a modern data acquisition system. Thus, instead of using the continuous Fourier transform, the discrete Fourier transform (DFT) is widely used in practice.

The DFT is an approximation of the continuous Fourier transform. In contrast to the continuous Fourier transform, the solutions are represented by a finite number of wave modes of discrete frequencies and thus, as an enormous advantage, one can use the fast Fourier transform (FFT) algorithm to compute the DFT and its inversion economically and quickly. Thus, the use of the FFT algorithm makes it possible to efficiently take into account as many spectral components as are needed up to the highest frequency of interest. Accordingly, the DFT/FFT-based SAM can provide very accurate solutions, while reducing the computation cost and time significantly.

It is worth mentioning that firstly, although the DFT is the approximation of the continuous Fourier transform, the DFT processing itself is exact in the sense that it does allow the time signal to be regained exactly at discrete times [127], and secondly, although one uses a computer to accomplish the DFT/FFT-based spectral analysis, it is certainly not a numerical method in the sense that the analytical descriptions of the Fourier transforms are used in the computation.

1.1.4 Spectral Element Method

As shown in Figure 1.1, the spectral element method (SEM) can be considered as the combination of the key features of the conventional FEM, DSM and SAM. The key features
The key features of each method can be summarized as follows:

1. **Key features of FEM.** Meshing (spatial discretization) and the assembly of finite elements.
2. **Key features of DSM.** Exactness of the dynamic stiffness matrix formulated in terms of a minimum number of DOFs.
3. **Key features of SAM.** Superposition of wave modes via DFT theory and FFT algorithm.

It is worth mentioning that the above key features of both FEM and SAM have been of little interest in most of the existing literature associated with the classical DSM.

In SEM, exact dynamic stiffness matrices are used as the element stiffness matrices for the finite elements in a structure. To formulate an exact dynamic stiffness matrix for the classical DSM, the dynamic responses of a structure are usually assumed to be the harmonic solutions of a single frequency. However, for the SEM, the dynamic responses are assumed to be the superposition of a finite number of wave modes of different discrete frequencies based on the DFT theory. Accordingly, the computation of the exact dynamic stiffness matrix must be repeated at all discrete frequencies up to the highest frequency of interest. By virtue of the exactness of the spectrally formulated dynamic stiffness matrix, one-element modeling will suffice for a regular structure member. The SEM is an element method, just like the conventional FEM. Thus, the mesh refining can also be applied in the SEM when any geometric or material discontinuities in the spatial domain of concern exist, and also when any externally applied forces exist. The details of the spectral element formulation procedure and some examples are provided in Chapter 3.

The spectrally formulated exact dynamic stiffness matrix to be used in the SEM in conjunction with the SAM is known as the “spectral element matrix” in this book and in much of the existing literature. In some of the literature, it is often called a “spectral finite element matrix” or “spectral dynamic stiffness matrix.” Accordingly, the spectral element method is also often called the “spectral finite element method” and, sometimes, the “continuum method.” The finite length of the structure element represented by the spectral element matrix is called the “spectral element,” as the terminology “finite element” is commonly used in the FEM. The terminologies “spectral element” and “finite element” will be used interchangeably throughout this book without introducing any unnecessary confusion. The frequency-domain nodal DOFs specified on a spectral element are known as “spectral nodal DOFs.”

It should be pointed out that unfortunately the same terminology “spectral element method” or “SEM” has also been used for a different class of finite element methods developed in 1984 by Anthony Patera at MIT [135]. In fact, the word “spectral” for Patera’s SEM is not time-wise, but space-wise.

The SEM is stiffness formulated. Thus, the spectral elements can be assembled to form a global system matrix equation for the whole problem domain by using exactly the same assembly techniques as used in the conventional FEM. The global system matrix equation is then solved for the global spectral nodal DOFs, of course, repeatedly at all discrete frequencies. Finally, we use the inverse-FFT (IFFT) algorithm to compute the time histories of dynamic responses (time-domain solutions).

### 1.1.5 Advantages and Disadvantages of SEM

By virtue of the apparent advantages of the exact dynamic stiffness matrix used in conjunction with the DFT/FFT-based SAM, the major advantages of the SEM (although not all can be listed
here) may be summarized as follow:

1. **Extremely high accuracy.** In theory, the SEM provides exact frequency-domain solutions, such as the eigensolutions (natural frequencies and modes) and the frequency response functions (system transfer functions). The SEM may also provide extremely accurate time-domain solutions (the time histories of dynamic responses) by efficiently taking into account as many high frequency wave modes as are required by using the FFT algorithm. Accordingly, the problems solved by using the spectral element model can be utilized as benchmark problems for evaluating the accuracy and performance of a newly developed solution method.

2. **Smallness of the problem size and DOFs.** This is true because one-element modeling suffices for the representation of a regular structure member of any size, without any structural and material discontinuities inside.

3. **Low computation cost.** For computing time-domain responses, this is certainly true due to the smallness of problem size as well as due to the use of the FFT algorithm. The quick computation of the time-domain responses also enables us to treat the wave propagation simulation more realistically. However, it is worth mentioning that the most significant part of time consumption in SEM is for computing eigenfrequencies by using an iteration method of root finding.

4. **Effective to deal with frequency-domain problems.** This is the situation because the SEM itself is a frequency-domain solution method. Accordingly the SEM can be successfully applied to the following situations: (i) when the characteristics of a system (e.g., constitutive equation or internal damping) are dependent on frequency (e.g., viscoelastic materials), and (ii) when the boundary conditions of a system are specified in the frequency domain (e.g., the impedance boundary conditions for the fluid-structure interaction problems).

5. **Effective to deal with the non-reflecting boundary conditions of the infinite- or semi-infinite-domain problems.** This is achieved by using the semi-infinite spectral element (often called through-off element), which can be formulated by simply removing the wave modes reflected from the boundary at the infinite from the dynamic shape functions.

6. **Locking-free method.** The SEM does not exhibit the shear locking problems often raised in the conventional FEM, because the exact wave solutions to the governing equations are used as the dynamic shape functions in the SEM.

7. **Effective to deal with digitized data.** This is true because the FFT algorithm used in SEM is an efficient tool to deal with digitized data as follows: (i) the experimental data measured digitally through the analogue-to-digital converters, and (ii) the excitation forces measured or specified in the forms of numerical values at regular intervals of time rather than in the forms of analytical functions.

8. **The system transfer functions** (the inverse of global dynamic stiffness matrix) are the by-products of the spectral element analysis. Thus, it is very straightforward to perform the inverse problems, such as the system identification (e.g., identification of system parameters, boundary conditions, structural joints, or structural damages) and the force identification.

Despite the aforementioned advantages of SEM, there are also some disadvantages as follows:

1. The exact spectral element formulation is possible for the problems where the exact wave solutions to the governing equations are available. However, unfortunately the exact wave solutions are not always available for most complex and multi-dimensional problems. Thus,
the exact spectral element models developed to-date have been mostly for one-dimensional (1-D) problems (e.g., rods and beams) or some multi-dimensional problems that can be transformed into the equivalent 1-D problems (e.g., Levy-type plates).

If the exact spectral element model is not available for a problem, the approximate spectral element modeling approach can be applied by adopting approximate dynamic shape functions obtained by using the wavenumbers computed from the approximate dispersion relation or by using another appropriate approximation method. As the assumptions possibly made for the approximate spectral element modeling will be less severe than those usually made for the conventional FEM, the approximate spectral element models may still provide very accurate solutions.

2. As the SEM is a frequency-domain analysis method based on DFT/FFT techniques, it cannot be directly applied to time-variant, non-linear systems for which the principle of superposition does not hold. Thus, for the case of non-linear systems, for instance, one may need to use an iteration method by treating non-linear terms as the pseudo-forces.

3. In the SEM, the time-domain solutions are post-processed by convolving the transfer functions with external loadings in the frequency domain via the IFFT algorithm. Thus, even though the SEM guarantees exact frequency-domain solutions, it is not true for the time-domain solutions, because errors due to aliasing or leakage are inevitable in practice. As the aliasing or leakage-induced errors depend on the FFT conditions, special care must be taken in determining the successful FFT conditions.

1.2 Historical Background

The history of spectral analysis, which is also known as the Fourier analysis or frequency-domain analysis, began with the pioneering work “Theorie analytique de la chaleur (The analytical theory of heat)” published in 1822 by Joseph Fourier. He showed how an infinite series of sine and cosine functions can be used to analyse heat conduction in solids. Owing to the distrust in the use of series then, Fourier’s method did not gain acceptability during his lifetime. Since then, works by Dirichlet, Riemann and other mathematicians have resolved any doubts about the validity of the Fourier series, and spectral analysis has followed two major pathways: the continuous Fourier transform and the discrete Fourier transform (DFT).

The continuous Fourier transform has the drawback that it involves the integration of a signal (function) specified analytically over the time period, which is not always easy to carry out. This is especially true when the signal is experimentally measured in a digitized data form. Thus, the continuous Fourier transform is usually transformed in a discrete form by replacing the integration by the summation of a finite number of signal values sampled at regular intervals of time. The discrete version of the continuous Fourier transform is the DFT and it is a very effective means for transforming a periodic signal sampled in the time domain into an expression of the data in the frequency domain and vice versa.

The major difficulty in using the DFT is the requirement of extensive computation time. Although some techniques and ideas to reduce the computation time appeared early in the twentieth century, it was not until 1965 when James W. Cooley (IBM researcher) and John W. Tukey (Princeton faculty member) developed a computational algorithm that is now known as the fast Fourier transform (FFT). The FFT is simply a remarkably efficient algorithm for computing the DFT and its inversion. The FFT algorithm reduces the number of arithmetical
operations for computing the DFT from the order $N^2$ to the order $N \log_2 N$, where $N$ is the number of samples. This is a dramatic reduction for large values of $N$: the reduction ratio is over 100 : 1 when $N = 1024$ and 400 : 1 when $N = 4096$. Historically it has been recognized that Carl Friedrich Gauss developed the same FFT algorithm around 1805, but without getting any attention until Cooley and Tukey rediscovered it, independently, 160 years later. The FFT rediscovery has made spectral analysis highly efficient, with widespread applications to digital signal processing and certain areas of engineering analysis. However, the applications of DFT/FFT-based spectral analysis to the dynamic analysis of structures are fairly recent and mostly limited to 1-DOF or multi-DOFs discrete systems. Extensive reviews on the subject and related applications can be found in the books by Ginsberg [49], Humar [58] and Newland [127].

In classical DSM, most researchers have been predominantly interested in formulating exact dynamic stiffness matrices to obtain more accurate natural frequencies and modes, rather than in developing it further in the form of the SEM by combination with the DFT/FFT-based SAM. As mentioned in the previous section, the exact dynamic stiffness matrix for the classical DSM is usually formulated from exact wave solutions obtained by assuming the harmonic solutions of a single frequency to the governing differential equations.

Over the last seven decades, there have been a large number of publications related to the formulation and applications of exact dynamic stiffness matrices. The 1941 work by Kolousek [70] is probably the first to derive the dynamic stiffness matrix for the Bernoulli–Euler beam. Przemieniecki [142] introduced the formulation of the frequency-dependent mass and stiffness matrices for both bar and beam elements in his book. In contrast to the conventional finite element mass and stiffness matrices, which result in the linear eigenproblems, the exact dynamic stiffness matrices result in transcendental eigenproblems, the coefficients of which are the transcendental functions of frequency. Thus, a drawback of the DSM, is that it is not an easy task to compute all natural frequencies (without missing any one within a specified frequency range of interest) accurately by solving the transcendental eigenvalue problems. In 1971, this difficulty was resolved by Wittrick and Williams [169] by developing the well-known Wittrick–Williams algorithm for automatic calculation of undamped natural frequencies [169]. The Wittrick–Williams algorithm has certainly enhanced the applicability of the DSM.

Since then numerous exact dynamic stiffness matrices have been developed for various structures. The extensive literature up to 1976 and 1983 can be found in Ref. [4] and Ref. [115], respectively. In addition, the extensive literature up to 1993, together with elegant theories for the dynamic stiffness and substructures, can be found in a book by Leung [108]. Although not all published literature can be cited herein, exact dynamic stiffness matrices have been developed mostly for the 1-D structures including the Timoshenko beams with or without axial force [26, 30, 31, 54, 159], extension-bending-torsion coupled beams [10, 16, 18, 19, 47, 52], non-uniform beams [17, 56], curved beams and springs [61, 78, 107, 136], Rayleigh–Timoshenko beams [4, 115], composite beams [20, 44], sandwich beams [14], aircraft wing [13], the beams on elastic foundation [167], axially moving string [77], and other structures [106, 110].

Although the history of the DSM spans about seven decades, it was not until 1978 when Beskos [124] introduced the fundamental concept of the SEM for the first time. He derived an exact dynamic stiffness matrix for the beam element and employed FFT for the dynamic analysis of plane frame-works. His work was further improved and generalized in his 1982 work [148], while claiming that his 1978 work [124] was the first to employ FFT for the dynamic analysis of plane frame-works in conjunction with the DSM. It was in 1988 that Doyle
at Purdue University [38] published his first work on the formulation of the spectral element for the longitudinal wave propagation in rods. He seemed to be the first to use the terminology “spectral element method” in his 1992 work [146] for the DFT/FFT-based spectral element analysis approach. Doyle and his colleagues have applied the SEM mostly to wave propagations in structures. A comprehensive list of the works by Doyle’s research group and other researchers up to 1997 can be found in a book by Doyle [40]. On the other hand, the present author and his students have extensively applied the SEM to various problems in structural dynamics and the outcome of this research [33, 67, 72, 80–82, 84–101, 103, 104, 130] is the major source of the material presented in this book.
Spectral Analysis of Signals

In mathematics, it is well known that an arbitrary periodic signal can be decomposed into many harmonic components and, in the reverse transform, the signal can be synthesized by superposing all its harmonic components. This is the basis of spectral (Fourier) analysis. For a signal given in functional form, we may evaluate its harmonic components by analytically evaluating the continuous Fourier transform integral. However, if the signal function is so complicated, we may decide to use numerical integration instead. In this case we need to know the signal values only at a discrete set of instances. A similar situation arises in most experimental measurements, because modern data acquisition systems based on analogue-to-digital converters store digitized data sampled at discrete, uniformly spaced intervals. The discrete Fourier transform (DFT) enables us to deal with such discrete sampled data from a signal. The important advantage of DFT theory is that it allows us to use the fast Fourier transform (FFT), which is an extremely efficient algorithm for computing the DFT. To formulate spectral element models, the exact solutions to the governing equations are in general represented in the spectral forms by using the DFT. Thus this chapter provides a brief review of the theories and applications associated with DFT and FFT.

2.1 Fourier Series

If \( x(t) \) is a continuous periodic function of time \( t \), with period \( T \), we can always represent it in the form of a Fourier series as

\[
x(t) = a_0 + 2 \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t)
\]

(2.1)

where \( a_0, a_n \) and \( b_n \) are constant Fourier coefficients given by

\[
a_0 = \frac{1}{T} \int_0^T x(t) \, dt
\]

\[
a_n = \frac{1}{T} \int_0^T x(t) \cos \omega_n t \, dt
\]

(2.2)

\[
b_n = \frac{1}{T} \int_0^T x(t) \sin \omega_n t \, dt
\]
In the above equations, $\omega_n = n\Delta \omega$ is the frequency of the $n$th harmonic and $\Delta \omega$ is the frequency spacing (fundamental frequency) defined by

$$\Delta \omega = \frac{2\pi}{T} \quad (2.3)$$

The cosine and sine functions can be decomposed by using the Euler identities as follows:

$$\cos \omega_n t = \frac{1}{2} (e^{i\omega_n t} + e^{-i\omega_n t})$$

$$\sin \omega_n t = \frac{i}{2} (e^{i\omega_n t} - e^{-i\omega_n t}) \quad (2.4)$$

where $i = \sqrt{-1}$ is an imaginary unit. By substituting Equation 2.4 into Equation 2.1, we get an expression as follows

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n - ib_n)e^{i\omega_n t} + \sum_{n=1}^{\infty} (a_n + ib_n)e^{-i\omega_n t} \quad (2.5)$$

Defining

$$X_n = a_n - ib_n \quad (2.6)$$

we can rewrite Equation 2.5 into a compact form

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{i\omega_n t} \quad (2.7)$$

where $X_n$ is the $n$th Fourier coefficient for a continuous periodic function $x(t)$. By substituting Equation 2.2 into Equation 2.6 and then by applying Equation 2.4, we obtain

$$X_n = \frac{1}{T} \int_0^T x(t)e^{-i\omega_n t} dt (n = 0, \pm 1, \pm 2, \ldots, \pm \infty) \quad (2.8)$$

Equations 2.7 and 2.8 constitute the Fourier transform pair for the continuous periodic function $x(t)$. Equation 2.7 is referred to as the synthesis equation and Equation 2.8 as the analysis equation for a continuous periodic function $x(t)$. In summary, the Fourier transform decomposes a periodic function into a discrete spectrum of its frequency components, and the inverse transform synthesizes a function from its spectrum of frequency components.

### 2.2 Discrete Fourier Transform and the FFT

#### 2.2.1 Discrete Fourier Transform (DFT)

Although $x(t)$ is a continuous periodic function of time, with period $T$, it is often the case that only sampled values of the function are available. This is true, for instance, when a time history $x(t)$ is experimentally measured in the form of digitized data taken at equally spaced instances by using a modern data acquisition system. Let $N$ be the number of samples in the
time domain and, as shown in Figure 2.1, assume that they are sampled at equally spaced time intervals given by

$$\Delta t = \frac{T}{N}$$  \hspace{1cm} (2.9)

We express the $N$ samples in a discrete time series $\{x_r\}$, where $x_r = x(t_r)$, $t_r = r\Delta t$ and $r = 0, 1, 2, \ldots, N - 1$. By using the discrete time series $\{x_r\}$, the integral in Equation 2.8 can be approximately replaced by the summation as

$$X_n = \frac{1}{T} \sum_{r=0}^{N-1} x_r e^{-j\omega_n t_r \Delta t}$$ \hspace{1cm} (2.10)

By using Equation 2.9, Equation 2.10 can be rewritten as

$$X_n = \frac{1}{N} \sum_{r=0}^{N-1} x_r e^{-j\omega_n t_r}$$ \hspace{1cm} (2.11)

Equation 2.11, the process of converting from discrete time series $\{x_r\}$ to frequency data is called the discrete Fourier transform (DFT). The frequency data $X_n$ are called the DFT coefficients.

As the discrete time series $x(t_r)$ are real, we can readily show from Equation 2.11 that

$$X_0 = \text{real}$$ \hspace{1cm} (a)

$$X_{N/2} = X_{-N/2} = \text{real}$$ \hspace{1cm} (b)

$$X_{-n} = X^*_n$$ \hspace{1cm} (c)

$$X_{N/2+n} = X_{-N/2+n} = X^*_{N/2-n}$$ \hspace{1cm} (d)

$$X_{N-n} = X^*_n = X^*_{n-N}$$ \hspace{1cm} (e)

$$X_{N+n} = X_n$$ \hspace{1cm} (f)
by using the following relationships

\[ e^{\pm j\omega_p qtr} = e^{\pm j\omega_p qtr} e^{\pm j\omega_p qtr} \quad (a) \]
\[ e^{\pm j\omega_p qtr} = e^{\pm j\omega_p qtr} \quad (b) \]
\[ e^{\pm j\omega_{N/2} qtr} = (-1)^r \quad (c) \]
\[ e^{\pm j\omega_{N} qtr} = 1 \quad (d) \]

The asterisk symbol (*) is commonly used in this book to denote the complex conjugate of a complex number.

The synthesis Equation 2.7 can also be approximated by truncating the summation index \( n \) as

\[ x_r \approx \sum_{n=-N/2}^{N/2-1} X_n e^{j\omega_n qtr} \quad (2.14) \]

We can rewrite the right-hand side of Equation 2.14 as

\[ x_r \approx \sum_{n=-N/2}^{-1} X_n e^{j\omega_n qtr} + \sum_{n=0}^{N/2-1} X_n e^{j\omega_n qtr} \quad (2.15) \]

By using Equations 2.12 and 2.13, the first summation of Equation 2.15 can be expressed as

\[ \sum_{n=-N/2}^{-1} X_n e^{j\omega_n qtr} = \sum_{k=0}^{N/2-1} X_{N/2+k} e^{j\omega_{N/2+k} qtr} = \sum_{k=0}^{N/2-1} X_{N/2+k} e^{j\omega_{N/2+k} qtr} = \sum_{n=N/2}^{N-1} X_n e^{j\omega_n qtr} \quad (2.16) \]

Substitution of Equation 2.16 into Equation 2.15 yields

\[ x_r \approx \sum_{n=0}^{N-1} X_n e^{j\omega_n qtr} \quad (2.17) \]

This is the inverse discrete Fourier transform (IDFT), which is the process of converting DFT coefficients into discrete time series data.

Equations 2.11 and 2.17 constitute a DFT pair. The range of the Fourier coefficients \( X_n \) is limited to \( n = 0 \) to \( N - 1 \) in order to maintain the symmetry of the DFT pair. Although the DFT pair, Equations 2.11 and 2.17, is the approximation of Equations 2.8 and 2.7, respectively, it does allow all discrete time series \( \{ x(t_r) \} \) to be regained exactly. In other words, the DFT processing is indeed exact. This important fact can be readily verified by showing that the discrete time series \( \{ x(t_r) \} \) are exactly regained by simply substituting \( X_n \) from Equation 2.11 into the right-hand side of Equation 2.17.