Dynamic sliding mode control design using attracting ellipsoid method

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Abstract

A methodology for the design of sliding mode controllers for linear systems subjected to matched and unmatched perturbations is proposed. It is considered that the control signal is applied through a first-order low-pass filter. The technique is based on the existence of an attracting (invariant) ellipsoid such that the convergence to a quasi-minimal region of the origin using the suboptimal control signal is guaranteed. The design procedure is given in terms of the solution of a set of Matrix Inequalities. A benchmark example illustrating the design is given.

1. Introduction

The design of controllers for dynamical systems in the presence of bounded external perturbations is one of the most important problems in the control theory. The Sliding Mode (SM) control is a very efficient methodology to control dynamic plants operating under uncertain conditions (see, Utkin, Guldner, and Shi (1999)). One of the main features of SM controllers is their insensitivity in the presence of matched perturbations (perturbations that lie on the same subspace than the control). The designing of a controller that effectively reduces the effect of the non-matched perturbations is still an open problem. A controller ensuring local practical stability for systems with matched and unmatched uncertainties is designed using a technique of eigenvalues assignment in Hui and Zak (1993), Edwards and Spurgeon (1998) study the design of the SM controller parameters by the minimization of a quadratic functional and the robust eigenvalue assignment. Hsu, Costa, and Cunha (2003) design a model-reference SM unit-vector control to a linear system with relative degree one. Andrade-Da Silva, Edwards, and Spurgeon (2009) present an SM Output Feedback Controller to stabilize an uncertain linear system in the presence of perturbations. The solution is presented as an optimization problem and solved using LMI’s (see, Boyd, Ghaoui, Feron, and Balakrishnan (1994), Poznyak (2008)). Castanos and Fridman (2006)

provide a controller minimizing the effect of unmatched perturbations by means of a combination of integral SM and H∞ techniques. A unit control is proposed to stabilize a perturbed linear system in the presence of unmatched perturbations by Choi (2008).

Dynamic SM controllers (see Sira-Ramirez (1993) and references therein), realized for example, by the inclusion of a first-order smoothing filter for the discontinuous control, allow the design of robust controllers in combination to smoothing the control signals. The effect of the introduction of this dynamics have been studied by Fridman (2001), Fridman (2002).

The invariant subspace approach (see Kurzhanski and Valyi (1996), Blanchini (1999) and the references therein) is a technique used for the design of controllers for perturbed systems. A particular technique applied for the suppression of bounded perturbations in linear systems is the invariant ellipsoid method (Nazin, Polyak, & Topunov, 2007). Polyakov and Poznyak (2009) present an innovative approach applying the invariant ellipsoid method to minimize the effect of unmatched perturbations on linear systems.

In this article a methodology for the design of the time constant, the surface and gains of a dynamic SM controller is presented. As a result the computed controller and filter parameters guarantee the global convergence of state and control to the suboptimal attracting ellipsoid around the origin. As far as the authors know, it is the first paper providing a methodology for the design of the parameters of a dynamic SM controller, represented by a first-order smoothing filter, to guarantee suboptimal solution to the global stabilization problem. A computational version of the method is given as the solution of a set of LMI’s.

2. Problem statement

Consider the following perturbed linear system

\[ \dot{x}(t) = Ax(t) + Bu(t) + Df(t, x) \]  

(1)
where \( x(t) \in \mathbb{R}^n \) is the state variable, \( u(t) \in \mathbb{R}^m \) is the control inputs vector, and \( f(t, x) \in \mathbb{R}^n \) is the vector of perturbations. The short notation \( x, u, f \) will be introduced for simplicity. Differential equations are understood in the Filippov sense (Filippov, 1988) in order to provide the possibility to use discontinuous signals in controls. Filippov solutions coincide with the usual solutions, when the right-hand sides are Lipschitzian. It is assumed also that all considered inputs allow the existence of solutions and their extension to the whole half-axis \( t \geq 0 \).

Let the sliding variable \( \sigma \in \mathbb{R}^m \) be given by

\[
\sigma = Cx + u, \quad C \in \mathbb{R}^{m \times n}.
\]

(2)

Suppose that the sliding mode controller is applied through the following dynamic

\[
\dot{u} = -H(u + M \text{Sign} \sigma) + \xi
\]

(3)

\[
M = \text{diag} [M_1, M_2, \ldots, M_m]
\]

(4)

then (9) to (11) ensure the convergence of \( \sigma \) to zero in finite time.

Proof. Consider \( V_\sigma = \frac{1}{2} \sigma^T \sigma \). Which derivative is \( \dot{V}_\sigma = \sigma^T (CAx + CB - H)u + CDF - HMsign(\sigma) + \xi \). Since \( |\sigma| \text{sign}\sigma = \sum_{i=1}^{n} |\sigma_i| \geq ||\sigma||. ||\sigma|| \leq ||V|| \), \( \forall \dot{V} \in \mathbb{R}^m \) the inequality \( \dot{V}_\sigma \leq 0 \) is satisfied if

\[
||CAx + (CB - H)u + CDF + \xi|| < ||H\bar{M}||
\]

(16)

where \( \bar{M} = (M_1, M_2, \ldots, M_m)^T \). Define \( Y = \xi, X = Cax + (CB - H)u + CDF \). Raising to the square both sides of (16) and introducing the \( A \)-Inequality (Poznyak, 2008) for the left hand side \( ||X + Y||^2 \leq X^T(i + \lambda)X + Y^T(i + \lambda)^{-1}Y \) valid for the defined \( X, Y \in \mathbb{R}^m \), and \( 0 < \lambda = \lambda^T \in \mathbb{R}^{m \times m} \). Inequality (16) is implied by:

\[
\begin{pmatrix}
\dot{x}^T \\
u^T
\end{pmatrix} \begin{pmatrix}
\mu^2 N^T(i + \lambda)N & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
x^T \\
u^T
\end{pmatrix} < \left( \sum_{i=1}^{m} h_i^2 \alpha_{ni} \right) I_n - \left( \sum_{i=1}^{m} \alpha_{ni} \right) I_m
\]

(17)

On the other hand, the perturbations satisfy the inequality (10). Using S-theorem (Poznyak, 2008), the Eq. (10) implies (17) if there exist \( \tau_1 \) such that the following inequalities are satisfied:

\[
\begin{pmatrix}
\sum_{i=1}^{m} h_i^2 \alpha_{ni} \\
0
\end{pmatrix} I_n - \frac{\tau_1}{f_0} Q_{li} \begin{pmatrix} 0 \\
0
\end{pmatrix} \geq 0
\]

(18)

Applying the Schur complements (see, for example Poznyak (2008)) we obtain inequality (11). Notice that while (11) ensures the convergence of \( \sigma \) to zero, the remaining inequalities (12)–(15) correspond to the controller restrictions (9). \( \square \)
Once the systems begins to slide on the surface, the equality \( \sigma = 0 \) is satisfied, it implies that the equivalent control becomes \( u = -C \dot{x} \). Define the extended variable \( z^T = [x^T \ u^T] \) and introduce the following Lyapunov function
\[
V_z = z^T P_z z, \quad P_z = \text{diag}(P_x, P_u).
\]

Theorem 2. Let \( (C^0, \alpha^0_1, H^0, \alpha^0_2, \alpha^0_3, \tau^0_1, \tau^0_2, \tau^0_4, \gamma^0, \beta^0, P^0_x, P^0_u) \) be any solution of the set of inequalities (11)–(15),
\[
\beta \geq \tau_2 \left( \sum_{i=1}^{m} \alpha_i \right) + \tau_4 + \tau_5 \tag{19}
\]

\[
F_1 = (A - BC)^T P_x + P_x (A - BC) + \gamma P_x
\]

\[
+ \tau_2 \left( \sum_{i=1}^{m} \alpha_i \right) I_m - \tau_3 P_x H - \tau_3 P_x H - \tau_3 I_m
\]

\[
F_2 = \gamma P_u + \tau_2 \left( \sum_{i=1}^{m} \alpha_i \right) I_m - \tau_3 P_u H - \tau_3 I_m
\]

the controller (3), (8) and (2) designed with this parameter selection guarantees the existence of an attracting invariant ellipsoid of \( z \) (\( \epsilon(\theta) = z^T \theta z \), \( \theta \)).

Proof. The derivative of \( V_z \) along \( z \) is given by the following equality
\[
\dot{V}_z = x^T P_x [(A - BC)x + \dot{x}] + [(A - BC)x + \dot{x}]^T P_x y
\]

\[
+ u^T P_u [\xi - H(u + M \text{sign} \sigma)] + \xi H(u + M \text{sign} \sigma) P_y u.
\]

Define the variable \( w^T = \left( x^T \ u^T \right) \cdot (M \text{sign} \sigma)^T \xi^T \). Adding and subtracting the term \( \tau_2 (M \text{sign} \sigma)^T \xi^T \) to Eq. (21), that can be written as:
\[
\dot{V}_z = w^T \bar{S}_1 w + \tau_2 \left( \sum_{i=1}^{m} \alpha_i \right)
\]

where
\[
\bar{S}_1 = \begin{pmatrix} F_3 & 0 & P_x D & 0 & 0 \\ 0 & F_4 & 0 & -P_u H & P_u \\ 0 & 0 & -P_u H & 0 & 0 \\ 0 & 0 & 0 & P_u & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
F_3 = (A - BC)^T P_x + P_x (A - BC) + \tau_2 \left( \sum_{i=1}^{m} \alpha_i \right) I_m
\]

\[
F_4 = \tau_2 \left( \sum_{i=1}^{m} \alpha_i \right) I_m - \tau_3 P_x H - \tau_3 P_u H.
\]

The variable \( z \) will be asymptotically ultimately bounded if the following inequality holds \( \dot{V}_z \leq -\gamma V_z + \beta \).

Equivalently
\[
\dot{V}_z + \gamma V_z \leq \beta.
\]

The ultimately bounded region can be represented as an attracting invariant ellipsoid defined by \( \lim_{t \to \infty} z^T P_z z \leq 1 \). In terms of the new variable \( w \), Eq. (23) can be written as
\[
\dot{V}_z + \gamma V_z = w^T \bar{S}_1 w + \tau_2 \left( \sum_{i=1}^{m} \alpha_i \right) \leq \beta
\]

where \( \bar{S}_2 = \text{diag}(P_x, P_u, \theta_0, 0_2, (q+2m) \times (q+2m)) \).

After a finite time transient the system trajectories reach the sliding surface, since that moment, the equality \( \sigma = u + C \dot{x} = 0 \) is satisfied. So, it is considered that:
\[
\|u + C \dot{x}\|^2 = w^T \bar{S}_4 w = 0
\]

As a result of the convergence of \( \sigma \) to zero, and applying the Finister Lemma (Poznyak, 2008) to Eqs. (24) and (25), we obtain that there exist \( \tau_3 \geq 0 \) such that:
\[
w^T (\bar{S}_1 + \gamma \bar{S}_2 - \tau_3 \bar{S}_4) w + \tau_2 \left( \sum_{i=1}^{m} \alpha_i \right) \leq \beta.
\]

Applying the S-Theorem (see Poznyak (2008)) to combine Eqs. (7), (10) and (26), we obtain that (23) is satisfied if there exist \( \tau_4, \tau_5 \geq 0 \) such that the inequality (20) is satisfied simultaneously with \( \beta \geq \tau_2 \left( \sum_{i=1}^{m} \alpha_i \right) \geq \tau_4 + \tau_5 \). This inequality is implied by (19).

Theorem 3. Let \( (C^0, \alpha^0_1, H^0, \alpha^0_2, \alpha^0_3, \tau^0_1, \tau^0_2, \tau^0_4, \gamma^0, \beta^0, P^0_x, P^0_u) \) be the solution of the optimization problem:

\[
\text{maximize} \quad \frac{\gamma}{\beta} \text{tr}(P_2).
\]

subject to (11)–(15), (19) and (20). The controller (3) designed with this selection of parameters ensures the existence of a quasi-minimal attracting invariant ellipsoid \( \epsilon(\bar{p}^6 P_x) \).

Proof. The attracting invariant ellipsoid takes the form \( \epsilon(\bar{p}^6 P_x) = z^T \bar{p}^6 P_x z \) the rest of the proof is a consequence of the ellipsoid definition.

4. Computational implementation

The above mentioned results provide the quasi-optimal solution to the stabilization problem. However, the resulting inequalities contain at least bilinear terms which increases the difficulty to find a computational solution. In this section the inequalities are modified to obtain an implementable form of the design process.

Let introduce the following variables \( X = P_x^{-1}, Y = CP_x^{-1}, R_u = P_u H, \alpha = \sum_{i=1}^{m} \alpha_i, \alpha_k = \sum_{i=1}^{m} \alpha_i, \bar{\alpha} = \sum_{i=1}^{m} \alpha_i \).

Theorem 4. Let \( (X^0, Y^0, R^0_u, \bar{\alpha}^0, \bar{\alpha}^0_k, \bar{\alpha}^0_k, \bar{\alpha}^0_k, \bar{\alpha}^0_k, \bar{\alpha}^0_k, \bar{\alpha}^0_k, \bar{\alpha}^0_k) \) be any solution of the set of inequalities:

\[
\begin{pmatrix} \tilde{W}_{11} & 0 & 0 & 0 & 0 & A^T \\ 0 & mh_{max} \bar{\alpha} G_2 & 0 & P_u B^T & -R_u \\ 0 & 0 & \tau_1 \bar{Q} & D^T & 0 \\ A & B P_u & 0 & \tau_3 X & 0 \\ 0 & 0 & -R_u & 0 & 0 \end{pmatrix} > 0
\]

\[
\tilde{W}_{11} = mh_{min} G_2 I_n - \frac{\tau_1}{\tau_0} Q_0,
\]
For the sliding surface, it is possible to introduce the following estimation
\[
\begin{pmatrix}
1 - \tau_6 Y^T & D & 0 & 0 \\
-\tau_6 Y & F_2 & 0 & -R_u & P_u \\
D^T & 0 & -\tau_6 Q_f & 0 & 0 \\
0 & -R_u & 0 & -\tau_6 I_m & 0 \\
0 & P_u & 0 & 0 & -\tau_6 K_f
\end{pmatrix} \leq 0
\]  
(28)

Third equation of (31) is obtained using Schur’s complements to the inequality \( P_u P_u \leq G_2 \). Using the last inequality, Eq. (27) implies (36).

Let define a new nonsingular matrix \( T_2 = \text{diag}(X, L) \). Multiplying on the left and on the right of (35) by \( T_2 \) we obtain
\[
\begin{pmatrix}
1 - \tau_6 X^{-1} & 0 & 0 & 0 \\
0 & L^{-1} & 0 & 0 \\
0 & 0 & \tau_6 Q_f & 0 \\
0 & 0 & 0 & \tau_6 I_m
\end{pmatrix} \geq 0
\]  
(30)

Using again the Schur’s complements we obtain (30).

Let define an other nonsingular matrix \( T_3 = \text{diag}(X, I_{m_1}) \). Multiplying on the left and on the right of Eq. (20) we obtain:
\[
\begin{pmatrix}
X F_1 & -\tau_6 X C^T & 0 & 0 & 0 \\
-\tau_6 C X & F_2 & 0 & -P_u H & P_u \\
D^T & 0 & -\tau_6 Q_f & 0 & 0 \\
0 & -H P_u & 0 & -\tau_6 I_m & 0 \\
0 & P_u & 0 & 0 & -\tau_6 K_f
\end{pmatrix} \leq 0.
\]  
(35)

The first and second inequalities of (31) are obtained from the Schur’s complements application to the inequality \( XX \leq G_1 \). As a consequence of the above defined matrix \( G_1 \) and using the inequality \( 0 \leq \tau_6 C^T C \), the inequality \( XX \leq G_1 \) is satisfied, hence, inequality (20) is implied by (28). Eqs. (32)–(34) are obtained from the restrictions of the controller gains and the corresponding definitions of the auxiliary variables. 

For each set of scalar parameters \( \bar{\alpha}_5, \bar{\alpha}_u, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4, \bar{\alpha}_5, \bar{\alpha}_6, \bar{\alpha}_7, \bar{\alpha}_8, \bar{\alpha}_9, \bar{\alpha}_{10} \), the conditions presented in Theorem 4 can be solved as a set of Linear Matrix Inequalities.

5. EXAMPLE

Consider the linear system \( \dot{x} = A x + B u + D f \) with the matrices \( A, B, C \) be defined as.
\[
A = \begin{bmatrix}
10 & 2 & 40 \\
3 & -2 & 20 \\
-5 & 3 & 100
\end{bmatrix}, \quad B = \begin{bmatrix}
3 \\
10 \\
6
\end{bmatrix},
\]
\[
D = \begin{bmatrix}
1 & 0 \\
0 & -1 \\
1 & 0
\end{bmatrix}.
\]

The perturbation \( f \) is given by
\[
f = \begin{bmatrix}
0.0028 \cos(0.4 t) - 0.0879 \sin(0.4 t) \\
0.0499 \cos(0.4 t) + 0.0049 \sin(0.4 t)
\end{bmatrix}
\]
and satisfies the following inequality
\[
f^T \begin{bmatrix}
130 & 15 \\
15 & 400
\end{bmatrix} f \leq 1 + \gamma^2 \begin{bmatrix}
0.02 & 0 & 0 & 0 \\
0 & 0.0001 & 0 & 0 \\
0 & 0 & 0 & 0.0001
\end{bmatrix} x.
\]  
(36)

Let us consider the sliding mode controller (3). The dynamic perturbation in the actuator is \( \xi = 0.05 \sin(0.9 t) \sin(0.4 t) - 0.0479 \sin(0.4 t) \), with constant \( K_f = 100 \). The controller restrictions are given by \( h_{\min} = 0.01 \). The controller gains \( H = 0.0135 \), \( \alpha = 5515.2, \alpha_8 = 5541345.8296, \alpha_9 = 6.5335 \) and sliding surface \( \sigma = -0.0015 \bar{x}_1 + 0.0024 \bar{x}_2 + 0.0027 \bar{x}_3 + u \), designed according to Theorem 4, were obtained with the selection of parameters \( \tau_1 = 0.000055457, \tau_2 = 0.00000084229, \tau_3 = 28920.4085, \tau_4 = 0.00123121, \tau_5 = 198337968, \tau_6 = 981009.4666, \gamma = 0.1238 \). The problem was solved with the quasi-optimal solution \( \tau P_f = 218.53 \).

Let the initial conditions for the system be \( x(0) = [3 - 2 1]^T \). The convergence of the state trajectories and the control...
signal are shown in Fig. 1, a zoom in on the convergence region is also presented in that Figure. Notice that even in the presence of perturbations, the trajectories converge to a small bounded region of the origin. In Fig. 2 the convergence of the trajectories to the quasi-optimal invariant ellipsoid $\varepsilon(\frac{1}{\beta} P_\beta)$ is presented.

6. Conclusions

The attracting invariant ellipsoid method is applied to the solution of the global stabilization problem of perturbed linear systems. The control signal is injected to the system through a first-order dynamic. The constants, surface and gains of the dynamic sliding mode controller are designed to reduce the effect of unmatched perturbations. The suboptimal solution to the stabilization problem is given as the computational solution of a set of Linear Matrix Inequalities. A benchmark example is presented to illustrate the workability of the method.

References


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Abstract—The super-twisting second-order sliding-mode algorithm is modified in order to design a velocity observer for uncertain mechanical systems. The finite time convergence of the observer is proved. Thus, the observer can be designed independently of the controller. A discrete version of the observer is considered and the corresponding accuracy is estimated.

Index Terms—Nonlinear observers, sliding modes.

I. INTRODUCTION

The design of observers for the mechanical systems with Coulomb friction is important for the following reasons:

• linear observers do not achieve adequate performance for such systems;

• model-based observers are usually restricted to the cases when the model is exactly known;

• high-gain differentiators [2] are not exact with any fixed finite gain and feature the peaking effect with high gains: The maximal output value during the transient grows infinitely as the gains tend to infinity (see, for example, [3], [5], [12], [15], and [16]).

The sliding mode observers are widely used due to the finite-time convergence, robustness with respect to uncertainties and the possibility of uncertainty estimation (see, for example, the bibliography in the recent tutorials [3], [5], and [12]). A new generation of observers based on the second-order sliding-mode algorithms has been recently developed. In particular, asymptotic observers [13] and the asymptotic observer for systems with Coulomb friction [1], [11] were designed based on the second-order sliding-mode. These observers require the proof of a separation principle theorem due to the asymptotic convergence of the estimated values to the real ones.

A robust exact differentiator [9] featuring finite-time convergence was designed as an application of the super-twisting algorithm [8]. Its implementation does not need the separation principle to be proved. These differentiators were, for example, successfully applied in [14], [4], and [10]. A new differentiator [7] was developed, based on it. Straightforward application of such a differentiator does not benefit from the knowledge of a mathematical model of the process. If such a model is known, or the system parameters and uncertainties can be estimated (which is common for the case of mechanical systems with Coulomb friction), it is reasonable to design a system-specific observer.

An observer is proposed in this paper, which reconstructs the velocity from the position measurements, using the modification of the second-order sliding-mode super-twisting algorithm [8] with finite-time convergence. The separation principle theorem is trivial in this case, and the observer can be designed separately from the controller. Only partial knowledge of the system model is required. The discrete version of the of the proposed observer is considered, and the corresponding accuracy of the proposed observer is estimated.

II. PROBLEM STATEMENT

The general model of second-order mechanical systems has the form

\[ M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + P(q) + G(q) + \Delta(t, q, \dot{q}) = \tau \]  

(1)

where \( q \in \mathbb{R}^n \) is a vector of generalized coordinates, \( M(q) \) is the inertia matrix, \( C(q, \dot{q}) \) is the matrix of Coriolis and centrifugal forces, \( P(q) \) is the Coulomb friction, which possibly contains delay terms depending on \( q \), \( G(q) \) is the term of gravitational forces, \( \Delta(t, q, \dot{q}) \) is an uncertainty term and \( \tau \) is the torque produced by the actuators. The control input \( \tau \) is assumed to be given by some known feedback function. Note that \( M(q) \) is invertible, since \( M(q) = M' \) is strictly positive definite. Also, other terms are supposed to be uncertain, but the corresponding nominal functions \( M_n(q), C_n(q, \dot{q}), P_n(q), G_n(q) \) are assumed known.

Introducing the variables \( x_1 = q, x_2 = \dot{q}, u = \tau \), the model (1) can be rewritten in the state-space form

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f(t, x_1, x_2, u) + \xi(t, x_1, x_2, u) \\
g(x) &= x_1 
\end{align*} \]  

(2)

where the nominal part of the system dynamics is represented by the function

\[ f(t, x_1, x_2, u) = -M^{-1}(x_1) \times [C_n(x_1, x_2)x_2 + P(x_2) + G_n(x_1) - u] \]
containing the known nominal functions \( M_n, C_n, G_n, P \), while the uncertainties are concentrated in the term \( \xi(t, x_1, x_2, u) \). The solutions to the system (2) are understood in Filippov’s sense [6]. It is assumed that the function \( f(t, x_1, x_2, U(t, x_1, x_2)) \) and the uncertainty \( \xi(t, x_1, x_2, U(t, x_1, x_2)) \) are Lebesgue-measurable and uniformly bounded in any compact region of the state–space \( x_1, x_2 \).

The task is to design a finite-time convergent observer of the velocity \( \hat{q} \) for the original system (1), when only the position \( q \) and the nominal model are available. In other words, the state \( x_2 \) of the system (2) is to be observed, while only the state \( x_1 \) is available. Only the scalar case \( x_1, x_2 \in \mathbb{R} \) is considered for the sake of simplicity. In the vector case, the observers are constructed in parallel for each position variable \( x_{1,j} \) in exactly the same way.

III. OBSERVER DESIGN

The proposed super-twisting observer has the form

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 + z_1 \\
\dot{x}_2 &= f(t, x_1, x_2, u) + z_2
\end{align*}
\]

where \( \dot{x}_1 \) and \( \dot{x}_2 \) are the state estimations, and the correction variables \( z_1 \) and \( z_2 \) are output injections of the form

\[
\begin{align*}
z_1 &= \lambda \|x_1 - \hat{x}_1\|^{1/2} \text{sign}(x_1 - \hat{x}_1) \\
z_2 &= \alpha \text{sign}(x_1 - \hat{x}_1).
\end{align*}
\]

It is taken for the definiteness that at the initial moment \( \dot{x}_1 = x_1 \) and \( \dot{x}_2 = 0 \). Taking \( x_1 = x_1 - \hat{x}_1 \) and \( x_2 = x_2 - \hat{x}_2 \) we obtain the error equations

\[
\begin{align*}
\dot{z}_1 &= \dot{x}_2 - \lambda \|x_1 - \hat{x}_1\|^{1/2} \text{sign}(\hat{x}_1) \\
\dot{z}_2 &= F(t, x_1, x_2, \hat{x}_2) - \alpha \text{sign}(\hat{x}_1)
\end{align*}
\]

where

\[
F(t, x_1, x_2, \hat{x}_2) = f(t, x_1, x_2, U(t, x_1, x_2)) - f(t, x_1, x_2, U(t, x_1, x_2)) + \xi(t, x_1, x_2, U(t, x_1, x_2)).
\]

Suppose that the system states can be assumed bounded, then the existence is ensured of a constant \( f^+ \), such that the inequality

\[ |F(t, x_1, x_2, \hat{x}_2)| < f^+ \]

holds for any possible \( t, x_1, x_2 \) and \( |\hat{x}_2| \leq 2 \sup |\hat{x}_2| \).

Remark 1: When the accelerations in the mechanical system are bounded, the constant \( f^+ \) can be found as the double maximal possible acceleration of the system. Moreover, the estimation constant \( f^+ \) does not depend on the nominal elasticity and control terms. Such assumption of the state boundedness is true too, if, for example, system (2) is BIBS stable, and the control input \( u = U(t, x_1, x_2) \) is bounded.

Let \( \alpha \) and \( \lambda \) satisfy the inequalities

\[
\begin{align*}
\alpha &> f^+ \\
\lambda &> \sqrt{\frac{2}{\alpha - f^+}} \frac{(\alpha + f^+)(1 + p)}{(1 - p)}
\end{align*}
\]

where \( p \) is some chosen constant, \( 0 < p < 1 \).

Theorem 1: Suppose that the parameters of the observer (3), (4) are selected according to (7), and condition (6) holds for system (2). Then, the variables of the observer (3), (4) converge in finite time to the states of system (2), i.e., \( (\hat{x}_1, \hat{x}_2) \rightarrow (x_1, x_2) \).

Proof: In order to prove the convergence of the state estimates to the real states, it is necessary to prove first the convergence of \( \hat{x}_1 \) and \( \hat{x}_1 \) to zero. Assume at first that (6) holds all the time (it will be proved further). As follows from (5), (6), estimation errors \( \hat{x}_1 \) and \( \hat{x}_2 \) satisfy the differential inclusion

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 - \lambda \|\hat{x}_1\|^{1/2} \text{sign}(\hat{x}_1) \\
\dot{\hat{x}}_2 &= \dot{x}_2 - \alpha \text{sign}(\hat{x}_1)
\end{align*}
\]

Here and further, all differential inclusions are understood in the Filippov sense, which means that the right-hand side is enlarged in some points in order to satisfy the upper semicontinuity property [6], in particular the second formula of (8) turns into \( \dot{\hat{x}}_2 \in [-\alpha - f^+ , \alpha + f^+] \) with \( \hat{x}_1 = 0 \). Note that the solutions of (8) exist for any initial condition and are infinitely extendible in time [6]. Computing the derivative of \( \hat{x}_1 \) with \( \hat{x}_1 \neq 0 \) obtain

\[
\hat{x}_1 \in [-f^+, f^+] - \frac{1}{2} \lambda \|\hat{x}_1\|^{1/2} + \alpha \text{sign}(\hat{x}_1).
\]

The trivial identity \( d|\hat{x}_1|/dt = \dot{x}_1 \text{sign} x \) is used here. Inclusion (9) does not “remember” anything on the real system, but can be used to describe the majorant curve drawn in Fig. 1. Note that at the initial moment \( \hat{x}_1 = 0 \) and \( \dot{x}_2 = x_2 - 0 = x_2 \). The trajectory enters the half-plane \( \hat{x}_1 > 0 \) with a positive initial value of \( x_2 \) and the half-plane \( \hat{x}_1 < 0 \), otherwise.

Let \( \hat{x}_1 > 0 \) then with \( \dot{\hat{x}}_1 > 0 \) the trajectory is confined between the axis \( \hat{x}_1 = 0, \hat{x}_1 = 0 \) and the trajectory of the equation \( \hat{x}_1 = -(\alpha + f^+) \); see Fig. 1 line (a). Let \( \hat{x}_1_M \) be the intersection of this curve with the axis \( \hat{x}_1 = 0 \). Obviously, \( 2(\alpha + f^+) \hat{x}_1_M = \hat{x}_2 \) where \( \hat{x}_2 > 0 \) is the value of \( \hat{x}_1 \) with \( \hat{x}_1 = 0 \). It is easy to see that for \( \hat{x}_1 > 0, \hat{x}_1 > 0 \)

\[
\hat{x}_1 \leq f^+ - \alpha \text{sign} \hat{x}_1 - \frac{1}{2} \lambda \|\hat{x}_1\|^{1/2} < 0.
\]

Thus, the trajectory approaches the axis \( \hat{x}_1 = 0 \). The majorant curve for \( \hat{x}_1 > 0, \hat{x}_1 > 0 \) is described by the equation (see Fig. 1)

\[
\hat{x}_1 = 2(\alpha - f^+) (\hat{x}_1_M - \hat{x}_1) \quad \text{for} \quad \hat{x}_1 > 0.
\]

The majorant curve for \( \hat{x}_1 > 0, \hat{x}_1 \leq 0 \) consists of two parts. In the first part the point instantly drops down from \( (\hat{x}_1_M, 0) \) to the point \( (\hat{x}_1_M, -2/\lambda (f^+ + \alpha \|\hat{x}_1\|^{1/2}) \) where, in the “worst case,” the right-hand side of inclusion (9) is equal to zero [see Fig. 1, line (b)]. The
second part of the majorant curve is the horizontal segment between the points \((\hat{x}_{1,M}, -(2)/(\lambda)(f^+ + \alpha \hat{x}_{1,M}^{1/2}) = (\hat{x}_{1,M}, \hat{x}_{1,M})\) and \((0, \hat{x}_{1,M})\) [see Fig. 1, line (c)].

Condition (7) implies that

\[
\left| \frac{\hat{x}_{1,M}}{\hat{x}_{1,0}} \right| < \frac{1 - p}{1 + p} < 1.
\]

Let us denote as \(\hat{x}_{1,0} = \hat{x}_{1,M} = \hat{x}_{1,1}, \hat{x}_{1,2}, \ldots, \hat{x}_{1,M} = \) the consequent crossing points of the system (5) trajectory starting at \((0, \hat{x}_{1,0})\) with the \(\hat{x}_1 = 0\) axis. Last inequality ensures the convergence of the state \((0, \hat{x}_{1,0})\) to \(\hat{x} = \hat{x}_1 = 0\) and, moreover, the convergence of \(\Sigma_{\overline{0}}^\infty |\hat{x}_{1,M}|\).

Consider the dynamics of \(\hat{x}_{2}\) to prove the finite-time convergence. Obviouly, \(\hat{x}_2 = \hat{x}_1\) at the moments when \(\hat{x}_1 = 0\) and, taking into account that

\[
\dot{\hat{x}}_2 = F(x_1, x_2, \hat{x}_2, u) - \alpha \operatorname{sign} \hat{x}_1
\]

obtain that

\[
0 < \alpha - f^+ \leq |\dot{\hat{x}}_2| \leq \alpha + f^+
\]

holds in a small vicinity of the origin. Thus

\[
|\dot{\hat{x}}_1| \geq (\alpha - f^+) t_i
\]

where \(t_i\) are the time intervals between the successive intersection of the trajectory with the axis \(\hat{x}_1 = 0\). Hence

\[
t_i \leq \frac{|\dot{\hat{x}}_1|}{(\alpha - f^+)}
\]

and the total convergence time is estimated by

\[
T \leq \sum \frac{|\dot{\hat{x}}_1|}{(\alpha - f^+)}.
\]

Therefore, \(T\) is finite and the estimated states converge to the real states in finite time.

The previous proof was based on inequality (6). As follows from the aforementioned consideration, sufficiently large \(f^+\) provides for \(|\dot{x}_2| \leq |\dot{x}_2(t_0)|\), where \(t_0\) is the initial time. It implies that \(|\dot{x}_2| \leq |\dot{x}_2(0)| + |x_2| \leq 2 \sup |x_2|\). Hence, the suggested choice of \(f^+\) is valid.

Remark 2: Finite-time convergence of the observer allows to design the observer and the control law separately, i.e., the separation principle is satisfied. The only requirement for its implementation is the boundedness of the function \(F(t,x_1, x_2, \hat{x}_2, u)\) in the operational domain. If the applied controller is known to stabilize the process, one of the admissible ways is to choose the observer dynamics fast enough to provide for the exact evaluation of the velocity before leaving some preliminarily chosen area, where the stabilization is assured. It is easily performed by simulation (see the following example).

Remark 3: The standard 2-sliding-mode-based differentiator [9] can be also implemented here to estimate the velocity. At the same time, the proposed observer requires smaller gains and is more accurate, since the elasticity term \(M^{-1}(q)G(q)\) does not influence the gain choice.

Remark 4: Another way to choose \(\alpha\) and \(\lambda\) is to take \(\alpha = a_1 f^+, \lambda = a_2 (f^+)^{1/2}\) with some predetermined proper \(a_1, a_2\). In particular, \(a_1 = 1.1, a_2 = 1.5\) is a valid choice [9].

The previous analysis is valid for the ideal version of the observer. Let \(f, x, z_1, z_2\) be measured at discrete times with the time interval \(\delta\), and let \(t_i, t_{i+1}\) be successive measurement times. Consider a discrete modification of the observer (the Euler scheme)

\[
\hat{x}_1(t_{i+1}) = \hat{x}_1(t_i) + (\hat{x}_2(t_i) + \lambda |x_1(t_i) - \hat{x}_1(t_i)|^{1/2}) \times \operatorname{sign} (x_1(t_i) - \hat{x}_1(t_i)) \delta
\]

\[
\hat{x}_2(t_{i+1}) = \hat{x}_2(t_i) + f(t_i, x_1(t_i), \hat{x}_2(t_i), u(t_i)) + \alpha \operatorname{sign} (x_1(t_i) - \hat{x}_1(t_i)) \delta
\]

where \(\hat{x}_1(t_i), \hat{x}_2(t_i)\) are the estimated variables.

Theorem 2: Suppose that the function \(f\) is uniformly bounded and condition (6) holds. Then, the observation algorithm (10) with parameters (7) ensures the convergence of the estimation errors to the domain \(|\hat{x}_1| \leq \gamma_1 \delta^2, |\hat{x}_2| \leq \gamma_2 \delta\) where \(\gamma_1, \gamma_2\) are some constants, depending on the observer parameters.

Proof: Let \(t \in [t_i, t_{i+1})\), where \(t_i, t_{i+1}\) are successive measurement times, \(t_{i+1} - t_i = \delta\), and \(t\) is the current time. The observer (10) may be rewritten in the continuous time as follows:

\[
\dot{\hat{x}}_1 = \hat{x}_2(t_i) + \lambda |x_1(t_i) - \hat{x}_1(t_i)|^{1/2} \operatorname{sign} (x_1(t_i) - \hat{x}_1(t_i))
\]

\[
\dot{\hat{x}}_2 = f(t, x_1(t_i), \hat{x}_2(t_i), u(t_i)) + \alpha \operatorname{sign} (x_1(t_i) - \hat{x}_1(t_i)).
\]

Hence, the errors satisfy the differential inclusion

\[
\dot{\hat{x}}_1 = \hat{x}_2(t_i) + x_2 - x_2(t_i) - \lambda |\hat{x}_1(t_i)|^{1/2} \operatorname{sign} (\hat{x}_1(t_i))
\]

\[
\dot{\hat{x}}_2 = [-f^+, f^+] - \alpha \operatorname{sign} (\hat{x}_1(t_i)).
\]

Let \(|f + \xi| \leq f_1^+\), then

\[
\dot{\hat{x}}_1 = \hat{x}_2(t_i) + x_2 - x_2(t_i) - \lambda |\hat{x}_1(t_i)|^{1/2} \operatorname{sign} (\hat{x}_1(t_i))
\]

\[
\dot{\hat{x}}_2 = [-f^+, f^+] - \alpha \operatorname{sign} (\hat{x}_1(t_i)).
\]

It may be considered as (8) with measurement errors. Indeed, let \(D\) be some compact region around the origin \(O\) of the space \(\hat{x}_1, \hat{x}_2\). As follows from the proof of Theorem 1, all trajectories of (8) starting in \(D\) converge in some finite time \(T\) to the origin \(O\). During this time they do not leave some larger homogeneous disk \(B_{R_0} = \{(\hat{x}_1, \hat{x}_2) : |\hat{x}_1|^{1/2} + |\hat{x}_2| \leq R_0\}\). Let \(M(R) = \max|x_2 - \lambda |\hat{x}_1|^{1/2} \operatorname{sign} (\hat{x}_1)| : (\hat{x}_1, \hat{x}_2) \in B_R\) due to the homogeneity property \(M(R) = m \) holds, where the constant \(m > 0\) can be easily calculated. Thus, obviously

\[
|\hat{x}_1(t_i) - \hat{x}_1(t_i)| \leq m R_0 \delta, \quad |\hat{x}_2(t_i) - \hat{x}_2(t_i)| \leq (f^+ + \alpha) \delta
\]

in \(B_{R_0}\) and, denoting \(f_1^+ = f^+ + f_1^+ + \alpha\), obtain that the trajectories of (12) satisfy the inclusion

\[
\dot{\hat{x}}_1 \in \hat{x}_2 + [-f_1^+, f_1^+] \delta - \lambda |\hat{x}_1| + [-2m, 2m] R_0 \delta^{1/2}
\]

\[
\times \operatorname{sign} (\hat{x}_1 + [-2m, 2m] R_0 \delta)
\]

\[
\dot{\hat{x}}_2 \in [-f^+, f^+] - \alpha \operatorname{sign} (\hat{x}_1 + [-2m, 2m] R_0 \delta)
\]

while \((\hat{x}_1, \hat{x}_2) \in B_{2R_0}\). With \(\delta\) being zero, the dynamics (13) coincides with (8), whose trajectories converge in finite time to the origin. Due to the continuous dependence of the Filippov solutions on the graph of the differential inclusion, with sufficiently small \(\delta\) the trajectories of (13) starting in \(D\) terminate in the time \(T\) in some small compact vicinity \(\hat{D} \subset D\) of the origin without leaving \(B_{2R_0}\) on the way. Let \(\Omega\) be the compact set \([6]\) of all points belonging to the trajectory segments starting in \(D\) and corresponding to the closed time interval \(\hat{T}, \hat{D} \subset \Omega\).

With \(\delta\) small enough \(\hat{D} \subset \Omega \subset D\), since the origin \(O\) is invariant for (13),
Obviously, $\Omega$ is an invariant set attracting the trajectories of (12) starting in $D$. Check now that it is a globally attracting set. Define a homogenous parameter-time-coordinate transformation

$$
\begin{align*}
t &\mapsto \eta t \\
(x_1, x_2, \dot{x}_1, \dot{x}_2) &\mapsto (\eta^2 x_1, \eta \dot{x}_2, \eta R_0, \eta \delta)
\end{align*}
$$

(14)

and let $G_\eta(x_1, x_2) = (\eta^2 x_1, \eta x_2)$. It is easily seen that (14) preserves (13), i.e., the trajectories are preserved. Choose such $\eta > 1$ that $G_\eta \Omega \subseteq D$, then the trajectories of the inclusion

$$
\dot{x}_1 \in x_2 + [-f^+, f^+] \eta \delta - \lambda [x_1 + [-2m, 2m] R_0 \eta^2 \delta]^{1/2} \times \text{sign} (x_1 + [-2m, 2m] R_0 \eta^2 \delta)
$$

$$
\dot{x}_2 \in [-f^+, f^+] \alpha \text{sign} (x_1 + [-2m, 2m] R_0 \eta^2 \delta)
$$

(15)

starting in $G_\eta D$ terminate following time $\eta T$ in $G_\eta \Omega \subseteq D$ without leaving $G_\eta B_{2\eta R_0} = B_{2\eta R_0}$ on the way. Comparing (13) and (15) obtain that (15) describes the solutions of (12) in $B_{2\eta R_0}$, but with redundantly enlarged “noise level” due to the replacement of $\delta$ by $\eta \delta > \delta$. Hence, the solutions of (12) satisfy (15) in $B_{2\eta R_0}$ Therefore, the trajectories of (12) starting in $G_\eta D$ terminate following time $\eta T$ in $G_\eta \Omega \subseteq D$ and proceed into $\Omega$ in time $T$. Representing the whole plane $x_1, x_2$ as $\mathbb{R}^2 \equiv \mathbb{G}^2 \mathbb{D}$ obtain the global finite-time convergence to the set $\Omega$.

IV. EXAMPLE

Consider a pendulum system with Coulomb friction and external perturbation given by the equation

$$
\ddot{\theta} + \frac{1}{M} \sin \theta \dot{\theta} - \frac{V_s}{M} \dot{\theta} - \frac{P_s}{M} \text{sgn}(\dot{\theta}) + v = \tau
$$

(16)

where the values $M = 1.1, g = 9.815, L = 0.9, J = ML^2 = 0.891, V_s = 0.18, P_s = 0.45$ were taken and $v$ is an uncertain external perturbation, $|v| \leq 1$. It was taken $v = 0.5 \sin 2t + 0.5 \cos 5t$ in simulation. Let it be driven by the twisting controller

$$
\tau = -30 \text{sgn}(\theta - \theta_d) - 15 \text{sgn}(\dot{\theta} - \dot{\theta}_d)
$$

(17)

where $\theta_d = \sin t$ and $\dot{\theta}_d = \cos t$ are the reference signals. The system can be rewritten as

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{J} \tau - \frac{g}{L} \sin x_1 - \frac{V_s}{J} x_2 - \frac{P_s}{J} \text{sgn}(x_2) + v.
\end{align*}
$$

Thus, the proposed velocity observer (see Remark 3) has the form

$$
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 + 1.5 (f^+)^{1/2} [x_1 - \hat{x}_1]^{1/2} \text{sign}(x_1 - \hat{x}_1) \\
\dot{\hat{x}}_2 &= \frac{1}{J_n} \tau - \frac{g}{L_n} \sin x_1 - \frac{V_s}{J_m} \hat{x}_2 + 1.1 f^+ \text{sign}(x_1 - \hat{x}_1)
\end{align*}
$$

where $M_n = 1, L_n = 1, J_n = M_n L_n^2 = 1, V_s = 0.2, P_s = 0.5$ are the “known” nominal values of the parameters, and $f^+$ is to be assigned. Assume also that it is known that the real parameters differ from the known values by not more than 10%. The initial values $\theta = x_1 = \hat{x}_1 = 0$ and $\dot{\theta} = x_2 = \hat{x}_2 = 0$ were taken at $t = 0$. Identifying $0$ and $2\pi$ obtain that $\theta$ belongs to a compact set (a ring). Thus, obviously, dynamic system (16) is BIBS stable. Easy calculation shows that the given controller provides for $|\tau| \leq 45$, and the inequality $|\theta| \leq 70$ is ensured, when the nominal values of parameters and their maximal possible deviations are taken into account. Taking $|x_2| \leq 70, |\dot{x}_2| \leq 140$ obtain that $|F| = |(1/J) \tau - (g/L) \sin x_1 - (V_s/J) x_2 - (P_s/J) \text{sgn}(x_2) + v - (1/J_n) \tau + (g/L_n) \sin x_1 + (V_s/J_m) \hat{x}_2| < 60 = f^+$. Therefore, the observer parameters $\alpha = 66$ and $\lambda = 11.7$ were chosen. Simulation adjustment (see Remark 1) shows that $f^+ = 6$ and the respective values $\alpha = 6.6$ and $\lambda = 1$ are sufficient. Note that the terms $(Mg/L) \sin x_1$ and $(1/J) \tau$ would be fully taken into account for the choice of the differentiator parameters [9] causing much larger coefficients to be used. The performance of the observer with the sampling interval $\delta = 0.00001$ is shown in Fig. 2. The finite-time convergence of the estimated velocity to the real one is demonstrated in Fig. 3, and...
Fig. 4. Graph of $\dot{x}_1$ versus $\dot{x}_2$.

Fig. 5. Error of $x_2$ estimation (detail) with the sampling interval $\delta = 0.0001$.

Fig. 6. Error of $x_2$ estimation (detail) with the sampling interval $\delta = 0.0001$.

Fig. 4 shows the convergence in the plane $\dot{x}_1$ vs $\dot{x}_2$. A detail of the estimation error graph is shown in Fig. 5. Fig. 6 demonstrates that the tenfold increase of the sampling time interval up to $\delta = 0.0001$ causes the proportional increase of the estimation error. This corresponds to Theorem 2.

V. CONCLUSION

The super-twisting second-order sliding-mode algorithm was modified in order to design a velocity observer for mechanical systems. The finite-time convergence of the observer is proved. Consequently, the separation principle is automatically satisfied, i.e., a controller and the observer can be separately designed. The gains of the proposed observer can be chosen ignoring the elasticity terms.

For the discrete realization of the observer the corresponding accuracy is estimated.

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