Performance Analysis of a Renewal Input Bulk Service Queue with Accessible and Non-Accessible Batches

V. Goswami¹ and P. Vijaya Laxmi²

¹School of Computer Application, KIIT University, Bhubaneswar, India
²Department of Applied Mathematics, Andhra University, Visakhapatnam, India

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Abstract: In this paper, we consider a single-server infinite- (finite-) buffer bulk-service queues. The interarrival and service times are respectively, arbitrarily and exponentially distributed. The customers are served by a single server in accessible or non-accessible batches of maximum size ‘b’ with a minimum threshold value ‘a’. We provide a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining interarrival time, to develop the steady-state queue length distributions at prearrival and arbitrary epochs. Finally, some numerical results have been presented.

Keywords: Accessible batch, bulk-service, non-accessible batch, supplementary variable.

1. Introduction

In queueing literature, bulk-service queueing models have gained a lot of attention due to their practical applicability in the fields of communication systems, lift operations, cargo loading and unloading problems etc. In such bulk-service queueing systems, to have the effective utilization of server, customers arriving one at a time must wait in the queue until a sufficient number of customers are accumulated in the queue. Further, the server has a pre-specified minimum and maximum threshold capacities of service. These and other variants of general bulk-service queues have been studied by many authors, see Baba [1], Chaudhry and Gupta [2], Chaudhry and Templeton [3], Gold and Tran-Gia [4], Hebuterne and Rosenberg [7], Medhi [9, 10], Neuts [11]. The concept of accessibility into batches during service has been considered by Goswami et al. [5], Gross and Harris [6], Kleinrock [8] and Sivasamy [12]. The infinite buffer queue with accessible and non-accessible batch service rule has been studied by Sivasamy [12], where the arrivals and service times are exponentially distributed. In discrete-time systems, the same type of model has been studied by Goswami et al. [5] with finite and infinite buffers. The finite buffer queues with general arrivals and bulk-service has been studied by Vijaya Laxmi and Gupta [14]. It may be noted that the general uncorrelated arrival process appears to be more appropriate and reasonable than exponential distribution, as the memoryless property of the arrival process does not always meets the needs of applications and also it can include exponential, deterministic, Erlang distributions etc. as special cases. Due to practical usage and easy computations, the finite buffer queues are getting more attention than the corresponding infinite buffer queues. The customers upon arrival, are queued in the buffer if the server is busy with full capacity. If the arriving customer finds that the buffer is full, then he is blocked and is considered to be lost. Therefore, one of the key index of performance is the probability of blocking which is kept at an optimum level so as to avoid the customers rejection. However, from a theoretical
viewpoint both (infinite- and finite-buffer) queueing models have importance.

The present paper focusses on the study of infinite- (finite-) buffer queue with accessible and non-accessible batches wherein inter-arrival time of arrival customers and service time of batches are, respectively, arbitrarily and exponentially distributed. The supplementary variable technique is used to develop the steady state equations treating the supplementary variable as the remaining inter-arrival time, and a recursive method has been developed to obtain the steady state distributions of the number in the system both at pre-arrival and arbitrary epochs. Numerical results have been presented in the form of tables and graphs. Here it may be stressed that with our method the pre-arrival epoch probabilities of the number in the system are obtained without using the embedded Markov chain technique. The queueing model presented above has applications in the field of communication systems, polling systems, cinema theaters and many other such related areas.

The rest of this paper is organized as follows. Section 2 presents the description of the queueing model, steady state equations using the supplementary variable technique and provides a recursive method to obtain the steady-state probability distributions of the number of customers in the system/queue at prearrival and arbitrary epochs for infinite- (finite-) buffer. In Section 3, some performance measures are given and Section 4 contains numerical results to show the effectiveness of the model parameters.

2. The Model Description and Solution

Let us consider an infinite- (finite-) buffer queue where the inter-arrival times of successive arrivals are independent and identically distributed (i.i.d.) random variables with cumulative distribution function $A(u)$, probability density function $a(u)$, $u \geq 0$, Laplace-Stiltjes (L.-S.) transform $A^*(\theta)$ and mean inter-arrival time $1/\lambda = -A^{(1)}(0)$, where $h^{(1)}(0)$ denotes the first derivative of $h(\theta)$ evaluated at $\theta = 0$. Service times are assumed to be exponentially distributed random variable with rate $\mu$. The customers (packets) are served (transmitted) by a single server in batches of maximum size 'b' with a minimum threshold value 'a'. However, if the number of customers in the queue is less than the minimum threshold value 'a', the server remains idle until the number of customers in the queue reaches 'a'. If 'b' more customers are present in the queue at service initiate epoch then only 'b' of them are taken into service. It is further assumed that the late entries can join a batch in course of ongoing service as long as the number of customers in that batch is strictly less than $d$ (called maximum accessible limit). At every departure epoch, that is, before initiating service of the next batch, the server may find the system in any one of the following three cases: (i) $0 \leq n \leq a - 1$, (ii) $a \leq n \leq d - 1$ and (iii) $n \geq d$. In case (i), the server cannot initiate service, it remains idle. In case (ii), the server takes the entire queue for batch service and admits the subsequent arrivals in the batch while the service is on, till the accessible limit $d$ is reached, and such a batch is called an accessible batch (AB). In case (iii), it takes $\min(n, b)$ customers for the service and does not allow further arrivals into the batch being served even if the current batch size is not 'b', that is, when the batch size is greater than or equal to $d$, the batch becomes non-accessible (NAB) for late arriving customers. The traffic intensity is given by $\rho = \lambda / b\mu$, which is less than one for infinite buffer queues.

The state of the system at time $t$ is described by the following random variables, namely

1. $N_s(t) =$ number of customers present in the system including those in service,
2. $N_q(t) =$ number of customers present in the queue not counting those in service,
customers. The traffic intensity is given by the number of customers in the system/queue at prearrival and arbitrary epochs for infinite- (finite-) buffer queues.

Let us define joint probabilities by

\[ P_{n,0}(u,t)du = P(N_s(t) = n, u < U(t) \leq u + du, \zeta(t) = 0), u \geq 0, 0 \leq n \leq d - 1, \]

\[ P_{n,1}(u,t)du = P(N_q(t) = n, u < U(t) \leq u + du, \zeta(t) = 1), u \geq 0, n \geq 0. \]

As we shall discuss the model in steady-state, that is, when \( t \to \infty \), the above probabilities will be denoted by \( P_n(i) \), \( i = 0, 0 \leq n \leq d - 1; i = 1, n \geq 0 \).

Let us define the Laplace transforms of \( P_n(i) \) as

\[ P_n^*(\theta) = \int_0^\infty e^{-\theta u} P_n(i)(u)du, \quad i = 0, 0 \leq n \leq d - 1; i = 1, n \geq 0. \]

So that,

\[ P_n(i) = P_n^*(0) = \int_0^\infty P_n(i)(u)du, \]

are the arbitrary epoch probabilities.

### 2.1. The Model with Infinite-Buffer

To obtain the queue length distribution at arbitrary epochs and performance measures of the system, we develop the difference equations using the remaining interarrival time as the supplementary variable. Relating the states of the system at two consecutive time epochs \( t \) and \( t + dt \), using definitions and probabilities defined above, we have in the steady-state

\[ -\frac{d}{du} P_{0,0}(u) = \mu \sum_{k=0}^{d-1} P_{k,0}(u) + \mu P_{0,1}(u), \quad (1) \]

\[ -\frac{d}{du} P_{n,0}(u) = \mu P_{n,1}(u) + a(u)P_{n-1,0}(0), 1 \leq n \leq a - 1, \quad (2) \]

\[ -\frac{d}{du} P_{n,0}(u) = -\mu P_{n,0}(u) + \mu P_{n+1,0}(u) + P_{n,0}(0), a \leq n \leq d - 1, \quad (3) \]

\[ -\frac{d}{du} P_{0,1}(u) = -\mu P_{0,1}(u) + \mu \sum_{k=d}^{b} P_{k,1}(u) + a(u)P_{d-1,0}(0), \quad (4) \]

\[ -\frac{d}{du} P_{n,1}(u) = -\mu P_{n,1}(u) + \mu P_{n+1,1}(u) + a(u)P_{n-1,1}(0), n \geq 1. \quad (5) \]

Multiplying equations (1)-(5) by \( e^{-\theta u} \) and integrating with respect to \( u \) from 0 to \( \infty \), yields

\[ -\theta P_{0,0}^*(\theta) = \mu \sum_{k=0}^{d-1} P_{k,0}(\theta) + \mu P_{0,1}^*(\theta) - P_{0,0}(0), \quad (6) \]
Adding equations (6)-(10), we get

\[
\sum_{n=0}^{d-1} P_{n,0}^* (\theta) + \sum_{n=0}^{\infty} P_{n,1}^* (\theta) = \frac{1 - A^*(\theta)}{\theta} \left[ \sum_{n=0}^{d-1} P_{n,1}(0) + \sum_{n=0}^{\infty} P_{n,1}(0) \right].
\]

Taking the limit as \( \theta \to 0 \) and using the normalization condition:

\[
\sum_{n=0}^{d-1} P_{n,0}(0) + \sum_{n=0}^{\infty} P_{n,1}(0) = 1,
\]

after simplification we get

\[
\sum_{n=0}^{d-1} P_{n,0}(0) + \sum_{n=0}^{\infty} P_{n,1}(0) = \lambda.
\]  

### 2.1.1. Steady State Distribution at Prearrival Epochs

To obtain the steady state distribution of number of customers in the system/queue at prearrival epochs, we first evaluate \( P_{n,0}(0) \) \((0 \leq n \leq d-1)\) and \( P_{n,1}(0) \) \((n \geq 0)\) from equations (6)-(10).

Denoting the displacement operator by \( E \), i.e., \( Ef_{i,1}(x) = f_{i+1,1}(x) \) for a sequence \( f_{i,1}(x) \), see Spiegel [13], the equation (10) can be written as

\[
(\mu - \mu E^b - \theta) P_{n,1}^* (\theta) = A^*(\theta)P_{n-1,1}(0) - P_{n,1}(0), \quad n \geq 1.
\]

Setting \( \theta = \mu - \mu E^b \) in equation (12), after algebraic manipulation, we obtain

\[
\left[ E - A^*(\mu - \mu E^b) \right] P_{n,1}(0) = 0, \quad n \geq 0.
\]

The solution of homogeneous difference equation (13) is

\[
P_{n,1}(0) = C r^n, \quad n \geq 0,
\]

where \( C \) (independent of \( n \)) may be a function of \( r \) and the queueing parameters. It can be proved using Rouche's theorem that the equation \( r = A^*(\mu - \mu E^b) \) has one real root inside the unit circle for \( \rho < 1 \).

Again from equation (12), using equation (14) and simplifying, we obtain

\[
(\mu - \mu E^b - \theta) P_{n,1}^* (\theta) = C(A^*(\theta) - r) r^{n-1}, \quad n \geq 1.
\]
Thus the solution of equation (15) is
\[
P_{n,1}(\theta) = \frac{Cr^{n-1}(A^*(\theta) - r)}{\mu - \mu r^b - \theta}, \quad n \geq 1.
\] (16)

Substituting \( \theta = \mu \), in (16), we get
\[
P_{n,1}(\mu) = \frac{Cr^{n-1-b}(r - A^*(\mu))}{\mu}, \quad n \geq 1.
\] (17)

Substituting \( \theta = \mu \), in equations (8)-(9) and using (17) gives
\[
P_{n,0}(0) = C \left[ K \omega^{d-n} - (r - A^*(\mu)) \sum_{i=n}^{d-2} r^{i+b} \omega^{i-n+1} \right], \quad a-1 \leq n \leq d-1,
\] (18)
where \( \omega = 1/A^*(\mu) \) and \( K = \omega \left[ 1 - (r - A^*(\mu))((r^{d-b-1})/(1-r)) \right] \).

Setting \( \theta = 0 \) in equation (16), we get
\[
P_{n,1} = \frac{Cr^{n-1}(1-r)}{\mu(1-r^b)}, \quad n \geq 1.
\] (19)

From equation (7) setting \( \theta = \mu \), and using (17),
\[
P_{n,0}(0) = C \left[ D - \frac{1-r}{1-r^b} \sum_{i=n}^{a-2} r^i \right], \quad 0 \leq n \leq a-2,
\] (20)
where \( D = C \left[ K \omega^{d-a} - (r - A^*(\mu))\omega r^{a-b-1}((1-(r\omega))^{d-a})/(1-r\omega) \right] \).

From equations (11), (14), (18) and (20), we get
\[
C = \lambda \left[ aD - \frac{1-r^{a-1}(a-ar+r)}{(1-r)(1-r^b)} + K \left( \frac{1-\omega^{d-a}}{1-\omega} \right) + \frac{1}{1-r} \right.
\]
\[
\left. - \frac{\omega(r - A^*(\mu))}{1-r\omega} \left( r^{a-b} \left( \frac{1-r^{d-a-1}}{1-r} \right) - \omega r^{a-b-1} \left( \frac{1-\omega^{d-a-1}}{1-\omega} \right) \right) \right]^{-1}.
\] (21)

Now, \( P_{n,0}(0) \) \((0 \leq n \leq d-1)\) and \( P_{n,1}(0) \) \((n \geq 0)\) are known, we can obtain prearrival epoch probabilities \( P_{n,0}^- \) \((0 \leq n \leq d-1)\) and \( P_{n,1}^- \) \((n \geq 0)\) using
\[
P_{n,i}^- = \frac{P_{n,i}(0)}{\sum_{n=0}^{d-1} P_{n,0}(0) + \sum_{n=0}^{\infty} P_{n,1}(0)} = \frac{P_{n,i}(0)}{\lambda}, \quad i = 0, 0 \leq n \leq d-1; \quad i = 1, n \geq 0.
\] (22)

Here \( P_{n,0}^- \) \((P_{n,1}^-)\) represents the probability that there are \( n \) customers present in the system (queue) prior to an arrival epoch of a customer when the server is idle or busy with an accessible (a non-accessible) batch.
2.1.2. Steady State Distribution at Arbitrary Epochs

To obtain the steady state probabilities at arbitrary epochs we develop relation between prearrival and arbitrary epoch probabilities. The state probabilities at arbitrary epochs $P_{n,1}$ ($n \geq 1$) are given by the equation (19).

Substituting $\theta = 0$ in equations (9) and (8), respectively, we get

$$P_{0,1} = \frac{C}{\mu} \left[ \frac{r^{d-1} - r^b}{1 - r^b} + K - 1 \right], \quad (23)$$

$$P_{n,0} = \frac{C}{\mu} \left[ \frac{r^{n-1}(1-r)}{1 - r^b} - \frac{\omega r^{n-1} (r - A^*(\mu))}{1 - r\omega} \left(1 - r + r^{d-\omega} \omega^{d-\omega-1} (1 - \omega)\right) + K \omega^{d-\omega-1} (\omega - 1) \right], \quad a \leq n \leq d - 1. \quad (24)$$

To obtain the arbitrary epoch probabilities $\{P_{n,0}\}_{1}^{d-1}$, we differentiate equation (7) with respect to $\theta$ and setting $\theta = 0$, we get

$$P_{n,0}^{(1)} = -\mu P_{n,1}^{(1)}(0) + P_{n-1,0}^{(1)}, \quad 1 \leq n \leq a - 1. \quad (25)$$

$P_{n,1}^{(1)}(0), n \geq 1$, can be obtained by differentiating (16) with respect to $\theta$, and substituting $\theta = 0$, we get

$$P_{n,1}^{(1)}(0) = \frac{C r^{n-1} (\lambda (1-r) - \mu (1-r^b))}{\lambda (\mu - \mu r^b)^2}, \quad n \geq 1. \quad (26)$$

So, from equations (25) and (26),

$$P_{n,0} = \frac{C}{\lambda} \left[ D - r^{n-1} \left\{ \frac{1-r^{a-n}}{1-r^b} + \frac{\mu (\lambda (1-r) - \mu (1-r^b))}{(\mu - \mu r^b)^2} \right\} \right], \quad 1 \leq n \leq a - 1. \quad (27)$$

Finally, we get $P_{0,0}$ using the normalization condition $\sum_{n=0}^{d-1} P_{n,0} + \sum_{n=0}^{\infty} P_{n,1} = 1$.

2.2. The Model with Finite Buffer Queue

This model differs from the model discussed in previous section in the sense that the maximum number of customers allowed in the queue (excluding those in service) at any time is $N$. Using the same arguments as mentioned in previous section, we have the same set of equations (1)-(4) in case of finite buffer as those for infinite buffer. The remaining equations are given by

$$-\frac{d}{du} P_{n,1}(u) = -\mu P_{n,1}(u) + \mu P_{n+1,1}(u) + a(u) P_{n-1,1}(0), \quad 1 \leq n \leq N - b, \quad (28)$$

$$-\frac{d}{du} P_{n,1}(u) = -\mu P_{n,1}(u) + a(u) P_{n-1,1}(0), \quad N - b + 1 \leq n \leq N - 1, \quad (29)$$

$$-\frac{d}{du} P_{N,1}(u) = -\mu P_{N,1}(u) + a(u) [P_{N-1,1}(0) + P_{N,1}(0)]. \quad (30)$$
Multiplying (28) to (30) by $e^{-\theta u}$ and integrating with respect to $u$ from 0 to $\infty$, yields

\begin{equation}
(\mu - \theta)P_{n,1}^*(\theta) = \mu P_{n+1,1}^*(\theta) + A^*(\theta)P_{n-1,1}(0) - P_{n,1}(0), \quad 1 \leq n \leq N - b, \tag{31}
\end{equation}

\begin{equation}
(\mu - \theta)P_{n,1}^*(\theta) = A^*(\theta)P_{n-1,1}(0) - P_{n,1}(0), \quad N - b + 1 \leq n \leq N - 1, \tag{32}
\end{equation}

\begin{equation}
(\mu - \theta)P_{N,1}^*(\theta) = A^*(\theta)[P_{N-1,1}(0) + P_{N,1}(0)] - P_{N,1}(0). \tag{33}
\end{equation}

Adding equations (6)-(9) and (31)-(33), we get

\begin{equation}
\sum_{n=0}^{d-1} P_{n,0}^*(\theta) + \sum_{n=0}^{N} P_{n,1}^*(\theta) = \frac{1 - A^*(\theta)}{\theta} \left[ \sum_{n=0}^{d-1} P_{n,1}(0) + \sum_{n=0}^{N} P_{n,1}(0) \right].
\end{equation}

Taking the limit as $\theta \to 0$, using the normalization condition:

\begin{equation}
\sum_{n=0}^{d-1} P_{n,0} + \sum_{n=0}^{N} P_{n,1} = 1,
\end{equation}

after simplification we get

\begin{equation}
\sum_{n=0}^{d-1} P_{n,0}(0) + \sum_{n=0}^{N} P_{n,1}(0) = \lambda. \tag{34}
\end{equation}

### 2.2.1. Steady State Distribution at Prearrival Epochs

As discussed in case of infinite buffer, the steady state distribution of number of customers in the system/queue at prearrival epochs, $P_{n,0}^{-} (0 \leq n \leq d - 1)$ and $P_{n,1}^{-} (0 \leq n \leq N)$ are given by

\begin{equation}
P_{n,i}^{-} = \frac{P_{n,i}(0)}{\sum_{n=0}^{d-1} P_{n,0}(0) + \sum_{n=0}^{N} P_{n,1}(0)} = \frac{1}{\lambda} P_{n,i}(0), \quad i = 0, 0 \leq n \leq d - 1; \quad i = 1, 0 \leq n \leq N. \tag{35}
\end{equation}

Now we first evaluate $P_{n,0}(0) (0 \leq n \leq d - 1)$ and $P_{n,1}(0) (0 \leq n \leq N)$ from equations (6)-(9) and (31)-(33) in the following manner. Setting $\theta = 0$ in (33), and $\theta = \mu$ in (33) and (32), we finally obtain

\begin{equation}
P_{n,1}(0) = \phi_n P_{N,1}, \quad N - b \leq n \leq N, \tag{36}
\end{equation}

where

\begin{equation}
\phi_n = \begin{cases} 
\frac{\mu A^*(\mu)}{1 - A^*(\mu)}, & n = N, \\
\mu \{A^*(\mu)\}^{n-1}, & N - b \leq n \leq N - 1.
\end{cases} \tag{37}
\end{equation}

Setting $\theta = \mu$ in (31) and equations (9)-(8), we get
\[ P_{n,1}(0) = \frac{1}{A^*(\mu)} \left\{ P_{n+1,1}(0) - \mu P_{n+b+1,1}^*(\mu) \right\}, \quad n = N - b - 1, N - b - 2, \ldots, 1, 0, \quad (38) \]

\[ P_{d-1,0}(0) = \frac{1}{A^*(\mu)} \left\{ P_{d,1}(0) - \mu \sum_{k=d}^{b} P_{k+1}^*(\mu) \right\}, \quad (39) \]

\[ P_{n,0}(0) = \frac{1}{A^*(\mu)} \left\{ P_{n+1,0}(0) - \mu P_{n+b+1}^*(\mu) \right\}, \quad n = d - 2, d - 3, \ldots, a - 1. \quad (40) \]

To obtain unknown quantities \( P_{n,1}^*(\mu) \) \((a \leq n \leq N)\) in the above equations, we differentiate (33)-(31), \((j + 1)\) times with respect to \( \theta \), and finally setting \( \theta = \mu \), we get

\[ P_{n,1}^{*(j)}(\mu) = \frac{-\mu A^{(j+1)}(\mu)}{(j+1)(1 - A^*(\mu))} P_{n,1}, \quad (41) \]

\[ P_{n,1}^{*(j)}(\mu) = \frac{-\phi_{n-1} A^{(j+1)}(\mu)}{(j+1)} P_{n,1}, \quad n = N - 1, N - 2, \ldots, N - b + 1, \quad (42) \]

\[ P_{n,1}^{*(j)}(\mu) = \frac{-1}{j+1} \left\{ \mu P_{n+b,1}^{*(j+1)}(\mu) + A^{(j+1)}(\mu) P_{n+1,1}(0) \right\}, \quad n = N - b, N - b - 1, \ldots, 2, 1. \quad (43) \]

Now substituting \( \theta = 0 \) in equation (7), we obtain

\[ P_{n,0}(0) = P_{n+1,0}(0) - \mu P_{n+1,1}, \quad n = a - 2, a - 3, \ldots, 1, 0. \quad (44) \]

Substituting \( \theta = 0 \) in (32)-(31), then \( P_{n+1,1}, \quad n = 0, 1, \ldots, a - 2 \), occurring in (44), are given by

\[ P_{n,1} = \frac{1}{\mu} \left[ P_{n-1,1}(0) - P_{n,1}(0) \right], \quad n = N - 1, N - 2, \ldots, N - b + 1, \quad (45) \]

\[ P_{n,1} = P_{n+b,1} + \frac{1}{\mu} \left[ P_{n-1,1}(0) - P_{n,1}(0) \right], \quad n = N - b, N - b - 1, \ldots, 2, 1. \quad (46) \]

So \( P_{0,0}(0) \) and \( P_{1,0}(0) \) are known in terms of \( P_{N,1} \). It may be noted that to evaluate \( P_{n,j} \) from (35) we do not require the value of \( P_{N,1}^* \), since it cancels out in the numerator and denominator of RHS of (35).

### 2.2.2. Steady State Distribution at Arbitrary Epochs

To obtain the system/queue length distribution at arbitrary epochs we develop relations between distributions of number of customers in the system/queue at prearrival and arbitrary epochs.

Setting \( \theta = 0 \) in (31)-(33) and equations (8)-(9), using (35), the arbitrary epoch probabilities are given by

\[ P_{N,1} = \rho b P_{N-1,1}^*, \quad (47) \]

\[ P_{n,1} = \rho b (P_{n-1,1}^* - P_{n,1}^*), \quad N - b + 1 \leq n \leq N - 1, \quad (48) \]
\[ P_{n,1} = P_{n+b,1} + \rho b (P_{n-1,1} - P_{n,1}), \quad n = N - b, N - b, ..., 1, \]  
(49)

\[ P_{0,1} = \sum_{k=d}^{b} P_{k,1} + \rho b (P_{d-1,0} - P_{0,1}), \]  
(50)

\[ P_{n,0} = P_{n,1} + \rho b (P_{n-1,0} - P_{n,0}), \quad a \leq n \leq d - 1. \]  
(51)

The arbitrary epoch probabilities \( P_{n,0} \) are given by differentiating equations (6)-(7) with respect to \( \theta \), and setting \( \theta = 0 \) we get

\[ P_{0,0} = -\left[ \mu \sum_{k=d}^{d-1} P_{k,0}^{(1)} (0) + \mu P_{0,1}^{(1)} (0) \right], \]  
(52)

\[ P_{n,0} = -\left[ \mu P_{n,1}^{(1)} (0) - P_{n-1,0}^{-} \right], \quad 1 \leq n \leq a - 1, \]  
(53)

where \( P_{k,0}^{(1)} (0) \) (\( a \leq k \leq d - 1 \)) and \( P_{n,1}^{(1)} (0) \) (\( 0 \leq n \leq a - 1 \)) can be obtained by differentiating (31)-(33) and equations (8)-(9) with respect to \( \theta \), and setting \( \theta = 0 \). They are given by

\[ P_{n,1}^{(1)} (0) = \frac{1}{\mu} \left[ P_{n,1} - P_{n-1,1} - P_{n,1}^{-} \right], \]  
(54)

\[ P_{n,1}^{(1)} (0) = \frac{1}{\mu} (P_{n,1} - P_{n-1,1}), \quad N - b + 1 \leq n \leq N - 1, \]  
(55)

\[ P_{n,1}^{(1)} (0) = P_{n+b,1}^{(1)} (0) + \frac{1}{\mu} (P_{n,1} - P_{n-1,1}), \quad 1 \leq n \leq N - b - 1, \]  
(56)

\[ P_{0,1}^{(1)} (0) = \sum_{k=d}^{b} P_{k,1}^{(1)} (0) + \frac{1}{\mu} (P_{0,1} - P_{d-1,0}), \]  
(57)

\[ P_{n,0}^{(1)} (0) = P_{n,1}^{(1)} (0) + \frac{1}{\mu} (P_{n,0} - P_{n-1,0}), \quad a \leq n \leq d - 1. \]  
(58)

### 2.2.3. Algorithm for Computing State Probabilities

To demonstrate the working schemes of the recursive method, we describe the algorithm for calculating the steady state probabilities. Given the values of \( \mu, a, d, b, N \) and the expression of the Laplace transform of the interarrival time distribution, namely \( A^*(\theta) \), the steps of the solution algorithm are stated as follows:

1. **Step 1:** Set \( P_{N,1} = 1 \).
2. **Step 2:** Compute \( \phi_n \) for \( N - b \leq n \leq N \) using (37).
3. **Step 3:** Compute \( P_{n,1} (0) \) for \( n = N, N - 1, ..., N - b \) using (36).
4. **Step 4:** For \( n = N, N - 1, ..., a \), and \( 0 \leq j \leq N - 1 \), compute \( P_{n,j}^{(1)} (\mu) \) from (41) to (43).
5. **Step 5:** Compute \( P_{n,1} (0) \) for \( n = N - b - 1, ..., 1, 0 \) from (38).
6. **Step 6:** Compute \( P_{d-1,0} (0) \) from (39).
7. **Step 7:** Compute \( P_{a,0} (0) \) for \( n = d - 2, ..., a - 1 \) from (40).
8. **Step 8:** Compute \( P_{n+1,1} \) for \( 0 \leq n \leq a - 2 \) using (45) to (46).
9. **Step 9:** Compute \( P_{d,0} (0) \) for \( n = a - 2, ..., 0 \) from (40).
10. **Step 10:** Compute \( P_{n,j}^{-} \) from (35).
Step 11: Compute $P_{n,0}$ for $0 \leq n \leq N$ and $P_{n,0}$ for $a \leq n \leq d - 1$ using (47) to (51).
Step 12: Compute $P_{n,1}^{(1)}(0)$ for $0 \leq j \leq a - 1$, and $P_{n,0}^{(1)}(0)$ for $a \leq n \leq d - 1$, from (54) to (58).
Step 13: Compute $P_{n,0}$ for $0 \leq n \leq a - 1$ from (52) and (53).

3. Performance Measures

In this section, we discuss some operating characteristics in queueing system. In the case of infinite buffer, the average number of customers in the queue ($L_q$) is given by

$$L_q = \sum_{n=0}^{a-1} nP_{n,0} + \sum_{n=0}^{\infty} nP_{n,1}.$$  

Average waiting time in the queue ($W_q$) can be obtained using Little's rule given by $W_q = L_q / \lambda$.

In the case of finite buffer, $L_q$ and $W_q$ are, respectively, given by

$$L_q = \sum_{n=0}^{a-1} nP_{n,0} + \sum_{n=0}^{N} nP_{n,1}, \quad W_q = L_q / \lambda',$$

where $\lambda' = \lambda(1 - PBL)$ is the effective arrival rate, and $PBL = P_{N,1}^-$ represents the probability of blocking.

4. Numerical results

This section presents numerical results in the form of tables and graphs. In Table 1 and Table 2, the results of the queue-length distributions at prearrival and arbitrary epochs are given for the interarrival time distributions: exponential, deterministic, Erlang and hyperexponential along with the performance measures at the bottom for infinite- and finite-buffer queues, respectively. All the results in the case of infinite-buffer match exactly with those for the finite-buffer, when $N$ is chosen sufficiently large. Further, these results match with those of $GI / M^{[b]} / 1 / N$ queue Vijaya Laxmi and Gupta [14] by taking $a = d = 1$.

![Figure 1. Effect of $N$ on $PBL$ for different arrival distributions.](image)
Performance Analysis of a Renewal Input Bulk Service Queue

Step 12: Compute $P_{BL}$ for \( n \) and $P_{GI}$ for \( N \), respectively. All the results in the case of infinite-buffer match exactly with those of Vijaya Laxmi and Gupta \cite{14} by taking $\lambda_1/1/\rho_1$.

This section presents numerical results in the form of tables and graphs. In Table 1 and Table 2, the results of the queue-length distributions at prearrival and arbitrary epochs are given for the interarrival time distributions: exponential, deterministic, Erlang and hyperexponential along with the performance measures at the bottom for infinite- and finite-buffer queues.

Table 1. Queue-length distributions at prearrival and arbitrary epochs for the infinite buffer queue.

<table>
<thead>
<tr>
<th>((n, r))</th>
<th>(P_{w,r})</th>
<th>(P_{n,r})</th>
<th>(P_{w,r})</th>
<th>(P_{n,r})</th>
<th>(P_{w,r})</th>
<th>(P_{n,r})</th>
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<tr>
<td>(0, 0)</td>
<td>0.080203</td>
<td>0.080203</td>
<td>0.266842</td>
<td>0.144145</td>
<td>0.013969</td>
<td>0.0092057</td>
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<td>(1, 0)</td>
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<td>0.089433</td>
<td>0.284241</td>
<td>0.276307</td>
<td>0.019563</td>
<td>0.017723</td>
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<tr>
<td>(2, 0)</td>
<td>0.097379</td>
<td>0.097379</td>
<td>0.174946</td>
<td>0.224960</td>
<td>0.024835</td>
<td>0.023101</td>
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<tr>
<td>(3, 0)</td>
<td>0.086851</td>
<td>0.086851</td>
<td>0.106924</td>
<td>0.137984</td>
<td>0.029803</td>
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<tr>
<td>(4, 0)</td>
<td>0.077284</td>
<td>0.077284</td>
<td>0.065407</td>
<td>0.084483</td>
<td>0.034485</td>
<td>0.032945</td>
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<tr>
<td>(5, 0)</td>
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<td>0.068630</td>
<td>0.034577</td>
<td>0.034577</td>
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<tr>
<td>(6, 0)</td>
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<td>0.060831</td>
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</table>

Figure 2. Effect of $a$ on $W_q$ with varying $d$.
Table 2. Queue-length distributions at prearrival and arbitrary epochs for the finite buffer queue.

<table>
<thead>
<tr>
<th>$(n, r)$</th>
<th>$P_{n,r}$</th>
<th>$P_{n,r}$</th>
<th>$P_{n,r}$</th>
<th>$P_{n,r}$</th>
<th>$P_{n,r}$</th>
<th>$P_{n,r}$</th>
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<td>0.007770</td>
<td>0.007770</td>
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<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
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</tr>
</tbody>
</table>

$PBL = 0.367299$, $L = 8.952499$, $W = 0.943310$

Figure 1 compares the buffer size ($N$) versus loss probability ($PBL$) for various interarrival time distributions with same mean $\lambda = 5.0$, $\mu = 0.8$ and $a = 3$, $d = 5$, $b = 7$. It can be seen that loss probability in case of $HE_2$ distribution is marginally higher as compared to deterministic, Erlang and exponential distributions whereas the deterministic distribution yields the lowest loss probability. We further observe that for all the distributions considered here, loss probability decreases as $N$ increases and finally reaches its minimum value zero as the model becomes an infinite-buffer queue. Figure 2 depicts the effect of $a$ on the average waiting-time ($W_q$) when interarrival time is exponential with $\lambda = 10$, $\mu = 1.2$, $b = 40$ and $N = 50$ for various values of $d$. It can be observed that for all values of $d$, the average waiting-time increases as $a$ increases but it is smaller when $d$ is large.

Figure 3 illustrates dependence of the blocking probability on the buffer size $N$ varying from 1 to 25 and the batch size $b$ varying from 4 to 12. The interarrival time is assumed to be deterministic with $\lambda = 2.4$, $\mu = 1.0$, $a = 2$ and $d = 3$. We observe that for fixed batch size the loss probability decreases as the buffer size increases. Further with fixed buffer size it increases when the batch size increases. Hence we can setup an admissible batch size and the sufficient buffer size in the system in order to have lower blocking probability. The effect...
of traffic load ($\rho$) on average waiting-time ($W_q$) is shown in Figure 4 for various interarrival time distributions with $\mu = 3.5, a = 4, d = 6, b = 10$. As one would intuitively expect, it is observed that the average waiting-time increases as traffic load increases but it is lowest in the case of deterministic distribution.

![Figure 4. Effect of $\rho$ on $W_q$ for different arrival distributions.](image)

5. Conclusion

This paper presents a single server bulk-service queues with accessible and non-accessible batch services for both infinite- and finite- buffers. The inter-arrival time of arrival customers and service time of batches are, respectively, arbitrarily and exponentially
distributed. A recursive method has been developed to obtain the steady-state queue length distributions at pre-arrival and arbitrary epochs. The results found in this paper may be useful in system design, telecommunication and other applications.

References


Authors’ Biographies:

V. Goswami is currently a Professor in the School of Computer Application, KIIT University, Bhubaneswar, India. She received her Ph.D. degree from Sambalpur University, India, in the year 1994 and then worked as post doctoral fellow at Indian Institute of Technology, Kharagpur for two years. Her research interests include continuous- and discrete-time queues. She has published research articles in INFORMS Journal on Computing, Computers and Operations Research, RAIRO Operations Research, Computers and Mathematics with Applications, Computers and Industrial Engineering, Applied Mathematical Modelling, Applied Mathematics and Computation, etc.

P. Vijaya Laxmi is an Assistant Professor in the Department of Applied Mathematics, Andhra University, Visakhapatnam, India. She did M.Sc and Ph.D. from Indian Institute of Technology, Kharagpur, India in 1995 and 2003, respectively. Her main areas of research interest are continuous and discrete-time queueing models and their applications. She has publications in various Journals like Operations Research Letters, Queueing Systems, Applied Mathematical Modelling, etc.