Solving Multiobjective Linear Programming Problem Using Interval Arithmetic

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Abstract

In the real world, we often encounter cases where the information/data items can’t be determined with certainty. Hence the value of the datum in the data is assessed using an interval. Meanwhile multi-objective linear programming model is more adequate to describe the problem in the real world. Thus, the multiobjective linear programming problem will be developed into a multiobjective interval linear programming to estimate these uncertainties. Interval programming is one of the tools to tackle uncertainty in mathematical programming models. In this paper, it will be presented the multiobjective linear programming problems with interval numbers as coefficients and values of its variables are also in the form of intervals. The problems will be solved by modified simplex method.

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1 Introduction

Linear programming is among the most widely and successfully used decision tools in the quantitative analysis of practical problems where rational decisions have to be made. The conventional linear programming model requires the parameters to be known as constants. In the real world, however, the parameters are seldom known exactly and have to be estimated. Classical sensitivity analysis allows a study of the effect on the solution of changes to single coefficients or very small groups of coefficients, but only to the extent that the optimal basis is not changed. Therefore, we interest to study interval...
linear programming where its the coefficients and variables are in the form of interval \[17\].

In [2], the authors generalize known concepts of the solution of the linear programming problem with interval coefficients in the objective function based on preference relations between intervals. \[3\], the authors propose a new approach in which some or all of the coefficients of the LP are specified as intervals, and find the best optimum and the worst optimum for the model, and the point settings of the interval coefficients that yield these two extremes. In \[17\], the authors solved linear programming problems with interval in coefficients and variables (ILP) by transformation ILP model into real linear program.. In \[18\], the authors solved ILP by modified simplex method. In \[19\], the authors solved ILP by transformation ILP model into biobjective programming. In \[14\], the authors provide an illustrated overview of the state of the art of interval programming in the context of multiple objective linear programming models. In \[13\], authors consider multiobjective linear programming with interval coefficients and solve it with respect to necessarily efficient points. But the authors take an extreme point only (not determine) for efficient necessary test. The extreme point is not necessary efficient solution, as described in \[14\] and \[5\].

In this paper, I solved multiobjective interval linear programming by modified BMS Interval Simplex Algorithm in \[19\]. We used software Pascal-XSC to implement the algorithm at real interval.

## 2 Interval Arithmetic

The basic definitions and properties of interval numbers (or interval) and interval arithmetic can be seen in \[1, 11\], and \[12\].

**Definition 2.1** A real interval vector \(\overline{x} \in I(R^n)\) is a set of the form \(\overline{x} = (x_i)_{i=1}^n\), where \(i = 1, 2, \ldots, n\) and \(x_i = [x_i, x_i^S] \in I(R)\).

**Definition 2.2** A real interval matrix \(\overline{A} \in I(M(R^n))\) is a set of the form \(\overline{A} = (a_{ij})_{n \times n}\), where \(i = 1, 2, \ldots, n\) and \(a_{ij} = [a_{ij}, a_{ij}^S] \in I(R)\).

Let \(\overline{x} = [x_I, x_S]\) and \(\overline{y} = [y_I, y_S]\), then

1. \(\overline{x} + \overline{y} = [x_I + y_I, x_S + y_S]\) (addition)
2. \(\overline{x} - \overline{y} = [x_I - y_S, x_S + y_I]\) (subtraction)
3. \(\overline{x} \cdot \overline{y} = [\min\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}, \max\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}]\) (multiplication)
Solving multiobjective linear programming problem

3961

4. \( x \backslash y = [x_I, x_S][1 \backslash y_S, 1 \backslash y_I] \) for \( 0 \notin y \). (division)

Let \( x, y \) and \( z \in I(R) \)

5. \( x + y = y + x, xy = yx \) (commutativity)

6. \( (x + y) + z = x + (y + z), (xy)z = x(yz) \) (associativity)

7. \( x(y + z) \subseteq xy + xz \) (subdistributivity)

8. \( a(x + y) = ax + ay, a \in R \).

Let \( x \) and \( y \) be two interval numbers. In \([1, 2, 6, 7, 10, 12]\) and \([16]\), the authors present the following ways of comparing them.

**Definition 2.3** 1. \( x \leq y \) if and only if \( x_S \leq y_I \)

2. \( x \leq y \) if and only if \( x_I \geq y_I \) and \( x_S \leq y_S \)

3. \( x \leq y \) if and only if \( x_I \leq y_I \) and \( x_S \leq y_S \)

4. (a) \( x \leq y \) if and only if \( x_I \leq y_I \) and \( m(x) \leq m(y) \)

   (b) \( x \leq y \) if and only if \( x_S > y_I \) and \( m(x) < m(y) \)

   (c) \( x \leq y \) if and only if \( m(x) \leq m(y) \) and \( w(x) \geq w(y) \)

   where \( m(x) = \frac{x_I + x_S}{2}, w(x) = \frac{x_S - x_I}{2} \)

5. (a) \( x \leq y \) if and only if \( x_I - \varepsilon \leq y_S \)

   (b) \( x \leq y \) if and only if \( x_S - \varepsilon \leq y_I \)

6. \( x \leq y \) if and only if \( x_I + x_S \leq y_I + y_S \)

3 Interval Linear Programming

ILP is a linear programming where all of coefficients and variables are intervals. Interval linear programming problem can be formulated as follows. Maximize (objective function)

\[
Z = \sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} [c_{jI}, c_{jS}] [x_{jI}, x_{jS}]
\]

subject to

\[
(1)
\]
\[ \sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n) \]

where \( c_j, a_{ij}, b_i, x_j \in I(R) \), and \( I(R) \) is the set of all interval numbers in \( R \).

**Theorem 3.1** [17] Suppose that we have an objective function given by

\[ Z = \sum_{j=1}^{n} [c_{jI}, c_{jS}] [x_{jI}, x_{jS}] \]

where \([x_{jI}, x_{jS}] \geq 0\). Then

\[ \sum_{j=1}^{n} \max \{c_{jS}x_{jI}, c_{jS}x_{jS}\} \geq \sum_{j=1}^{n} \min \{c_{jI}x_{jI}, c_{jI}x_{jS}\} \]

for any given interval vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \).

**Theorem 3.2** [17] Suppose that we have an objective function given by

\[ Z = \sum_{j=1}^{n} [c_{jI}, c_{jS}] [x_{jI}, x_{jS}] \]

where \([x_{jI}, x_{jS}] \leq 0\). Then

\[ \sum_{j=1}^{n} \min \{c_{jS}x_{jI}, c_{jS}x_{jS}\} \leq \sum_{j=1}^{n} \max \{c_{jI}x_{jI}, c_{jI}x_{jS}\} \]

for any given interval vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \).

## 4 Multiobjective Interval Linear Programming

A multiobjective linear programming problem simultaneously optimizes some objectives subject to the given constraints. In general, the problem has no optimal solution that could optimize all objectives simultaneously. And concept of optimal solution gives rise to the concept of non dominated solutions, for which no improvement in any objective function is possible without sacrificing at least one of the other objective functions. This multiobjective linear programming is formulated as follows:

Maximize \( Z_1(x) = c_1x \)

Maximize \( Z_2(x) = c_2x \)

\[ \ldots \]
Solving multiobjective linear programming problem

Maximize \( Z_r(x) = c_r x \)
subject to

\[ x \in X = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} \]

where \( x \) is an \( n \)-dimensional vector of decision variables, \( Z_1(x), Z_2(x) \) and \( Z_r(x) \) are \( r \) distinct linear objective functions of the decision vector \( x \), \( c_1, c_2 \) and \( c_r \) are \( n \)-dimensional cost vectors, \( A \) is an \( m \times n \) constraint matrix, and \( b \) is an \( m \)-dimensional constant vector. An optimal (efficient) solution to this problem is Pareto optimal and is defined precisely as follows.

**Definition 4.1** [8] A solution \( x \in X \) is said to be an efficient solution of equation (2) if for any \( y \in X \) satisfying \( Z_1(y) \geq Z_1(x) \) and \( Z_2(y) \geq Z_2(x) \), we have \( Z_1(y) = Z_1(x) \) and \( Z_2(y) = Z_2(x) \).

**Definition 4.2** [15] Multiple objective functions can be combined into a single composite one to be maximized by summing objectives with positive weights on maximizes and negative weights on minimizes. If the composite is to be minimized, weights on maximize objectives should be negative, and those on minimizes should be positive.

**Theorem 4.3** [15, 4] If a single weighted-sum objective model derived from multiobjective optimization as in Definition 4.2 produces an optimal solution, the solution is an efficient point of the multiobjective model.

Based on formulation interval linear programming problems, multiobjective linear programming explain above, multiobjective interval linear programming problem (MOILP) can be formulated as follows.

\[
Maximize Z_1 = \sum_{j=1}^{n} c_{1j} x_j
\]

\[
Maximize Z_2 = \sum_{j=1}^{n} c_{2j} x_j
\]

\[ \ldots \]

\[
Maximize Z_r = \sum_{j=1}^{n} c_{rj} x_j
\]

subject to

\[
\sum_{j=1}^{n} a_{ij} x_j \leq b_i, (i = 1, 2, \ldots, m)
\]

where \( c_{1j}, c_{2j}, \ldots, c_{rj}, a_{ij}, b_i, x_j \in I(\mathbb{R}) \).
5 BMSM Interval Simplex Algorithm

In [19], the authors used BMS Interval Simplex Algorithm to solve ILP. In this paper I modified BMS Interval Simplex Algorithm to solve MOILP. I modified the algorithm in order that the algorithm can be used to determine non dominated solutions of MOILP. The modified algorithm can be written as in the following algorithm

Algorithm 5.1 Data: $A, c_1, c_2, \ldots, c_1, \text{ and } b$ in the MOILP model, where $x \geq 0$ and $N$ (maximum transformation)

1. case true of
   - model has no solution :
     1.1. stop! write message “no solution”
   - model is unbounded :
     1.2. stop! write message “unbounded”
   - model has nondominated solution
     1.3. stop! write message “nondominated solution”
   - default :
     1.4. Objective function transformation.
     1.5. Check an initial feasible solution.

2. if no initial basic feasible solution.
   then
   2.1. stop! the model has no solution to the current transformation and go to 3
   else!
   2.2. Make set of nondominated solution
   2.3. Selection of the entering variable
   2.4. Determine values for the coefficients objective function for Dominance Test
   2.5. If all coefficient’s values positive or zero.

   - then
     2.5.1. Set of nondominated solution the current transformation been got and go to 3
   - else
     2.5.2. Choose the entering variable with the most negative coefficient’s values.
     2.5.3. Selection of the leaving variable.
     2.5.4. Compute vector right hand side.
     2.5.5. Compute value vector column pivot.
     2.5.6. If all vector’s values negative or zero.
2.5.6.1. The model has no bounded solution to the current transformation and go to 3.

2.5.6.2. Choose the leaving variable with the minimum ratio test.
2.5.6.3. Updating the inverse basis matrix.
2.5.6.4. Determine the value of basic variable.
2.5.6.5. Determine new basic feasible solution.
2.5.6.6. Dominance Test.
2.5.6.7. If the new solution is nondominated

* then
2.5.6.7.1. Updating the set of nondominated solution.
2.5.6.7.2. go to 2.4.

* else
2.5.6.7.3. go to 2.5.2

3. if number of iteration equal to N

• then Stop !

• else go to 1.4

6 Numerical Example

Let us consider the following example of MOILP, Ida in [14], (where variables are real interval.

Maximize \( Cx \)

subject to \( Ax \leq b \)

\( x \geq 0, x \in I(\mathbb{R}^n). \)

\[
\begin{pmatrix}
[3,4] & [0,1] & [1,2] & [1,2] & [0,1] & [-2,-1] & [-2,-1]
\end{pmatrix}
\]

\[
A = \begin{bmatrix}
1 & 2 & 1 & 1 & 2 & 1 & 2 \\
-2 & -1 & 0 & 1 & 2 & 0 & 1 \\
-1 & 0 & 1 & 0 & 2 & 0 & -2 \\
0 & 1 & 2 & -1 & 1 & -2 & -1
\end{bmatrix}
\]

\[
b = \begin{bmatrix}
16 \\
16 \\
16 \\
16
\end{bmatrix}
\]
Implementation the algorithm to Pascal-XSC, I can have result non-dominated solution: \(x = ([0,0], [0,0], [0,0], [16,16], [0,0], [0,0])\). In [13, 14], authors test the necessary efficiency of the extreme point \((0, 0, 32/2, 16/3, 0, 0, 0)\) (not determine), then the authors obtain that the extreme point is not necessary efficient. In [5], author can have result solution \(x = (0, 0, 0, 16, 0, 0, 0)\) and author has proved that the solution is necessary efficient. My solution better than the solution in [13, 14] and equal to the solution in [5]. In [13, 14, 5] used a real valued variables, whereas I use a real interval valued variables, and I use the method to process interval directly.

An example, Lee in [9], the authors obtain two efficient solutions. Using the algorithm BMSM, I also obtain two efficient solutions \(((49.99,50], [0,0], [0,0], [0,0])\) and \(((0,0], [0,0], [99.98,100], [0,0])\). And one of my solutions includes a solution on [9].

Example 7.13, Wiecek in [4], the authors obtain three efficient solutions. Using the algorithm BMSM, I also obtain three efficient solutions \(((0,0], [1,1], [0,0])\), \(((1,1], [0,0], [0,0]),\) and \(((0,0], [1,1], [5,5]))\). My solutions is equal to results obtained by Ehrgott [4], although in [4] used a real valued variables. If Example 7.13, Wiecek in [4] is modified by replacing the values of the coefficients, real number become real interval. Median interval are equal to the coefficients values. Using the algorithm BMSM, I obtain six efficient solutions \(((0,1.4], [0,0], [0,4.7]), ([0,1.4], [0,0], [0,0]), ([0,0], [0,1.4], [0,0]), ([0,0], [0,1.4], [0.66,5.2]), ([0,0], [0,1.4], [0,4.7]),\) and \(((0,1.4], [0,1.4], [0.66,4.7]))\). And three of my solutions includes three solution on [4]. This is an advantage possessed by calculation using interval.

## 7 Conclusions

I has modified BMSD Interval Simplex Algorithm become BMSM Interval Simplex Algorithm. BMSM Interval Simplex Algorithm can be used to determine some non dominated solutions of the multiobjective interval linear programming problems. Multiobjective interval linear programming problems can be solved with execute real interval directly use software Pascal-XSC.

## References


Solving multiobjective linear programming problem


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