

Mathematical Model of oxygen Transport in micro vessels of Human Body with special role of Blood

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ABSTRACT

This article is a Mathematical Study of oxygen transport in human body. We can clearly see that diffusion is an inefficient means of oxygen transport. We also observe the role of blood in the transport of the oxygen in the human body in the form of that erythrocyte which is the main carrier of transport of oxygen. We also modeled an expression for partial pressure as well as pressure gradient.

Keywords

Mathematical Modeling, Diffusion Process, partial pressure, pressure gradient.

1. INTRODUCTION

Mathematical Modeling plays an important role in every field of knowledge viz. Ecology, Physiology, Sociology, Biology etc., we consider here an application of Mathematical Modeling to oxygen transport in the human body. Symbolically basic Mathematical Modeling may be represented in figure-1. [1]

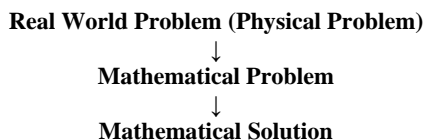


Figure-1: Basic Mathematical Modeling

Mathematical Modeling in terms of differential equations arises when the situation modeled involves some continuous variable(s) varying with respect to some other continuous variable(s) and we have some responsible hypothesis about the rates of change of dependent variable(s) with respect to independent variable(s). If there are a number of dependent continuous variables and only one independent variables and only one independent variable, the hypothesis may give a mathematical Model in terms of system of first or higher order ordinary differential equations [2]. This study also generates the system of ordinary differential equations. The purpose of this paper is to study mathematical model of human blood. In order to develop a mathematical model it is necessary to model many aspects of the blood in the human body. The interdisciplinary field of applied mathematical modeling in human physiology has developed tremendously during the last decade and continues to develop. One of the reasons for this development is researchers' improved ability to gather data. The amount of physiological data obtained from various experiments is growing exponentially due to faster sampling methods and better methods for obtaining both invasive and non-invasive data. In addition, data have a much better resolution in time and space than just a few years ago. For example, some of the non-invasive measurements using MRI (magnetic resonance imaging) can provide information of

blood velocity as a function of time and three spatial coordinates in both the heart and in arteries with a diameter of only a few millimeters. This large amount of data obtained from advanced measurement techniques constitutes a giant collection of potential insight. Statistical analysis may discover correlations, but may fail to provide insight into the mechanisms responsible for these correlations. However, combined with mathematical modeling of the dynamics new insights into physiological mechanisms may be revealed. The large amount of data can make the models give not only qualitative but also quantitative information of the function they predict and they may also be used to suggest new experiments. We think that such models are necessary for improving the understanding of the function of the underlying physiology, and in the long term mathematical models may help in generating new mathematical and physiological theories.

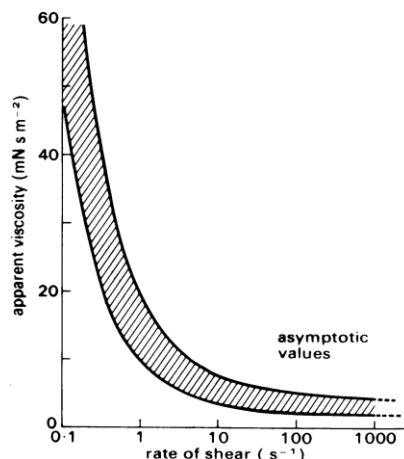


Figure2: The apparent viscosity as a function of the shear rate in human blood. When the shear rate is about 1000 s^{-1} the non-Newtonian behavior becomes insignificant, and the apparent viscosity approaches an asymptotic value ranging from $0.03\text{--}0.04 \text{ g}/(\text{cm s}) (= 3\text{--}4 \text{ mN s m}^{-2})$. [4]

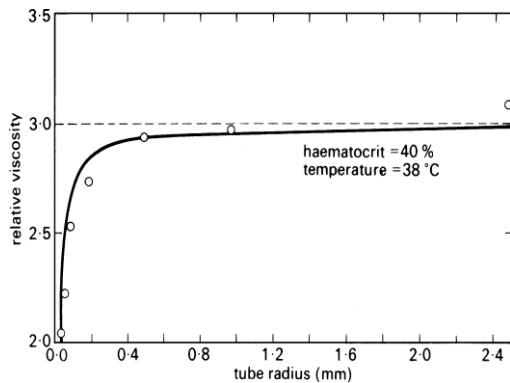


Figure 3: The relationship between the apparent viscosity of blood relative to the plasma viscosity and the diameter of the tube in which the blood is flowing. The tube is assumed to be cylindrical. [4].

Liesch, D. and Moravec, S. [5] Investigated the flow of a shear thinning blood, analog fluid in pulsatile flow through arterial branch model and observed large differences in velocity profiles relative to these measured with Newtonian fluids having the high shear rate viscosity of the analog fluid. Rodkiewicz et al. (1990)[6] used several different non-Newtonian models for simulations of blood flow in large arteries and they observed that there is no effect of the yield stress of blood on either the velocity profiles or the wall shear stress. Out of these a large number of numerical studies concentrate on the transport phenomena in the microcirculation. Tang, D. et al [7] Analyzed blood flow in carotid arteries with stenosis. Sharma, G.C. et al [8] considered a mathematical analysis of blood flow through arteries using finite element Galerkin approaches. Khaled and Vafai [9] Studied flow and heat transport in porous media using mass diffusion and different convective flow models such as Darcy and the Brinkman models, Energy transport in tissue is also analyzed. In the micro-circulation, it is no longer possible to think of the blood as a homogeneous fluid; it is essential to treat it as a suspension of red cells in plasma. The reason being that even the largest vessels of the micro-circulation are only approximately 15 cells in diameters. Also, as discussed earlier in this chapter, viscosity starts dominating the mechanical behavior, leading to low Reynolds numbers. Typical Reynolds numbers in 100 μm arteries are about 0.5. We can conclude that blood is generally a non-Newtonian fluid, but it is reasonable to regard it as a Newtonian fluid when modeling arteries with diameters larger than 100 μm . For very small vessels it is not easy to reach conclusions as to the Newtonian nature of blood because some effects tend to decrease the viscosity and others tend to increase the viscosity. The latter influence on viscosity is due to a small flow, which increases the viscosity significantly, as well as to the fact that cells often become stuck at constrictions in small vessels. However, a cell becoming stuck happens most often in the capillaries. The net effect of all these influences on blood viscosity is that it is reasonable to assume that the overall viscous effects in the small vessels are approximately equivalent to those that occur in the larger vessels. The diameters of blood vessels range over several orders of magnitude. This variation may impose a problem for modeling purposes, but that problem can be overcome by dividing the arteries into several groups: large arteries, small arteries, arterioles and capillaries. This distinction is somewhat arbitrary, but can be justified by the different properties of the vessels as they gradually become smaller. The main purpose of the respiratory system is to transport

oxygen and carbon dioxide between the atmosphere and the tissue and organs in the body. Oxygen is a necessity for life and a human being consumes approximately 260 ml/min at rest [10]. The oxygen is delivered from the atmosphere to the organs and tissue via the lungs and the blood circuit. Carbon dioxide is a waste product of oxidative metabolism, and is carried by the blood in the opposite direction, from the tissue to the lungs, where it is removed by ventilation. The carbon dioxide elimination rate at rest is about 160 ml/min [10]. Since carbon dioxide dissolved in blood forms carbonic acid, which affects the pH value of the blood, the removal of carbon dioxide plays an important role in the acid-base balance in the blood. The respiratory cycle starts in the atmosphere outside the body. By inspiration oxygen enters the lungs, as 21% by volume of atmospheric air consists of oxygen. During inspiration air enters the lung where it mixes with the air already in the lung. The upper airways and the lungs form a tree structure, i.e. the pulmonary tree, connecting the atmosphere with the alveoli, which are small air-filled sacs. From the alveoli oxygen diffuses across a membrane into the blood of the pulmonary capillaries, by this diffusion the content of oxygen in the alveoli is reduced, and hence the expiratory air contains only 16% oxygen.

2. MATHEMATICAL MODELING PROCESS

We all know that oxygen is a very important for human life and if oxygen transport breaks in the human body this leads to very fatal stage and may cause death. Oxygen transport in the human body evolves a complex networks of small capillaries as many studies reveals in the literature as described in the introduction part of the paper. Many theories have been arrived. This capillaries network is as complex as the traffic transport in a metro city. Like roads in city capillaries have their different names as shown in Figure-4, some may be long and some small in their size, length etc. A study has been done by Rushmer, R. F. [11] in which describes the shape and size of different capillaries evolves in the human body for oxygen transport. We, model the oxygen transport phenomenon in the body via diffusion process i.e. micro circulation. According the Merriam-Webster Diffusion may be written in simple way as follows “the process whereby particles of liquids, gases, or solids intermingle as the result of their spontaneous movement caused by thermal agitation and in dissolved substances move from a region of higher to one of lower concentration.” We use this basic concept of diffusion in this paper. In this direction Fick's First Law of Diffusion is

$$J = -D \frac{\partial C}{\partial x} \Rightarrow J = -D \frac{dC}{dx}$$

Eq. (1)

Where C is the concentration, x is the position, and D is the diffusivity, or diffusion constant, of the substance. As described above that erythrocyte is the main carrier of oxygen and erythrocyte is the main component of blood therefore one may consider that blood is main component in the transport of oxygen in the human body. In the study of oxygen transport the concept of partial pressure is equally important. Again Merriam-Webster describes the partial pressure as “the pressure exerted by a (specified) component in a mixture of gases.” In this direction, Henry's Law

$$C = \beta P \text{ Eq. (2)}$$

Defines the concentration, C , of a gas in some medium, like tissue, in terms of partial pressure, P , the solubility of the medium and Hills equation for total saturation is given by

$$S(P) = \frac{P^n}{P^{n_{1/2}} + P^n} \quad \text{Eq. (3)}$$

Where, $P_{1/2}$ denotes the partial pressure when the saturation is exactly $\frac{1}{2}$. We will use these basic laws i.e. Henry's Law and Hills equation for further modeling process.

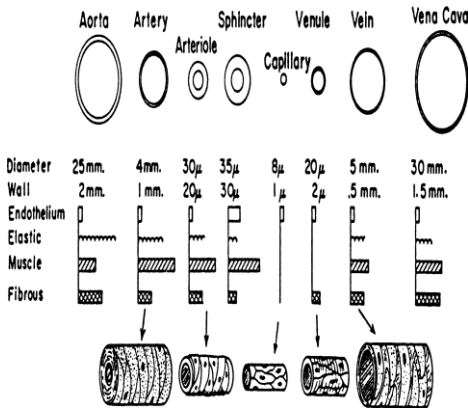


Figure 4: Cross-sections of arteries and veins. The vessels have an inner layer of endothelial cells and an outer layer composed of fibers with a varying degree of muscle and elastic fibers. The figure shows the relative content for each group of vessels [11]

For Human Tissue we can take in general, the values of Diffusion constant and the solubility of the medium respectively as follows-

$$D = 2.0 \times 10^{-5} \text{ cm}^2 / \text{seconds}$$

$$\beta = 3.1 \times 10^{-5} \text{ cm}^3 \frac{\text{O}_2}{\text{cm}^3 \cdot \text{mmHg}}$$

Now we have, the Three Dimensional Poisson equation is as follows:

$$\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \right) = \frac{M}{D\beta} \quad \text{Eq. (4)}$$

Where M is a suitable parameter. Now, if we take $M=0$, then, we have:

$$\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \right) = 0 \quad \text{Eq. (5)}$$

If, we consider one dimensional case, then Eq. (5) converted to:

$$\frac{d^2 P}{dx^2} = 0 \Rightarrow P = d_1 x + d_2 \quad \text{Eq. (7)}$$

Where d_1 and d_2 are integration constants. For human tissue we can impose the initial conditions $x = 0$ cm and

$P=100$ mmHg. and $x=0.1$ cm, $P=0$ mmHg. These initial conditions give us $d_1 = -1000$ and $d_2 = 100$ Eq. (2)

$$\text{gives } C = \beta(d_1 x + d_2)$$

$$\frac{dc}{dx} = \beta d_1 \Rightarrow \frac{dc}{dx} = 3.1 \times 10^{-2} \text{ unit}$$

hence diffusion flux in this one dimensional case is

$$J_x = 6.2 \times 10^{-7} \frac{\text{cmO}_2}{\text{sec}}$$

It means if the initial partial pressure is $P=100$ mmHg Oxygen

diffuses $0.0062 \mu\text{m sec}^{-1}$. This is a very short distance, hence one may conclude that diffusion is an inefficient means of transport, it is a clear-cut picture of relation between diffusion and transport. Using Poisson's equation again, with $M = 1/600 \text{ cm}^3 \text{O}_2 / \text{cm}^3 \cdot \text{s}$, given an initial partial pressure of 100 mmHg at $x = 0$ cm, we can easily calculate the distance that oxygen will diffuse until the diffusion flux and the partial pressure are both zero. The microcirculation is so important in supplying nutrients to the body. Vessels must always be extremely close to each other, or diffusion will not be an adequate means of supplying oxygen to all the necessary parts of the body. In order to formulate a model of oxygen transport that is more representative of actual phenomenon in the human body, we must consider three dimensions. As shown in figure 4 the different cross sections arteries and veins and consider the cylindrical shape of the capillaries of the body through which the oxygen transport takes place.

In one dimensional model, we know that-

$$\nabla^2 C = \partial^2 C / \partial r^2$$

In three-dimensions, in polar co-ordinates, we have

$$\nabla^2 C = \frac{\partial^2 C}{\partial r^2} + \frac{\partial C}{r \partial r} + \frac{1}{r^2} \frac{\partial^2 C}{\partial \theta^2} \quad \text{Eq. (8)}$$

$$\frac{\partial P}{\partial r} = \frac{rM}{2D\beta} + \frac{c_1}{r}$$

Eq.(9), Or,

$$r \frac{\partial P}{\partial r} = \frac{r^2 M}{2D\beta} + c_1$$

Eq. (9a)

Where c_1 is integration constant. Eq. (9) is an expression for pressure gradient. Integration of Eq. (9) will lead to expression,

$$P = \frac{r^2 M}{4D\beta} + c_1 \log r + c_2$$

Eq. (10),

Where c_2 is again integration constant. In general, we must have the P as the form

$$P = P_0 + \frac{M}{2D\beta} \left[\frac{(r-r_c)^2}{2} - r_i^2 (\log r - \log r_c) \right]$$

Eq.(11), Or

$$P = P_0 + \frac{M}{2D\beta} \left[\frac{(r-r_c)^2}{2} - r_i^2 (\log r / \log r_c) \right]$$

Eq. (12)

In this equation, we assume, p_0 is the initial partial pressure at the center of the cylinder, r_0 is the radius of the capillary, r_t is the radius of the tissue surrounding the capillary and r is the total radius of the cylindrical capillary. The parameters viz. initial pressure, radius of the capillary, radius of the tissue surrounding the capillary etc. are in general variables so the this type of transport phenomenon is to complicated to formulate however, we have derived in general an expression for partial pressure as well as pressure gradient. For one set of values this may work. From the preceding models, we can clearly see that diffusion is an inefficient means of oxygen transport. This is why the microcirculation is so important. Without it, oxygen could not be transported to all parts of the body.

3. CONCLUSION

In this paper, we only analyzed models of oxygen transport with initial partial pressure of 100 mmHg and an oxygen consumption rate of either 0 or 1/600 cm³O₂/cm³. Really, these values can change depending on certain conditions. For example, the partial pressure of oxygen will be less in people with lung disease or people at high altitudes. Oxygen consumption can change significantly depending on where oxygen is being supplied or how much work a person is doing. In skeletal muscle $M \approx 1/6000$ at rest but increases to 1/600 at moderate exercise, and to 4/600 or high levels of exercise. Mathematical models of oxygen transport can be much more sophisticated than the models analyzed above. Obviously, oxygen transport in the human body is much more complicated than simple diffusion. In this paper we also give the basic properties of human blood. Micro-vascular networks often change their structure when they are growing and in response to functional demands, such as changes in metabolic requirements. From the preceding models, we can clearly see that diffusion is an inefficient means of oxygen transport. We also observe the role of blood in the transport of the oxygen in the human body in the form of that erythrocyte is the main carrier of transport of oxygen. We also modeled an expression for partial pressure in the form of Eq. (11) or Eq. (12) pressure gradient in Eq. (9)

4. ACKNOWLEDGMENTS

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