ON NON-FORMAL SIMPLY CONNECTED MANIFOLDS

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Abstract. We construct examples of non-formal simply connected and compact oriented manifolds of any dimension bigger or equal to 7.

1. Introduction

An oriented compact manifold of dimension at most 2 is formal. On the other hand, if the dimension is 3 or more, there are examples which are non-formal, e.g., nilmanifolds which are not tori [4].

If we turn our attention to simply connected manifolds, we know that a simply connected oriented compact manifold of dimension at most 6 is formal [6, 5, 3]. The natural question already raised in [3] is whether there are examples of non-formal simply connected oriented compact manifolds of dimension $d \geq 7$.

Clearly, the question is reduced to the cases $d = 7$ and $d = 8$. For if we have a non-formal simply connected manifold M of dimension d, then $M \times S^{2n}$ is a non-formal simply connected manifold of dimension $d + 2n$, for any $n \geq 1$.

From now on let $d = 7$ or $d = 8$. By the results of [3], if a d–dimensional connected and compact oriented manifold M is 3–formal then it is formal. Therefore, the non-formality of M has to be detected in the 3–stage of its minimal model. Moreover if $H^1(M) = 0$ then M is automatically 2–formal, so the non-formality is due to the kernel of the cup product map $\cup : H^2(M) \otimes H^2(M) \to H^4(M)$. The easiest way to detect the non-formality is thus to have a non-trivial Massey product of cohomology classes of degree 2.

The method of construction of d–dimensional simply connected manifolds that we will use is the following: take a non-formal compact nilmanifold X of dimension d with a non-trivial Massey product of cohomology classes of degree 1. Multiply these cohomology classes by

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some cohomology classes so that we get a non-trivial Massey product of cohomology classes of degree 2. Then perform a suitable surgery of X to kill the fundamental group such that the non-trivial Massey product survives. This will give the sought example.

In [1] Babenko and Taimanov have already given examples of non-formal simply connected manifolds of any *even* dimension bigger or equal to 10. The relevant property of their examples is that they are symplectic manifolds. They ask whether there exist examples of non-formal simply connected *symplectic* manifolds of dimension 8. Unfortunately, our examples do not have a symplectic structure, at least in an obvious way.

2. The 8-dimensional example

Let H be the Heisenberg group, that is, the connected nilpotent Lie group of dimension 3 consisting of matrices of the form

$$a = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. Then a global system of coordinates x, y, z for H is given by $x(a) = x$, $y(a) = y$, $z(a) = z$, and a standard calculation shows that a basis for the left invariant 1–forms on H consists of $\{dx, dy, dz - xdy\}$. Let Γ be the discrete subgroup of H consisting of matrices whose entries are integer numbers. So the quotient space $N = \Gamma\backslash H$ is a compact 3–dimensional nilmanifold. Hence the forms dx, dy, $dz - xdy$ descend to 1–forms α, β, γ on N and

$$d\alpha = d\beta = 0, \quad d\gamma = -\alpha \wedge \beta.$$

The non-formality of N is detected by a non-zero triple Massey product

$$\langle [\alpha], [\beta], [\alpha] \rangle = [2\,\alpha \wedge \gamma].$$

Now let us consider $X = N \times \mathbb{T}^5$, where $\mathbb{T}^5 = \mathbb{R}^5/\mathbb{Z}^5$. The coordinates of $\mathbb{R}^5$ will be denoted x_1, x_2, x_3, x_4, x_5. So $\{dx_i | 1 \leq i \leq 5\}$ defines a basis $\{\delta_i | 1 \leq i \leq 5\}$ for the 1–forms on $\mathbb{T}^5$. By multiplying the classes α and β by some of the δ_i, we get a non-zero triple Massey product of cohomology classes of degree 2 for X,

$$\langle [\alpha \wedge \delta_1], [\beta \wedge \delta_2], [\alpha \wedge \delta_3] \rangle = [2\,\gamma \wedge \alpha \wedge \delta_1 \wedge \delta_2 \wedge \delta_3]. \tag{1}$$

Our aim now is to kill the fundamental group of X by performing a suitable surgery construction. Let C_1 the image of $\{(x, 0, 0) | x \in \mathbb{R}\} \subset H$ in $N = \Gamma\backslash N$ and let C_2 be the image of $\{(0, y, \xi) | y \in \mathbb{R}\}$ in N, where ξ is a generic real number. Then $C_1, C_2 \subset N$ are disjoint embedded circles such that $p(C_1) = \mathbb{S}^1 \times \{0\}$, $p(C_2) = \{0\} \times \mathbb{S}^1$. The projection $p(x, y, z) = (x, y)$ describes N as a fiber bundle $p : N \to \mathbb{T}^2$ with fiber $\mathbb{S}^1$. Actually, N is the total space of the unit circle bundle of the line bundle of degree 1 over the 2–torus. The fundamental group of N is therefore

$$\pi_1(N) \cong \Gamma = \langle \lambda_1, \lambda_2, \lambda_3 | [\lambda_1, \lambda_2] = \lambda_3, \ \lambda_3 \text{ central} \rangle, \tag{2}$$

where λ_3 corresponds to the fiber, λ_1 and λ_2 correspond to the homotopy classes $\lambda_1 = [C_1]$ and $\lambda_2 = [C_2]$. The fundamental group of $X = N \times \mathbb{T}^5$ is

$$\pi_1(X) = \pi_1(N) \oplus \mathbb{Z}^5. \tag{3}$$

Consider the following submanifolds embedded in X:

$$\begin{aligned} T_1 &= C_1 \times \mathbb{S}^1 \times \{0\} \times \mathbb{S}^1 \times \{0\} \times \mathbb{S}^1, \\ T_2 &= C_2 \times \{0\} \times \mathbb{S}^1 \times \{0\} \times \mathbb{S}^1 \times \mathbb{S}^1, \end{aligned}$$

which are 4-dimensional tori with trivial normal bundle. Consider now another 8-manifold Y with an embedded 4-dimensional torus T with trivial normal bundle. Then we may perform the *fiber connected sum* of X and Y identifying T_1 and T, denoted $X\#_{T_1=T}Y$, in the following way: take (open) tubular neighborhoods $\nu_1 \subset X$ and $\nu \subset Y$ of T_1 and T respectively; then $\partial\nu_1 \cong \mathbb{T}^4 \times \mathbb{S}^3$ and $\partial\nu \cong \mathbb{T}^4 \times \mathbb{S}^3$; take an orientation reversing diffeomorphism $\phi : \partial\nu_1 \stackrel{\cong}{\to} \partial\nu$; the fiber connected sum is defined to be the (oriented) manifold obtained by gluing $X - \nu_1$ and $Y - \nu$ along their boundaries by the diffeomorphism ϕ. In general, the resulting manifold depends on the identification ϕ, but this will not be relevant for our purposes.

Lemma 1. *Suppose Y is simply connected. Then the fundamental group of $X\#_{T_1=T}Y$ is the quotient of $\pi_1(X)$ by the image of $\pi_1(T_1)$.*

Proof. Since the codimension of T_1 is bigger or equal than 3, we have that $\pi_1(X - \nu_1) = \pi_1(X - T_1)$ is isomorphic to $\pi_1(X)$. The Seifert-Van Kampen theorem establishes that $\pi_1(X\#_{T_1=T}Y)$ is the amalgamated sum of $\pi_1(X - \nu_1) = \pi_1(X)$ and $\pi_1(Y - \nu) = \pi_1(Y) = 1$ over the image of $\pi_1(\partial\nu_1) = \pi_1(T_1 \times \mathbb{S}^3) = \pi_1(T_1)$, as required. □

We shall take for Y the sphere $\mathbb{S}^8$. We embed a 4-dimensional torus $\mathbb{T}^4$ in $\mathbb{R}^8$. This torus has a trivial normal bundle since its tangent bundle is trivial (being parallelizable) and the tangent bundle of $\mathbb{R}^8$ is also trivial. After compatifying $\mathbb{R}^8$ by one point we get a 4-dimensional torus $T \subset \mathbb{S}^8$ with trivial normal bundle.

In the same way, we may consider another copy of the 4-dimensional torus $T \subset \mathbb{S}^8$ and perform the fiber connected sum of X and $\mathbb{S}^8$ identifying T_2 and T. We may do both fiber connected sums along T_1 and T_2 simultaneously, since T_1 and T_2 are disjoint. Call

$$M = X\#_{T_1=T}\mathbb{S}^8\#_{T_2=T}\mathbb{S}^8$$

the resulting manifold. By Lemma 1, $\pi_1(M)$ is the quotient of $\pi_1(X)$ by the images of $\pi_1(T_1)$ and $\pi_1(T_2)$. This kills the $\mathbb{Z}^5$ summand in (3) and it also kills λ_1 and λ_2 in (2). Therefore $\pi_1(M) = 1$, i.e., M is simply connected.

3. Non-formality of the constructed manifold

Our goal is now to prove that M is non-formal. We shall do this by proving the non-vanishing of a suitable triple Massey product. More specifically, let us prove that the Massey product (1) survives to M. For this, let us describe geometrically the cohomology classes $[\alpha\wedge\delta_1]$, $[\beta\wedge\delta_2]$ and $[\alpha\wedge\delta_3]$. Consider the following three codimension 2 submanifolds of X:

$$\begin{aligned} B_1 &= p^{-1}(\mathbb{S}^1\times\{a_1\})\times\{b_1\}\times\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1, \\ B_2 &= p^{-1}(\{a_2\}\times\mathbb{S}^1)\times\mathbb{S}^1\times\{b_2\}\times\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1, \\ B_3 &= p^{-1}(\mathbb{S}^1\times\{a_3\})\times\mathbb{S}^1\times\mathbb{S}^1\times\{b_3\}\times\mathbb{S}^1\times\mathbb{S}^1, \end{aligned}$$

where the a_i and b_i are generic points of $\mathbb{S}^1$. It is easy to check that $B_i\cap T_j=\emptyset$ for all i and j. So B_i may be also considered as submanifolds of M. Let η_i be the 2–forms representing the Poincaré dual to B_i in X. By [2], η_i are taken supported in a small tubular neighborhood of B_i. Therefore the support of B_i lies inside $X-T_1-T_2$, so we also have naturally $\eta_i\in\Omega^2(M)$. Note that in X we have clearly that $[\eta_1]=[\alpha\wedge e_1]$, $[\eta_2]=[\beta\wedge e_2]$ and $[\eta_3]=[\alpha\wedge e_3]$, where e_i are differential 1–forms on $\mathbb{S}^1$ cohomologous to δ_i and supported in a neighborhood of $b_i\in\mathbb{S}^1$. Thus $[\eta_1]=[\alpha\wedge\delta_1]$, $[\eta_2]=[\beta\wedge\delta_2]$ and $[\eta_3]=[\alpha\wedge\delta_3]$.

Lemma 2. *The triple Massey product $\langle[\eta_1],[\eta_2],[\eta_3]\rangle$ is well-defined on M and equals to $[2\,\gamma\wedge\alpha\wedge e_1\wedge e_2\wedge e_3]$.*

Proof. Clearly

$$(\alpha\wedge e_1)\wedge(\beta\wedge e_2)=d\gamma\wedge e_1\wedge e_2,$$

where the 3–form $\gamma\wedge e_1\wedge e_2$ is supported in a neighborhood of $N\times\{b_1\}\times\{b_2\}\times\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1$, which is disjoint from T_1 and T_2. Hence $\gamma\wedge e_1\wedge e_2$ is well-defined as a form in M. Also

$$(\beta\wedge e_2)\wedge(\alpha\wedge e_3)=-d\gamma\wedge e_2\wedge e_3,$$

where $-\gamma\wedge e_2\wedge e_3$ is also well-defined in M. So the triple Massey product

$$\langle[\eta_1],[\eta_2],[\eta_3]\rangle=[2\,\gamma\wedge\alpha\wedge e_1\wedge e_2\wedge e_3]$$

is well-defined in M. □

Finally let us see that this Massey product $\langle[\eta_1],[\eta_2],[\eta_3]\rangle=[2\,\gamma\wedge\alpha\wedge e_1\wedge e_2\wedge e_3]$ is non-zero in

$$\frac{H^5(M)}{[\alpha\wedge e_1]\cup H^3(M)+H^3(M)\cup[\alpha\wedge e_3]}.$$

To see this, consider $B_4=p^{-1}(\{a_4\}\times\mathbb{S}^1)\times\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1\times\{b_4\}\times\{b_5\}$, for generic points a_4,b_4,b_5 of $\mathbb{S}^1$. Then the Poincaré dual of B_4 is defined by a 3–form $\beta'\wedge e_4\wedge e_5$ supported near B_4, where β' is Poincaré dual to $p^{-1}(\{a_4\}\times\mathbb{S}^1)$ and $[\beta']=[\beta]$, $[e_4]=[\delta_4]$ and $[e_5]=[\delta_5]$.

Again this 3–form can be considered as a form in M. Now for any $[\varphi], [\varphi'] \in H^3(M)$ we have

$$([2\,\gamma \wedge \alpha \wedge e_1 \wedge e_2 \wedge e_3] + [\alpha \wedge e_1 \wedge \varphi] + [\beta \wedge e_3 \wedge \varphi']) \cdot [\beta' \wedge e_4 \wedge e_5] = -2,$$

since the first product gives 2; to compute the second product, we notice that the 5–form $\alpha \wedge \beta' \wedge e_1 \wedge e_4 \wedge e_5$ is exact in M because $\alpha \wedge \beta' \wedge e_1 \wedge e_4 \wedge e_5 = -d\gamma' \wedge e_1 \wedge e_4 \wedge e_5$ in X, with $\gamma' = \gamma + f\,\alpha$ for some function f on N, and $\gamma' \wedge e_1 \wedge e_4 \wedge e_5$ is well-defined on M; and for the third product, $\alpha \wedge \beta' \wedge e_3 \wedge e_4 \wedge e_5$ is also exact in M. Therefore we have proved the following

Theorem 3. *M is a compact oriented simply connected non-formal 8–manifold.*

4. The 7-dimensional example

A compact oriented simply connected non-formal manifold M' of dimension 7 is obtained in an analogous fashion to the construction of the 8–dimensional manifold M. We start with $X' = N \times \mathbb{T}^4$ and consider the 3-dimensional tori

$$\begin{aligned} T_1' &= C_1 \times \mathbb{S}^1 \times \{0\} \times \mathbb{S}^1 \times \{0\}, \\ T_2' &= C_2 \times \{0\} \times \mathbb{S}^1 \times \{0\} \times \mathbb{S}^1. \end{aligned}$$

Define

$$M' = X' \#_{T_1'=T'} \mathbb{S}^7 \#_{T_2'=T'} \mathbb{S}^7$$

where T' is an embedded 3–torus in $\mathbb{S}^7$ with trivial normal bundle. Then M' is a non-formal simply connected manifold. To prove the non-formality, consider the codimension 2 submanifolds

$$\begin{aligned} B_1' &= p^{-1}(\mathbb{S}^1 \times \{a_1\}) \times \{b_1\} \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \\ B_2' &= p^{-1}(\{a_2\} \times \mathbb{S}^1) \times \mathbb{S}^1 \times \{b_2\} \times \mathbb{S}^1 \times \mathbb{S}^1 \\ B_3' &= p^{-1}(\mathbb{S}^1 \times \{a_3\}) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \{b_3\} \times \mathbb{S}^1 \end{aligned}$$

and the 2–forms η_i' Poincaré dual to B_i. Then $\langle [\eta_1'], [\eta_2'], [\eta_3'] \rangle = [2\,\gamma \wedge \alpha \wedge e_1 \wedge e_2 \wedge e_3]$. This triple Massey product is non-zero in

$$\frac{H^5(M')}{[\alpha \wedge e_1] \cup H^3(M') + H^3(M') \cup [\alpha \wedge e_3]},$$

by using the same argument as before with $B_4' = p^{-1}(\{a_4\} \times \mathbb{S}^1) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \{b_4\}$.

Note that it is in this last step where the similar argument for the 6–dimensional case breaks down, since if we drop the last factor all throughout the argument, then the submanifold $B_4'' = p^{-1}(\{a_4\} \times \mathbb{S}^1) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ would not be disjoint from the two tori where the surgery is taken place.

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