The application of spectral representations in coordinates of complex frequency for digital filter analysis and synthesis

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1. Introduction

The suitability of using one or another spectral representation depends on the type of signal to be analysed and problem to be solved, etc. (Kharkevich, 1960, Jenkins, 1969). Thus, the spectral representations, based on Fourier transform, are widely applied for linear circuit and frequency filter analysis for sinusoidal and periodical input signals (Siebert, 1986, Atabekov, 1978). However, using these spectral representations for a filter analysis of non-stationary signals would not be so simple and visually advantageous (Kharkevich, 1960).
In the majority of cases input signals of automation and measurement devices have an analogue nature, and can be represented as a set of semi-infinite or finite damped oscillatory components. In the case of IIR filter impulse functions the representation uses this set of damped oscillatory components. Impulse functions of FIR filters representation are also based on this set of damped oscillatory components, but with the difference of a finite duration of the impulse functions. Thus, the generalized signal and impulse function of analog filters have similar mathematical expressions. In this case it is reasonable to use the Laplace transform instead of the Fourier transform, because the Laplace transform operates with complex frequency, and its damped oscillatory component is a base function of the transform (Mokeev, 2006, 2007, 2009a).
The application of the spectral representations based on Laplace transform, or in other words, the spectral representations in complex frequency coordinates, enables to simplify significantly calculations of stationary and non-stationary modes and get efficient methods of filter synthesis (Mokeev, 2006). It also extends the application area of the complex amplitude method, including use of this method for analysis of stationary and non-stationary modes of analog and digital filters (Mokeev, 2007, 2008b, 2009a).

2. Mathematical description of filters

2.1 Mathematical description of input signals

It should be considered in frequency filter simulation, that input signals of digital automation and measurement devices have an analogue nature. Therefore, an analog filter-prototype is theoretically perfect. In the majority of cases filter signals and impulse functions can be described by a set of semi-infinite or finite damped oscillatory components.

The mathematical expression of the generalized complex continuous and discrete input signal can be briefly represented in the following way

$$\dot{x}(t) = \dot{\mathbf{X}}^{\mathrm{T}} e^{\mathbf{P}(\mathbf{C}t-\mathbf{t})} - \dot{\mathbf{X}}^{'\mathrm{T}} e^{\mathbf{P}(\mathbf{C}t-\mathbf{t}^{'})}, \tag{1}$$

$$\dot{x}(k) = \dot{\mathbf{X}}^{\mathrm{T}} Z(\mathbf{P},\mathbf{C}k-\mathbf{K}) - \dot{\mathbf{X}}^{'\mathrm{T}} Z(\mathbf{P},\mathbf{C}k-\mathbf{K}^{'}), \tag{2}$$

where $\dot{\mathbf{X}} = \left[\dot{X}_n\right]_N = \left[X m_n e^{-j\varphi_n}\right]_N$ and $\dot{\mathbf{X}}^{'} = \left[\dot{X}_n^{'}\right]_N = \left[\dot{X}_n e^{p_n(t_n^{'}-t_n)}\right]_N$ – are complex amplitude vectors of two input signal components, $\mathbf{p} = [p_n]_N = [-\beta_n + j\omega_n]_N$ – is complex frequency vector, $\mathbf{t} = [t_n]_N$, $\mathbf{t}^{'} = \left[t_n^{'}\right]_N$, $\mathbf{K} = [K_n]_N$, $\mathbf{K}^{'} = \left[K_n^{'}\right]_N$ – are vectors, which elements define a time delay of input signal components, $\mathbf{P} = \mathrm{diag}(\mathbf{p})$ – is square matrix $N \times N$ with the vector $\mathbf{p}$ on the main diagonal, $\mathbf{C}$ – is unit vector, T – is discrete sampling step, $Z(p,k) = e^{pkT}$.

The use of the complex generalized input signal (1) enables to get more compact form of the signal expression. The transition to real signal

$$x(t) = \mathrm{Re}(\dot{x}(t)),\ x(k) = \mathrm{Re}(\dot{x}(k)).$$

When $\dot{\mathbf{X}}^{'} = \mathbf{0}$ и $\mathbf{t} = \mathbf{0}$ $(\mathbf{K} = \mathbf{0})$, the input signal is represented by a set of continuous (discrete) semi-infinite damped oscillatory components.

Particular cases of n-th damped oscillatory component at $t_n = 0$

$$\dot{x}_n(t) = \dot{X}_n e^{p_n t},\ x_n(t) = \mathrm{Re}(\dot{x}_n(t)) = X m_n e^{-\beta_n t} \cos(\omega_n t - \varphi_n),$$

are semi-infinite sinusoidal ($p_n = j\omega_n$) and constant ($p_n = 0$) components, exponential component ($p_n = -\beta_n$), component in the form of a delta function ($X_{mn} = \beta_n$, $p_n = -\beta_n$, $\beta_n \to \infty$).

Compound signals of different forms, including compound periodical and quasi-periodic signals, non-stationary signals and signals with compound envelopes can be synthesized on the basis of the collection of components mentioned above.

The most frequently used semi-infinite or finite signals with compound envelopes in radio engineering are described by the following model

$$\dot{x}(t) = \dot{X}(t) e^{p_1 t},\ x(t) = \mathrm{Re}(\dot{x}(t)),$$

or in general case it would be

$$\dot{x}(t) = \dot{\mathbf{X}}(t)^{\mathrm{T}} e^{\mathbf{P}(\mathbf{C}t-\mathbf{t})},\ x(t) = \mathrm{Re}(\dot{x}(t)). \tag{3}$$

Examples of signal mathematical expression, represented by mathematical model (3) and model (1), are shown in the Table 1. In this case signal models (1) and (3) enable to describe not only radio signal (item 1 and 2), but real signals of measurement and automation devices. The example for a signal of intellectual electronic devices of electric power systems as the set of sequentially adjacent finite component groups, each one of those corresponds to defined operation mode of the electric power system, is represented in the item 3, Table 1.

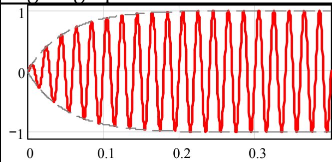
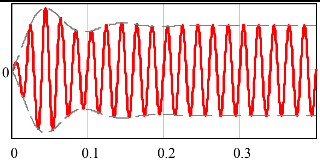
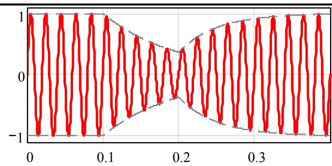
№	Mathematical description	Signal graph
1.	$\dot{X}(t)=1-e^{-\beta_1 t}$, $p_1=j\omega_1$, $\beta_1=20$, $\omega_1=314$ $\dot{\mathbf{X}}=[1\quad -1]^{\mathrm{T}}$, $\mathbf{p}=[p_1\quad p_2]^{\mathrm{T}}$, $p_1=j\omega_1$, $p_2=-\beta_1+j\omega_1$	
2.	$\dot{X}(t)=1-e^{-\beta_1 t}\cos(0{,}2\omega_1 t)$, $p_1=j\omega_1$, $\beta_1=20$, $\omega_1=314$ $\dot{\mathbf{X}}=\left[e^{-j0{,}5\pi}\quad -0{,}5e^{-j0{,}5\pi}\quad -0{,}5e^{-j0{,}5\pi}\right]^{\mathrm{T}}$, $\mathbf{p}=\left[j\omega_1\quad -\beta_1+j1{,}2\omega_1\quad -\beta_1+j0{,}8\omega_1\right]^{\mathrm{T}}$	
3.	$\dot{X}(t)=1(t)-1(t-\tau_1)+e^{-b_1(t-\tau_1)}-k_1e^{-b_1(t-\tau_1)}+1(t-\tau_2)-k_2e^{-b_2(t-\tau_2)}$, $p_1=j\omega_1$, $k_1=e^{-b_1\tau_1}$, $k_2=e^{-b_1(\tau_2-\tau_1)}$	
	$\dot{\mathbf{X}}=[1\quad 1\quad 1\quad -k_2]^{\mathrm{T}}e^{-j0.5\pi}$, $\dot{\mathbf{X}}'=[1\quad k_1\quad 0\quad 0]^{\mathrm{T}}e^{-j0.5\pi}$, $\mathbf{p}=[p_1\quad p_2\quad p_1\quad p_3]^{\mathrm{T}}$, $\mathbf{t}=[0\quad \tau_1\quad \tau_2\quad \tau_2]^{\mathrm{T}}$, $\mathbf{t}'=[\tau_1\quad \tau_2\quad \infty\quad \infty]^{\mathrm{T}}$	
	$\beta_1=10$, $\beta_2=20$, $\omega_1=314$, $\tau_1=0{,}1$, $\tau_2=0{,}02$, $p_1=j\omega_1$, $p_2=-\beta_1+j\omega_1$, $p_3=-\beta_1+j\omega_1$	

Table 1. Input signal models

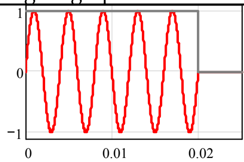
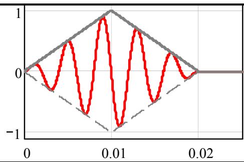
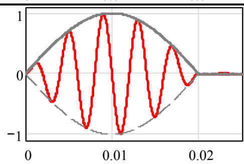
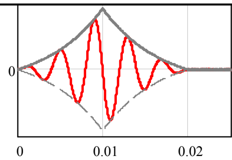
№	Mathematical description	Signal graph
1.	rectangular pulse $\dot{X}(t)=1(t)-1(t-\tau_1)$, $p_1=0$, $\tau_1=0.02$, $\dot{\mathbf{X}}=[1]$, $\dot{\mathbf{X}}'=[-1]$, $\mathbf{p}=[0]$, $\mathbf{t}=[0]$, $\mathbf{t}'=[\tau_1]$ rectangular radio pulse $\dot{X}(t)=1(t)-1(t-\tau_1)$, $p_1=j\omega_1$, $\omega_1=1571$ $\dot{\mathbf{X}}=[1]$, $\dot{\mathbf{X}}'=[-1]$, $\mathbf{p}=[j\omega_1]$, $\mathbf{t}=[0]$, $\mathbf{t}'=[\tau_1]$	
2.	triangular pulse $\dot{X}(t)=t\cdot 1(t)-2(t-\tau_1)1(t-\tau_1)+(t-\tau_2)1(t-\tau_2)$, $p_1=0$, $\tau_1=0{,}01$, $\tau_2=0{,}02$ $\dot{\mathbf{X}}=\dot{\mathbf{X}}'=\left[\beta^{-1}\quad -\beta^{-1}\right]^{\mathrm{T}}100$, $\mathbf{p}=[0\quad -\beta]^{\mathrm{T}}$, $\beta\to 0$ $\mathbf{t}=[0\quad \tau_1]^{\mathrm{T}}$, $\mathbf{t}'=[\tau_1\quad \tau_2]^{\mathrm{T}}$	
3.	sine pulse $\dot{X}(t)=\sin(\omega_2 t)\cdot 1(t)+\sin(\omega_2(t-\tau_1))1(t-\tau_1)$, $p_1=0$, $\tau_1=0{,}02$, $\omega_2=157{,}1$, $\dot{\mathbf{X}}=[1]$, $\dot{\mathbf{X}}'=[1]$, $\mathbf{p}=[j\omega_2]$, $\mathbf{t}=[0]$, $\mathbf{t}'=[\tau_1]$	
4.	exponential pulse $\dot{\mathbf{X}}=\left[1\quad -1\quad e^{\beta\tau_1}\right]^{\mathrm{T}}$, $\beta=150$, $\tau_1=0{,}01$, $\tau_2=0{,}02$, $\dot{\mathbf{X}}'=\left[e^{\beta\tau_1}\quad 1\quad e^{-\beta(\tau_2-2\tau_1)}\right]^{\mathrm{T}}$, $\mathbf{p}=[\beta\quad 0\quad -\beta]^{\mathrm{T}}$, $\mathbf{t}=[0\quad 0\quad \tau_1]^{\mathrm{T}}$, $\mathbf{t}'=[\tau_1\quad \tau_2\quad \tau_2]^{\mathrm{T}}$	

Table 2. Video pulse and radio pulse models

The model (1) also makes it possible to describe the majority of impulse signals, which are widely applicable in radio engineering. Examples of some impulse signals are shown it the Table 2. Therefore, the generalized mathematical model (1) enables to describe a big variety of semi-infinite or finite signals.

As it is shown below, the compound finite signal representations in the form of the set of damped oscillatory components significantly simplifies the problem solving of the signal passage analysis through the frequency filters, by using the analysis methods based on signal and filter spectral representations in complex frequency coordinates (Mokeev, 2007, 2008b).

2.2 Mathematical description of filters

Analysis and synthesis of filters of digital automation and measurement devices are primarily carried out for analog filter-prototypes. The transition to digital filters is implemented by using the known synthesis methods. However, this method can only be applied for IIR filters, as a pure analog FIR filter does not exist because of complications of its realization. Nevertheless, implementation of this type of analog filters is rational exclusively as they are considered "perfect" filters for analog signal processing and as filter-prototypes for digital FIR filters (Mokeev, 2007, 2008b).

When solving problems of digital filters analysis and synthesis, one will not take into account the AD converter errors, including the errors due to signal amplitude quantization. This gives the opportunity to use simpler discrete models instead of digital signal and filter models (Ifeachor, 2002, Smith, 2002). These types of errors are only taken into consideration during the final design phase of digital filters. In case of DSP with high digit capacity, these types of errors are not taken into account at all.

The mathematical description of analog filter-prototypes and digital filters can be expressed with the following generalized forms of impulse functions:

$$\dot{g}(t) = \dot{\mathbf{G}}^{\mathrm{T}} e^{\mathbf{q}t} - \dot{\mathbf{G}}^{'\mathrm{T}} e^{\mathbf{Q}(\mathbf{C}t-\mathbf{T})}, \quad g(t) = \mathrm{Re}\left(\dot{g}(t)\right), \tag{4}$$

$$\dot{g}(k) = \dot{\mathbf{G}}^{\mathrm{T}} Z(\mathbf{q},k) - \dot{\mathbf{G}}^{'\mathrm{T}} Z(\mathbf{Q},\mathbf{C}k-\mathbf{N}), \quad g(k) = \mathrm{Re}\left(\dot{g}(k)\right). \tag{5}$$

Therefore, for analog and digital filter description it is sufficient to use vectors of complex amplitudes of two parts of complex function:
$\dot{\mathbf{G}} = \left[\dot{G}_m\right]_M = \left[k_m e^{-j\phi_m}\right]_M$ and $\dot{\mathbf{G}}^{'} = \left[\dot{G}_m^{'}\right]_M = \left[\dot{G}_m e^{\rho_m T_m}\right]_M$, vector of complex frequencies $\mathbf{q} = \left[\rho_m\right]_M = \left[-\alpha_m + jw_m\right]_M$ and vectors $\mathbf{T} = \left[T_m\right]_M$ и $\mathbf{N} = \left[N_m\right]_M$, which define the duration (length) of the filter pulse function components; $\mathbf{Q} = \mathrm{diag}(\mathbf{q})$ – is a square matrix $M \times M$ with the vector $\mathbf{q}$ on the main diagonal.

Adhering to the mathematical description of the FIR filter impulse function mentioned above (4), the IIR filter impulse functions are a special case of analogous functions of FIR filters at $\dot{\mathbf{G}}^{'} = \mathbf{0}$.

Recording the mathematical description of filters in such a complex form has advantages: firstly, the expression density, and secondly, correlation to two filters at the same time, which allows for ensured calculation of instant spectral density module and phase on given complex frequency (Smith, 2002).

The transfer function of the filter (4) with the complex coefficients is

$$\underline{K}(p) = \dot{\mathbf{G}}^{\mathrm{T}} \left[\frac{1}{p - \rho_m} \right]_M - \dot{\mathbf{G}}^{'\mathrm{T}} \left[\frac{1}{p - \rho_m} e^{-pT_m} \right]_M , \tag{6}$$

The transfer function $\underline{K}(p)$ is an expression of the complex impulse function (6), therefore it has along with the complex variable p complex coefficients, defined by the vectors $\dot{\mathbf{G}}$, $\dot{\mathbf{G}}^{'}$ and $\mathbf{q}$. A filter with the transfer function $\underline{K}(p)$ correlates with two ordinary filters, which transfer functions are $\mathrm{Re}\left(\underline{K}(p)\right)$ and $\mathrm{Im}\left(\underline{K}(p)\right)$. In this case the extraction of the real and imaginary parts of $\underline{K}(p)$ can be applied only to complex coefficients of the transfer function and has no relevance for the complex variable p.

As it appears from the input signal models (1) and filter impulse functions (4), there is a similarity between their expressions of time and frequency domains. Filter impulse functions based on the model (4) may have a compound form, including the analogous ones referred to above in Tables 1 and 2.

The similarity of mathematical signal and filter expressions: firstly, allow to use one compact form for their expression as a set of complex amplitudes, complex frequencies and temporary parameters. Secondly, it significantly simplifies solving problems of mathematical simulation and frequency filter analysis.

The digital filter description (5) can be considered as a discretization result of analog filter impulse function (4). Another known transition (synthesis) methods can be also applied, if they are revised for use with analogue filters-prototypes with a finite-impulse response (Mokeev, 2008b).

2.3 Methods of the transition from an analog FIR filter to a digital filter

The mathematical description of digital FIR filters at $M = 1$ is given in the Table 3, these filters were obtained on the basis of the analog FIR filter (item 0) by use of three transformed known synthesis methods: the discrete sampling method of the differential equation (item 1), as well as the method of invariant impulse responses (item 2) and the method of bilinear transformation (item 3).

№	Differential or difference equation	Impulse function	Transfer or system function
0.	$\frac{d\dot{y}(t)}{dt} - \rho_1 \dot{y}(t) = \dot{G}_1 x(t) - \dot{G}_1^{'} x(t - \mathrm{T}_1)$	$\dot{g}(t) = \dot{G}_1 e^{\rho_1 t} - \dot{G}_1^{'} e^{\rho_1 (t - \mathrm{T}_1)}$	$\underline{K}(p) = \frac{1}{p - \rho_1}\left(\dot{G}_1 - \dot{G}_1^{'} e^{-p\mathrm{T}_1}\right)$
1.	$\nabla \dot{y}_k - \rho_1 \dot{y}_k = \dot{G}_1 x_k - \dot{G}_2 x_{k-N_1}$	$g_k = k_{11}\left(\dot{G}_1 z_{11}^k - \dot{G}_1 z_{11}^{k-N_1}\right)$	$K(z) = \frac{k_{11} z}{z - z_{11}}\left(\dot{G}_1 - \dot{G}_2 z^{-N_1}\right)$
2.	$\nabla \dot{y}_k - a_0 \dot{y}_k = \dot{G}_1 x_k - \dot{G}_2 x_{k-N_1}$	$g_k = k_{12}\left(\dot{G}_1 z_1^k - \dot{G}_1 z_1^{k-N_1}\right)$	$K(z) = \frac{k_{12} z}{z - z_1}\left(\dot{G}_1 - \dot{G}_2 z^{-N_1}\right)$
3.	-	-	$K(z) = k_{13} \frac{z + 1}{z - z_{13}}\left(\dot{G}_1 - \dot{G}_2 z^{-N_1}\right)$

Table 3. Methods of the transition from an analog FIR filter to a digital FIR filter

Note: The double subscripts are given for the parameters that do not coincide. The second number means the sequence number of the transition method.

1. $k_{11} = T$, $z_{11} = 1/(1 - T\rho_1)$, $N_1 = T_1 / T$, complex frequency $\rho_{11} = \ln(z_{11})/T$;

2. $z_1 = e^{\rho_1 T}$, $a_0 = (z_1 - 1)/T$, $k_{12} = T$;

3. $k_{13} = T/(2-\rho_1 T)$, $z_{13} = (2+\rho_1 T)/(2-\rho_1 T)$, complex frequency $\rho_{13} = \ln(z_{13})/T$.

In cases of the first and third methods the coincidence of impulse function complex frequencies of digital filter and analog filter-prototype is possible only if $T \to 0$. The second method ensures the entire concurrence of complex frequencies of an analogue filter-prototype and a digital filter in all instances. The later is very important, when the filter is supposed to be used as a spectrum analyzer in coordinates of complex frequency.

The features of transition from a digital (discrete) filter, considering finite digit capacity influence of microprocessor, including cases for filters with integer-valued coefficients, are considered by the author in the research .

One of the most important advantages of the considered above approach to mathematical description of FIR filters is obtaining FIR filter fast algorithms (Mokeev, 2008a, 2008b).

2.4 Overlapping the spectral and time approach

The impulse function (3) corresponds to the following differential equation

$$\frac{d\dot{\mathbf{y}}(t)}{dt} = \mathbf{A}\dot{\mathbf{y}}(t) + \mathbf{B}x(t) + \mathbf{D}x(\mathbf{C}t - \mathbf{T}), \tag{7}$$

where $\mathbf{A} = \mathrm{diag}(\mathbf{q})$, $\mathbf{B} = \dot{\mathbf{G}}$, $\mathbf{D} = \mathrm{diag}(\dot{\mathbf{G}}')$; $y(t) = \mathrm{Re}(\mathbf{C}^{\mathrm{T}}\dot{\mathbf{y}}(t))$ is a output signal of the filter.

In case of FIR filter ($\mathbf{D} = \mathbf{0}$) the expression (7) is conform to one of known forms of state space method. Thus, the application of mentioned spectral representations allows to combine the spectral approach with the state space method for frequency filter analysis and synthesis (Mokeev, 2008b, 2009b).

If one places the expression of generalized impulse characteristic (4) to the expression of convolution integral, one will get the following expression of the filter output signal

$$\dot{y}(t) = \int_{\mathbf{C}t-\mathbf{T}}^{t} x(\tau)\dot{\mathbf{G}}^{\mathrm{T}} e^{\mathbf{q}(t-\tau)} d\tau . \tag{8}$$

If a generalized input signal (1) is fed into the filter input, simple input-output relations (Mokeev, 2008b) can be gained on the base of the expression (8).

The expression (8) can be transformed into the following form

$$\dot{y}(t) = \sum_{m=1}^{M} \dot{G}_m X_T(\rho_m) e^{\rho_m t} ,$$

where $X_T(p,t) = \int_{t-T}^{t} x(\tau) e^{-p\tau} d\tau$ - is the instant spectrum of input signal in coordinates of complex frequency.

Therefore, the elements of the vector $\dot{\mathbf{y}}(t)$ are defined by solving M-number of independent equations (7), each one of those can be interpreted as a value of instant (FIR filter) or current (IIR filter) Laplace spectrum in corresponding complex frequency of filter impulse function component.

The expression (7) is a generalization of one of state space method forms, and at the same time directly connected with the Laplace spectral representations. So, one can view the overlapping time approach (state space method) and frequency approach in complex frequency coordinates.

On the base of analogue filter-prototype (7) descriptions, a mathematical expression of digital filters can be obtained, by use of the known transition (synthesis) methods, applied to FIR filters (Mokeev, 2008b). In this case fast algorithms for FIR filters are additionally synthesized.

2.5 Features of signal spectrum and filter frequency responses in complex frequency coordinates

To illustrate the features of signal spectrums and filter frequency responses in coordinates of complex frequency, the fig. 1 shows amplitude-frequency response schematics of IIR filter and a spectral density module of input signal, if the following conditions apply: the filter represents a series of low-pass second-order and first-order filters, and can be described by complex amplitude vector $\dot{\mathbf{G}} = \begin{bmatrix} 9,63e^{-j2,336} & 6,67 \end{bmatrix}^{\mathrm{T}}$ and complex frequency vector $\mathbf{q} = \begin{bmatrix} -150 + j640 & -400 \end{bmatrix}^{\mathrm{T}}$; the input signal consists of an additive mixture of an unit step, exponential component, semi-infinite sinusoidal component and damped oscillatory component, and can be compactly described by complex amplitude vector $\dot{\mathbf{X}} = \begin{bmatrix} 1 & e^{j\pi} & 2e^{j0,25\pi} & 2 \end{bmatrix}^{\mathrm{T}}$ and complex frequency vector $\mathbf{p} = \begin{bmatrix} 0 & -120 & j300 & -40 + j500 \end{bmatrix}^{\mathrm{T}}$.

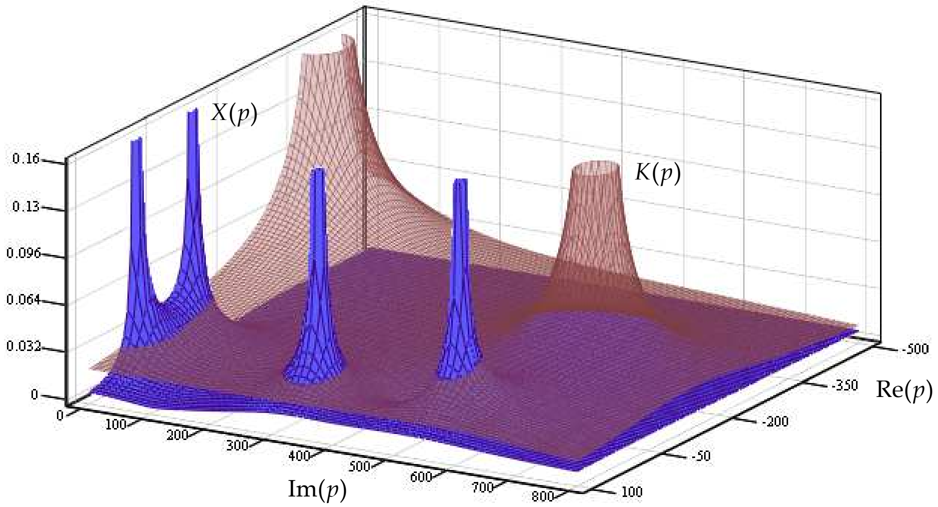


Fig. 1. 3D amplitude signal spectrum and filter amplitude-frequency response

The 3D amplitude-frequency response (fig. 1) of the filter and signal spectrum module shows, that complex frequencies of filter and input signal impulse functions have clearly defined peaks.

This means, a 3D signal spectrum in complex frequency coordinates contains a continuous spectrum along with four discrete lines on complex frequencies of input signal components. The signal spectral densities on the mentioned complex frequencies are proportional to delta

function. Values of the transfer function on the mentioned complex frequencies of input signal define a variation law of forced filter output signal components concerning input signal components (Mokeev, 2007, 2008b). The rest of spectral regions characterize the transient process in the filter due to step-by-step change of the input signal at the time zero.
A filters amplitude-frequency response is also three-dimensional and is represented by a continuous spectrum and two discrete lines on complex frequencies of impulse function components. In this case the values of the input signal representation of the above mentioned complex frequencies, define a variation law of free components in relation to filter impulse function components (Mokeev, 2007).

3. Filter analysis

3.1 Analysis methods based on features of signal and filter spectral representations in complex frequency coordinates

Three methods of frequency filter analysis are suggested from the time-and-frequency representations positions of signals and linear systems in coordinates of complex frequency (Mokeev, 2007, 2008b).
The first method is based on the above considered features of signal spectrums and filter frequency responses in complex frequency coordinates, and it allows for the determination of forces and free filter components, by the use of simple arithmetic operations.
The other two methods are based on applied time-and-frequency representations of signals or filters in coordinates of complex frequency. In this case instead of determining forced and free components of the output filter signal, it is enough to consider the filter dynamic properties by using only one of the mentioned component groups.
Based on time-and-frequency representations of signals and linear systems in coordinates of complex frequency, the known definition by Charkevich A.A. (Kharkevich, 1960) for accounting the dynamic properties of linear system is generalized:

1. the signal is considered as current or instantaneous spectrum, and the system (filter) – only as discrete components of frequency responses in coordinates of complex frequency;
2. the signal is characterized only by discrete components of spectrum, and the system (filter) – by time dependence frequency responses.

Analysis methods for analog and digital IIR filters in case of semi-infinite input signals, similar to (1), are considered below. These methods of filter analysis can be simply applied to more complicated cases, for instance, to FIR filter (4) analysis at finite input signals (Mokeev, 2008b).

3.2 The first method of filter analysis: complex amplitude method generalization

The first method is a complex amplitude method generalization for definition of forced and free components for filter reaction at semi-infinite or finite input signals.
The advantages of this method are related to simple algebraic operations, which are used for determining the parameters of linear system reaction (filter, linear circuit) components to input action described by a set of semi-infinite or finite damped oscillatory components.
Here, the expressions for determining forced and free components of analog and digital IIR filter reaction to a signal, fed to filter input as a set of continuous or discrete damped

oscillatory components, i.e. for the generalized signal (1) and (2) at $\dot{\mathbf{X}}' = \mathbf{0}$, are given as examples on fig. 2 and 3.

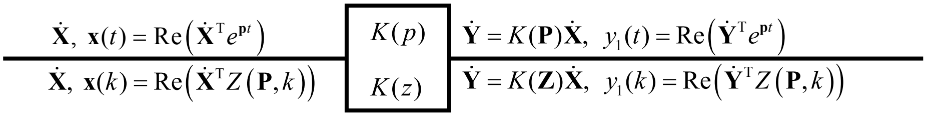


Fig. 2. Determining the forced components of an IIR filter output signal

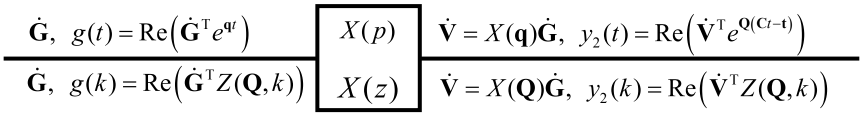


Fig. 3. Determining the free components of an IIR filter output signal

The following notations are used in the expressions on fig. 2 and fig. 3: $X(p)$ or $X(z)$, that are the representations of the input signal without regard for phase shift of signal components $\mathbf{Z} = e^{\mathbf{q}T}$.

The example for determining the reaction (curve 1) of analog and digital (discrete) third-order filter (condition in item 3.1), and the total forced (curve 2) and free (curve 3) components is shown on the fig. 4. Using Matlab and Mathcad for determining the forced and free components of an output signal, only complex amplitude vectors of an input signal and filter impulse function , as well as the complex frequency vectors of an input signal and filter are needed to be specified. The remaining calculations are carried out automatically.

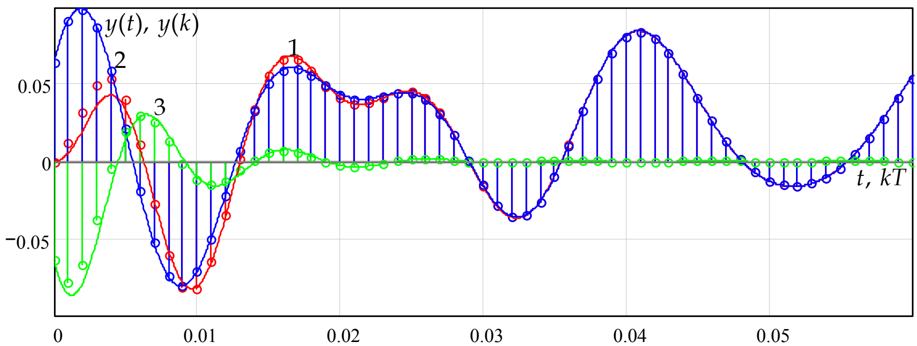


Fig. 4. Determining the forced and free components of an output signal

The input-output expressions presented on fig. 2 and fig. 3 can be applied also to FIR filters and finite signals (Mokeev, 2008b).

3.3 The second method: filter as a spectrum analyzer

The second method is based on interpreting a filter as an analyzer of current or instantaneous spectrum of an input signal in coordinates of complex frequency (Mokeev, 2007, 2008b).

If one converts the expression for an IIR filter complex impulse function (4) into an expression of convolution integral, the result will be the dependence for a filter output signal:

$$\dot{y}(t) = \int_0^t x(\tau)\dot{g}(t-\tau)d\tau = \dot{\mathbf{G}}^{\mathrm{T}} X(\mathbf{Q},t)e^{\mathbf{q}t} \,, \tag{9}$$

where $X(p,t) = \int_0^t x(\tau)e^{-p\tau}d\tau$ - is the current spectral density of an input signal, using Laplace transform.

On the base of the expression (9) the calculations for determining a filter output signal components are gained and represented on the fig. 5.

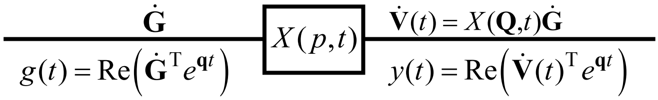


Fig. 5. Determining the IIR filter reaction

As concluded from the expression above, an IIR filter output signal depends on values of the current Laplace spectrum of an input signal on filter impulse function complex frequencies. Thus, a FIR filter is an analyzer of a signal instantaneous spectrum in a coordinates of complex frequency.

3.4 The third method: diffusion of time-and-frequency approach to transfer function

The time-and-frequency approach in the third analysis method applies to a filter transfer function, i.e. time dependent transfer function of the filter is used.

If one places the expression for a complex semi-infinite input signal (1) into the expression for convolution integral, one will obtain the following dependence

$$\dot{y}(t) = \int_0^t \dot{x}(\tau)g(t-\tau)d\tau = \dot{\mathbf{X}}^{\mathrm{T}} K(\mathbf{P},t)e^{\mathbf{p}t} \,,$$

where $K(p,t) = \int_0^t g(\tau)e^{-p\tau}d\tau$ - is time dependent transfer function of filter.

Then the input-output dependence for an IIR filter (4), when it is fed to semi-infinite input signal, can be compactly presented in the following way (fig. 6).

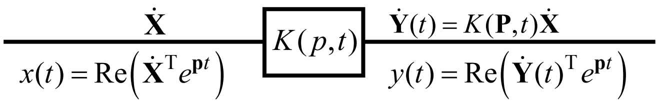


Fig. 6. Filter reaction determination

Thus, a function modulus $K(p_n,t)$ value on the complex frequency of n-th input signal component describes the variation law of n-th component envelope of filter output signal.

The function argument characterizes phase change of the later mentioned output signal component. Since the transient processes in filter are completed, the complex amplitude $\dot{Y}_n(t)$ will coincide with the complex amplitude of the forced component $\dot{Y}_n$.

In that case, filter amplitude-frequency and phase-frequency functions will be a three-variable functions, i.e. it is necessary to represent responses in 4D space. For practical visualization of frequency responses the approach, based on use of three-dimensional frequency responses at complex frequency real or imaginary partly fixed value, can be applied.

Let us consider the example from the item 3.1. The plot, shown on fig. 7 , is proportional to the product $|K(-\beta_4 + j\omega, t)|e^{-\beta_4 t}$. This plot on the complex frequency $p_4 = -\beta_4 + j\omega_4$ is equal to the envelope (curve 1 and 2) of filter reaction (curve 3) on the fourth component's input action for the filter input signal.

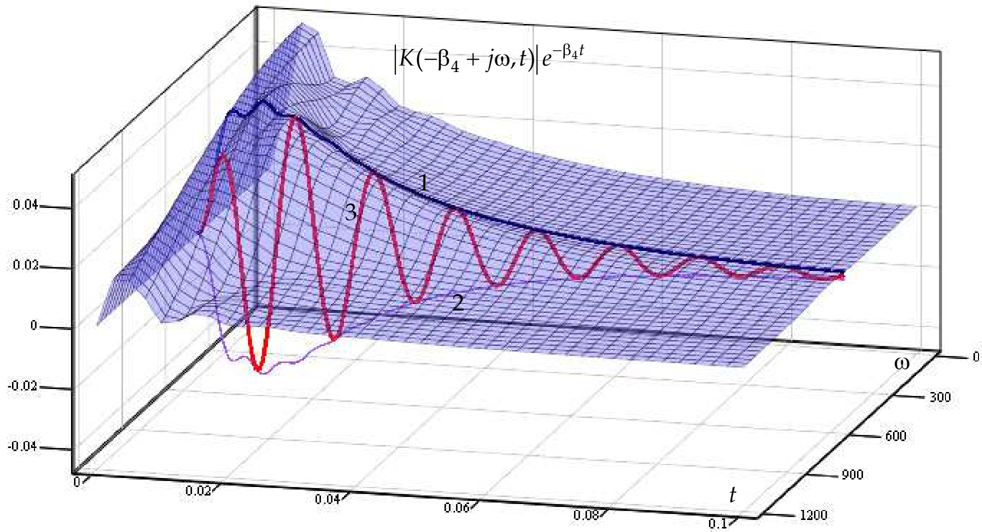


Fig. 7. Plot of the function $|K(-\beta_4 + j\omega, t)|e^{-\beta_4 t}$

The advantages of these suggested analysis methods, comparing to the existing ones for specified generalized models of input signals and frequency filters, consist in calculation simplicity, including solving problems of determining the performance parameters of signal processing by frequency filters.

4. Filter synthesis

4.1 IIR filter synthesis

The application of spectral representations in complex frequency coordinates allows to simplify significantly solving problems of filter synthesis for generalized signal model (1).

Let us consider robust filter synthesis, which have low sensitivity to change of useful signal and disturbance parameters (Sánchez Peña, 1998). In other words, robust filters must ensure the required signal performance factors at any possible variation of useful signal and disturbance parameters, influencing on their spectrums. If one takes into account only two main performance factors of signals: speed and accuracy, it will be enough to assure fulfillment of requirements, connected to limitations for filter transfer function module on complex frequency of useful signal and disturbance components (Mokeev, 2009c).

Thus, filter synthesis problem, instead of setting the requirements to particular frequency response domains (pass band and rejection band), comes to form the dependences for filter transfer function on complex frequencies of input signal components. To ensure the required performance signal factors, it is necessary to consider possible variation ranges of mentioned complex frequencies.

The synthesis will be carried out with increasing numbers of impulse function components (4) till the achievement of the specified performance signal factors.

The block diagram, shown on fig. 8, illustrates the synthesis of optimal analogue filter-prototype.

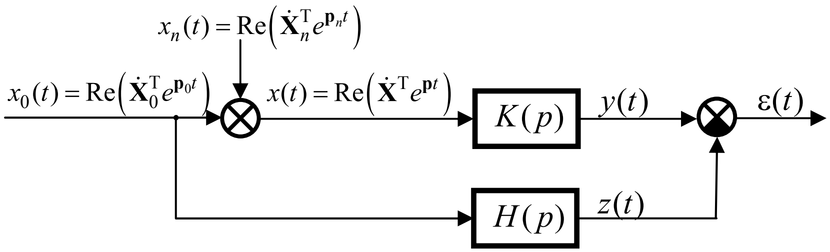


Fig. 8. Block diagram of optimal filter

The useful signal $x_0(t)$ and the disturbance $x_n(t)$ on the graph _ are completely determined by complex amplitude vectors $\dot{\mathbf{X}}_0$, $\dot{\mathbf{X}}_n$ and complex frequency vectors $\mathbf{p}_0$, $\mathbf{p}_n$. The vectors of complex amplitudes and input signal frequencies are characterized as $\dot{\mathbf{X}} = \left[\dot{\mathbf{X}}_0 \quad \dot{\mathbf{X}}_n\right]^{\mathrm{T}}$, $\mathbf{p} = \left[\mathbf{p}_0 \quad \mathbf{p}_n\right]^{\mathrm{T}}$. In case of the value of the transformation operator $H(p) = 1$, the error vector-function is $\varepsilon(t) = y(t) - x_0(t)$, in the rest of cases : $\varepsilon(t) = y(t) - z(t)$.

Limitations on forced component level for IIR filter are set by the limitations on filter amplitude-frequency response in complex frequency coordinates. Therefore, the problem of fulfillment of signal processing accuracy requirements in filter operation stationary mode is completely solved, and the filter speed τ will be determined by transient process duration in the filter, i.e. by free component damping below the permissible level (less than acceptable error of signal processing).

Free components damping can be approximately determined by the sum of their envelopes.

Thus, filter synthesis at specified structure comes to determination of its parameters, at which the specified requirements to frequency responses in complex frequency coordinates are ensured, and to ascertain the minimum time for signal processing performance requirements guaranteeing. One more suggested method, that enables to simplify optimal filter estimation, is related to use of time dependent filter transfer function $K(p,t)$.

For searching the optimal solution it is reasonable to apply the realization in Optimization Toolbox package, a part of MATLAB system of nonlinear optimization procedure methods with the limitations to a filter transfer function value on specified complex frequencies of input signal components and filter speed.
Order of filter synthesis, according to specified block diagram (fig. 8), consists in the following. Type and filter order are given on the basis of features of solving problem, target function and restrictions on filter frequency response values in complex frequency coordinates are formed based on ensuring of signal processing performance required parameters. Then filter parameters are calculated with use of optimization procedures. In case of the found solution does not meet signal processing performance requirements, the order of filter should be raised and filter parameters should be found again.
Let us consider an example of analogue filter-prototype synthesis to separate the sine signal against a disturbance background in the exponential component form.
To extract the useful signal and eliminate the disturbance, acceptable speed can be only be obtained with use of second-order and higher order filters. Let us consider second-order high-pass filter synthesis.
The main phases of IIR filter synthesis for selection industrial frequency useful signal against a background of exponential disturbance are presented in table 4.

№	Name	Conditions
1.	Input signal $x(t) = X_{m1}\cos\omega_1 t - X_2 e^{-\beta_2 t}$	limits of useful signal frequency variation $\omega_1 = 2\pi(45 \div 55)$ rad/s, maximum disturbance level $X_2 = X_{m1}$, changing size of damping coefficient $\beta_2 = 0 \div 200$ s^{-1}
2.	Signal processing performance requirements	1. acceptable error in signal processing: automation function $\varepsilon_1 \le 0{,}1$ (5 %), metering function $\varepsilon_2 \le 0{,}01$ (1 %), 2. speed: $\tau_1 \le 20$ мс (5%), $\tau_2 \le 40$ ms (1%), 3. acceptable overshoot level: $\le$ 10%
3.	Requirements to filter amplitude-frequency response in complex frequency coordinates	1. section $p = j\omega$: $\lvert K(j\omega_0)\rvert = 1$, $\omega_0 = 100\pi$ rad/s, $1-\varepsilon_2 \le \lvert K(j(\omega_0 \pm \Delta\omega))\rvert \le 1+\varepsilon_2$, $\Delta\omega = 10\pi$ rad/s 2. section $p = -\gamma$: $\lvert K(-\gamma)\rvert e^{-\gamma\tau_1} \le \varepsilon_1$, $\lvert K(-\gamma)\rvert e^{-\gamma\tau_2} \le \varepsilon_2$
4.	Transfer function of second-order high-pass filter	$K(p) = \dfrac{0{,}874p^2}{p^2 + 224p + 221^2}$

Table 4. IIR filter synthesis

The amplitude-frequency responses in the sections $p = j\omega$ and $p = -\gamma$ (at $\tau_1 = 0{,}02$ s) are represented on fig. 9. On fig. 9 along with filter amplitude-frequency response the limitations on filter amplitude-frequency response values, according to the requirements in table 4 item 3, are shown. Amplitude-frequency response value out of mentioned restrictions zone conventionally is ≤ 1. As follows from the fig. 9, the synthesized filter completely meets the requirements of signal processing accuracy at frequency change ±5 Hz in power system.

The plot of transient process in second-order high-pass filter at signal feeding (table 4 point 1) is presented on fig. 10. The transient process durations are 11 ms (that is 10% of acceptable error), 15 ms (5%) and 33 ms (1%) at any exponential component damping coefficient value from the specified range $\beta = 0 \div 200$ s^{-1}.

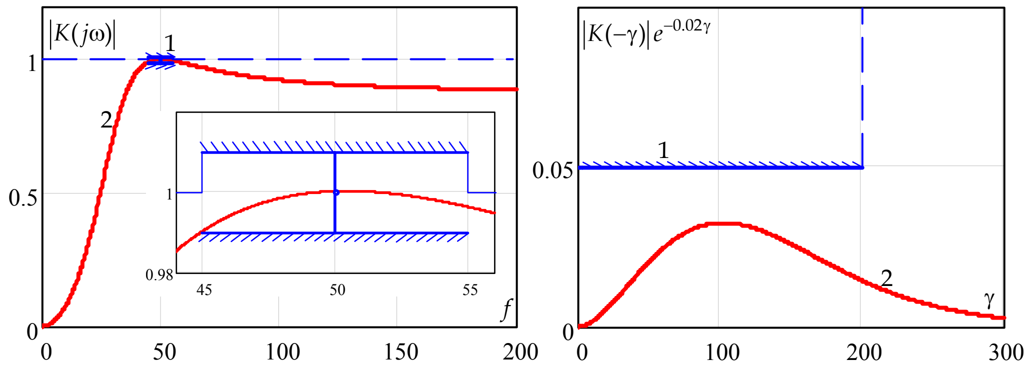


Fig. 9. Filter amplitude-frequency response in the sections $p = j2\pi f$ and $p = -\gamma$

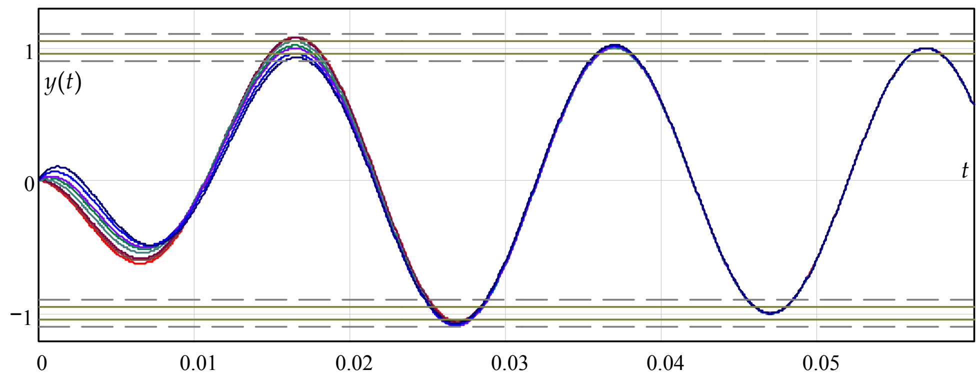


Fig. 10. Filter output signal

Therefore, synthesized second-order high-pass filter has low sensitivity to exponential component damping coefficient variation and to power system frequency deviation.
This example clearly illustrates the advantages of using the Laplace transform spectral representations for frequency filter synthesis. Applying these representations in combination with multidimensional optimization methods with the contingencies enables to perform frequency filter synthesis for problems, that were unsolvable at traditional spectral representations usage (Mokeev, 2008b). For instance, for the problem of filter synthesis for separation of the following signals: constant and exponential signals, two exponential signals with non-overlapping damping coefficient change ranges, sinusoidal and damped oscillatory components with equal or similar frequencies.
The mentioned above synthesis method can be also effectively apply for typical signal filtering problems, including problems of useful signal extraction against the white noise. In

that, the white noise realizations can be described by the special case of generalized signal model (1) as a set of time-shifted fast damping exponents of different digits. Initial values and appearance time of the mentioned exponential components are random variables, which variation law ensures the white noise specified spectral characteristics. This white noise model allows to approach filter synthesis on the basis of the signal spectral representation features (1) in complex frequency coordinates and to guarantee the required combinations of signal processing speed and accuracy (Mokeev, 2008b).

4.2 FIR filter synthesis

Comparing to IIR filter synthesis, synthesis of FIR filters is significantly simpler due to easier control over transient processes duration in filter. In case of compliance with the restrictions on amplitude-frequency response values on input signal complex frequencies (1), filter speed will be determined by the length of its impulse response.

As examples of synthesis, let us consider averaging FIR filter synthesis for intellectual electronic devices (IED) of electric power systems. Block diagram of the most widespread signal processing algorithm is given on Fig. 11.

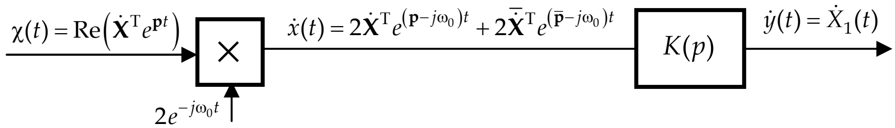


Fig. 11. Block diagram for signal processing

There is the input-output dependence for the considered algorithm

$$\dot{X}_1(t) = \int_{t-T}^{t} \chi(\tau) e^{j\omega_0 \tau} w(t-\tau) d\tau = \int_{t-T}^{t} \dot{x}(\tau) w(t-\tau) d\tau .$$

This expression corresponds to short-time Fourier transform on the frequency ω_0 .

Frequency filtering efficiency depends much to a large extent on the choice (synthesis) of time window $w(t)$, or on filter impulse function, that is equivalent for averaging filter.

Let us consider input signal as a set of complex amplitudes and exponential disturbance frequencies, industrial frequency useful signal ω_1 and higher harmonics

$$\dot{\mathbf{X}} = \begin{bmatrix} X_0 & \dot{X}_1 & \dot{X}_2 & \dot{X}_3 & \ldots & \dot{X}_N \end{bmatrix}^{\mathrm{T}}, \ \mathbf{p} = \begin{bmatrix} -\beta_0 & j\omega_1 & j2\omega_1 & j3\omega_1 & \ldots & j4\omega_1 \end{bmatrix}^{\mathrm{T}} . \qquad (10)$$

If one separates the exponential component and denotes the vector for harmonic complex amplitudes by $\dot{\mathbf{X}}_1$, the filter input signal can be presented in the following way

$$\dot{x}(t) = 2X_0 e^{(-\beta - j\omega_0)t} + 2\dot{\mathbf{X}}_1^{\mathrm{T}} e^{j(\mathbf{n}\omega_1 - \omega_0)t} + 2\bar{\dot{\mathbf{X}}}_1^{\mathrm{T}} e^{-j(\mathbf{n}\omega_1 + \omega_0)t} ,$$

where the vector $\bar{\dot{\mathbf{X}}}_1$ consists of conjugate to the vector $\dot{\mathbf{X}}_1$ elements.

When nominal frequency of power system is $\omega_1 = \omega_0$,

$$\dot{x}(t) = 2X_0 e^{(-\beta - j\omega_0)t} + \dot{X}_1 + \bar{\dot{X}}_1 e^{j2\omega_0 t} + \sum_{n=2}^{N} \left(\dot{X}_n e^{j(n-1)\omega_0 t} + \bar{\dot{X}}_n e^{-j(n+1)\omega_0 t} \right) .$$

Thus, averaging FIR filter at $\omega_1 = \omega_0$ must ensure the separation of constant component $\dot{X}_1$ and elimination of damped oscillatory component, sinusoidal component with double to industrial frequency, related to useful signal transform , and also of higher harmonics with frequencies multiple of ω_0. In case of $\omega_1 \neq \omega_0$, useful input signal of averaging filter will be a low-frequency sine signal with the frequency $\omega_1 - \omega_0$.

In filter synthesis the following signal parameters should be taken into account: the exponential disturbance damping coefficient changes in signal $x(t)$, power system frequency and related to it useful signal and disturbance changes, which influence on signal spectral composition and useful signal-disturbance ratio.

Let us consider averaging FIR filters synthesis for PMU (Phasor Measurement Units) devices and compare the gained results with averaging FIR filters, applied in one of the best PMU – Model 1133A Power Sentinel, made by American company Arbiter (Gustafson, 2009).

In this PMU one of the following time windows can be implemented: Raised cosine, Hann, Hamming, Blackman, Bartlett, Rectangular, Flat Top, Kaiser, Nutall 4-term, at any filter length, which can be from one to several periods of industrial frequency $T_0 = 2\pi / \omega_0$.

First let us find the solutions without consideration of exponential disturbance elimination, as it is accepted in the most of PMU (Phadke, 2008). The filter must guarantee less than 40 ms speed and 0.2 accuracy class.

Let us accept the following parameters for FIR filter generalized impulse function:

$$\dot{\mathbf{G}} = \dot{\mathbf{G}}' = [G_0 \quad G_1 \quad G_2 \quad G_3 \quad G_4]^T, \quad \mathbf{q} = [0 \quad jw_1 \quad j2w_1 \quad j3w_1 \quad j4w_1]^T,$$

$$\mathbf{T} = [T_1 \quad T_1 \quad T_1 \quad T_1 \quad T_1]^T, \quad T_1 = 2\pi / w_1 \quad .$$

This special case corresponds to so-called generalized cosine time window (Smith, 2002). This type of window will be further described by the set of only two parameters: $\dot{\mathbf{G}}$ and T_1.

Optimization procedure and target function choice of is a nontrivial problem. In general, in case of several synthesis purposes (criteria), it is complicated to get a rigorous optimal solution. Therefore, the found solutions for averaging FIR filters, should be considered as suboptimal.

Let us consider averaging FIR filters synthesis with use of nonlinear multivariable method, based on function of The Optimization Toolbox extends of MATLAB system. The found solutions at different filter lengths are given in table 5 and on fig. 12.

№	$\dot{\mathbf{G}}$	T_1, c
1.	$98{,}8842[0{,}2601 \quad -0{,}4843 \quad 0{,}2325 \quad -0{,}0231 \quad 0]^T$	0,0401
2.	$101{,}0814[0{,}2827 \quad -0{,}5148 \quad 0{,}1983 \quad -0{,}0058 \quad -0{,}0016]^T$	0,0350
3.	$70{,}027[0{,}3989 \quad -0{,}4976 \quad 0{,}1015 \quad -0{,}0021 \quad -0{,}0001]^T$	0,0358
4.	$73{,}505[0{,}4535 \quad -0{,}4953 \quad 0{,}0547 \quad 0 \quad -0{,}0034]^T$	0,0300
5.	$77{,}691[0{,}5108 \quad -0{,}4819 \quad 0{,}0204 \quad 0{,}0014 \quad -0{,}0145]^T$	0,0252
6.	$82{,}7152[0{,}5397 \quad -0{,}4651 \quad 0{,}0072 \quad 0 \quad -0{,}0121]^T$	0,0224

Table 5. Averaging FIR filter parameters

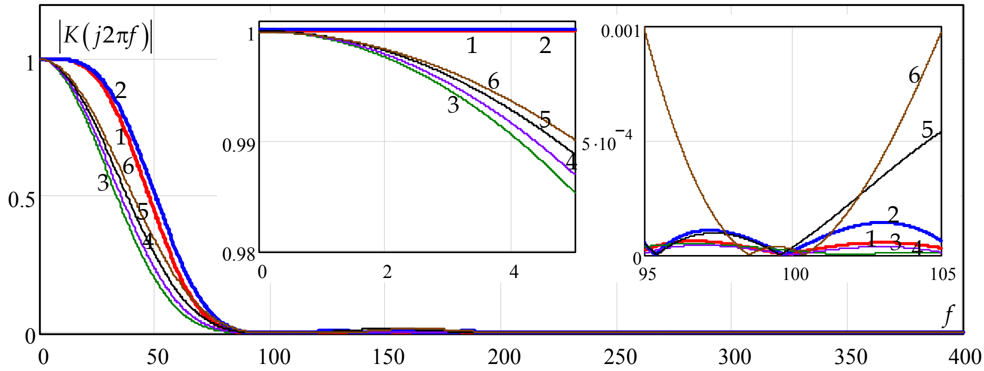


Fig. 12. Amplitude-frequency responses of FIR filters

The impulse functions (time windows) of synthesized filters are presented on fig. 13.

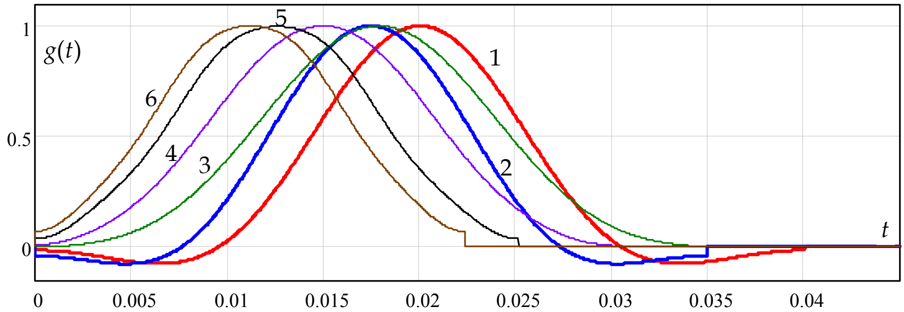


Fig. 13. Time windows of averaging FIR filters

As follows from the fig. 12, filters 1 and 2 have significantly better metrological performances, than averaging FIR filters PMU 1133A.

Filters 3÷6 are used in algorithms of IED signal processing, which do not need ensuring of amplitude-frequency responses stability over the range 0÷5 Hz, according to specified accuracy class (Mokeev, 2009c). Besides that, the more amplitude-frequency responses will be reduced with the frequency growth and the more harmonic elimination with the frequency about 100 Hz there will be, the more exactly the power system frequency will be determined

$$\omega_1(t) = \omega_0 - \frac{\frac{dX_{c1}(t)}{dt} X_{s1}(t) - \frac{dX_{s1}(t)}{dt} X_{c1}(t)}{X_{m1}^2(t)}, \tag{11}$$

where $\dot{X}_1(t) = X_{c1}(t) + jX_{s1}(t)$.

Although the mentioned algorithms of signal processing are adaptive, stationary filters are used in them. Signal processing error, connected to amplitude-frequency response deviation from 1 over the range from 0 to 5 Hz, can be easily compensated due to frequency measurement according to (11).

Let us do synthesis of averaging filter with use of FIR filter generalized model (4) at $M=2$, according to the requirements in table 6.

№	Name	Conditions
1.	Changing sizes of filter input signal parameters (10)	$\omega_1 = 2\pi(45 \div 55)$ rad/s, $\varphi_1 = 0 \div 2\pi$, $X_0 = (0 \div 1) X_{m1}$, $\beta_0 = 20 \div 200\ \text{s}^{-1}$, $X_0 = (0 \div 1) X_{m1}$, $X_n = (0 \div 0{,}5) X_{m1}$, $n \ge 2$
2.	Signal processing performance requirements	1. Acceptable error: $\varepsilon_1 \le 0{,}001$, $\varepsilon_2 \le 0{,}0015$ (0,15 %), additional error at power system frequency deviation: $\varepsilon_2 \le 0{,}0015$ (0.15 %), additional error at $X_0 = X_{m1}$, $\beta_0 = 20 \div 200\ \text{s}^{-1}$ and $t \ge T_1$: $\varepsilon_3 \le 0{,}03$ (3 %), 2. speed: $T_1 \le 0{,}06$ s, $\tau_1 \le 0{,}04$ s 3. acceptable overshoot level: $\le$ 10%
3.	Requirements to filter amplitude-frequency responses in complex frequency coordinates	1. section $p = j\omega$: $\lvert K(0) \rvert = 1$, $1-\varepsilon_{12} < \lvert K(j\Delta\omega) \rvert < 1+\varepsilon_{12}$, $\lvert K(j2\omega_0) \rvert \le \varepsilon_1$, $\lvert K(j(2\omega_0 \pm \Delta\omega)) \rvert < \varepsilon_{12}$, $\lvert K(j(2\omega_0 \pm \Delta\omega)n) \rvert < 0{,}5\varepsilon_{12}$, $n \ge 3$ where $\Delta\omega = 10\pi$ rad/s, $\varepsilon_{12} = \varepsilon_1 + \varepsilon_2$ 2. section $p = -\gamma + j\omega_0$: $2\lvert K(-\gamma + j\omega_0) \rvert e^{-\gamma T_1} \le \varepsilon_3$, $2\lvert K(-\gamma + j\omega_0) \rvert e^{-\gamma \tau_1} \le 0.05$

Table 6. Averaging FIR filter synthesis

The lengths of all finite damped oscillatory components of filter impulse functions will be considered as equal. Using different efficiency functions, two averaging FIR filters with practically identical frequency responses were obtained:

$$\dot{\mathbf{G}}_1 = \left[80{,}48e^{j4{,}232} \quad 37{,}93e^{j0{,}5887}\right]^{\mathrm{T}},\ \mathbf{q}_1 = \left[-22{,}99 + j62{,}30 \quad -23{,}26 + j186{,}9\right]^{\mathrm{T}},$$
$$\mathbf{T}_1 = \left[T_{11} \quad T_{11}\right]^{\mathrm{T}},\ T_{11} = 0{,}051\ \text{c},\ g_1(t) = \operatorname{Re}\left(\dot{\mathbf{G}}_1^{\mathrm{T}} e^{\mathbf{q}_1 t} - \dot{\mathbf{G}}_1^{'\mathrm{T}} e^{\mathbf{q}_1 (t - T_{11})}\right); \tag{12}$$

$$\dot{\mathbf{G}}_2 = \left[42{,}26e^{j6{,}024} \quad 38{,}36e^{j2{,}938}\right]^{\mathrm{T}},\ \mathbf{q}_2 = \left[-4{,}668 + j42{,}69 \quad -23{,}28 + j178{,}7\right]^{\mathrm{T}},$$
$$\mathbf{T}_2 = \left[T_{21} \quad T_{21}\right]^{\mathrm{T}},\ T_{21} = 0{,}050\ \text{c},\ g_2(t) = \operatorname{Re}\left(\dot{\mathbf{G}}_2^{\mathrm{T}} e^{\mathbf{q}_2 t} - \dot{\mathbf{G}}_2^{'\mathrm{T}} e^{\mathbf{q}_2 (t - T_{21})}\right). \tag{13}$$

Filter amplitude-frequency responses and their impulse responses (curve 1 and 2) are shown on the fig. 14 and fig. 15. The averaging filters impulse responses as opposed to ones, considered above (fig. 13), are asymmetrical. Therefore, the filters with mirror-inverse impulse responses (curve 3 and 4) will have the same amplitude-frequency responses in the sections $p = j2\pi f$, i.e. $g_3(t) = g_1(T_{11} - t)$ and $g_4(t) = g_2(T_{21} - t)$. However, filter amplitude-frequency responses with the numbers 3 and 4 in the section $p = -\gamma + j\omega_0$ significantly differ from the analogous amplitude-frequency responses of filters – ancestors (filters 1 and 2).
Thus, the principal conclusion follows from the above: the use of filter traditional amplitude-frequency responses (the section $p = j2\pi f$) for aperiodic signals analysis is not effective.

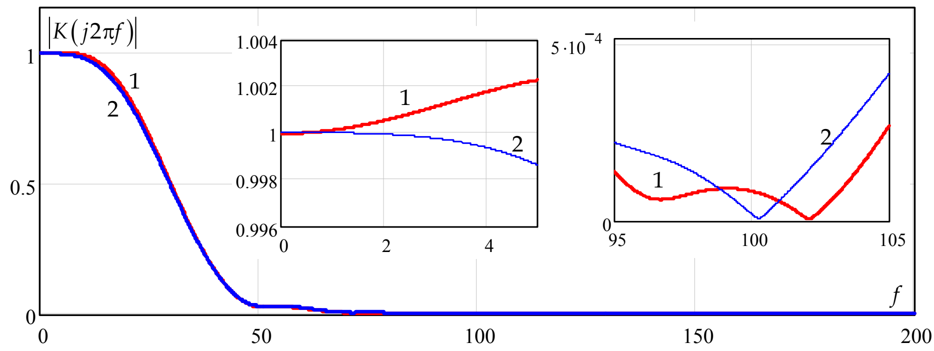


Fig. 14. Filter amplitude-frequency response in the section $p = j2\pi f$

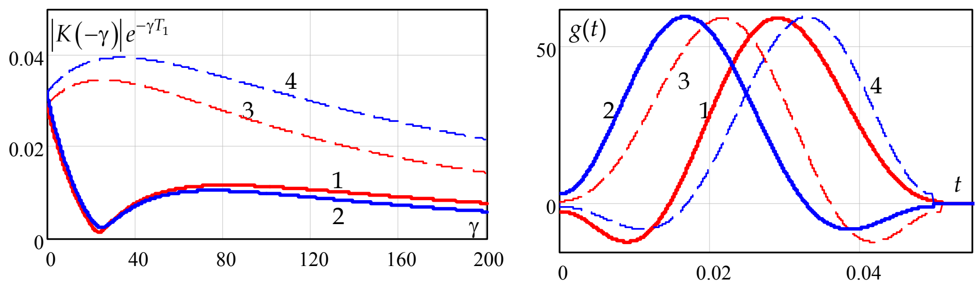


Fig. 15. Amplitude-frequency response in the section $p = -\gamma + j\omega_0$ and impulse responses

The principal difference filter 1 from filter 2 consists in the following: in the first case (filter 1) oscillatory nature of transient process will be observed in the beginning, in the second case it will occur after transient process completes in the filter. As it follows from the fig. 16, the combined use of filters 1 and 2 with practically identical amplitude-frequency response enables to reveal the transient processes in filters (curve 3).

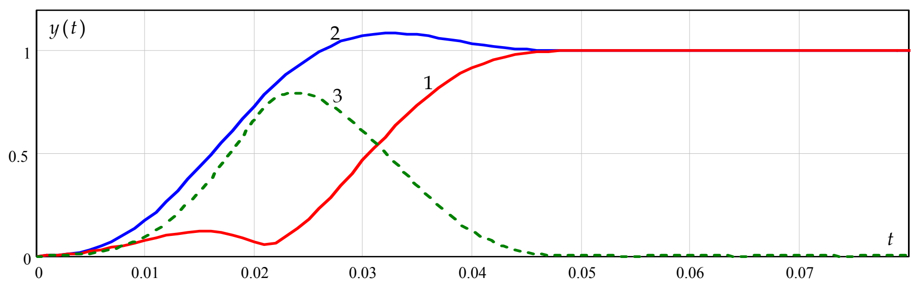


Fig. 16. Output signals of FIR filter

Synthesized filters ensure the combination of signal processing high speed and accuracy, have a low sensitivity to power system frequency deviation and to disturbance spectrum change, and significantly exceed filters, used in PMU 1133A.

The following regularities of time windows for averaging FIR filters can be defined on example of filter synthesis for special case

1. in case of using the cosine time windows and/or time windows (4) at harmonic input signals the form of the synthesized windows is similar to symmetrical "bell-shaped" or in the form of "hat" (fig. 13);
2. in case of using the general time windows (4) at necessity of aperiodic disturbance elimination the windows with clearly defined asymmetrical form (fig. 15) are obtained.

Therefore, the fact can be stated, that for processing of compound semi-infinite or finite aperiodic input signals it is reasonable to use the FIR filter impulse functions (4). Considering the relation between filters and wavelet transforms Koronovskii, 2005, Lyons, 2004), the conclusion about reasonability of mother and father wavelets synthesis, based on the expression (4), can be made. The transition from the mathematical description of analogue filter-prototype to digital filter is carried out by one of the following known methods with the consideration of analog FIR filter specifics (Mokeev, 2008b).

5. Fast algorithms synthesis of FIR filters and spectrum analyzers

5.1 FIR filter fast algorithms synthesis, based on generalized model of analogue filter-prototype impulse function

The advantage of using the analogue filter-prototypes with finite impulse response is direct synthesis of FIR filter realization fast (recursive) algorithms, according to the chosen modified transition method under the table 3.

The fast (recursive) algorithm for general case, using the first or second synthesis methods, is given below

$$\dot{\mathbf{y}}(k) = \mathbf{A}x(k) - \mathbf{B}x(\mathbf{C}k - \mathbf{N}) + \mathbf{D}\dot{y}(k-1)\,,\; y(k) = \mathrm{Re}\left(\mathbf{C}^{\mathrm{T}}\dot{\mathbf{y}}(k)\right). \tag{14}$$

where for the first method $\mathbf{A} = \left[\dot{G}_m T\left(1 - T\rho_m\right)^{-1}\right]_M$, $\mathbf{B} = \left[\dot{G}'_m T\left(1 - T\rho_m\right)^{-1}\right]_M$, $\mathbf{D} = \left[\left(1 - T\rho_m\right)^{-1}\right]_M$, $\mathbf{C} = [1]_M$, $\mathbf{N} = [N_m]_M$; for the second method $\mathbf{A} = \left[\dot{G}_m T\right]_M$, $\mathbf{B} = \left[\dot{G}'_m T\right]_M$, $\mathbf{D} = \left[e^{\rho_m T}\right]_M$.

The block scheme of FIR filter (14) fast (recursive) algorithm is represented on the fig. 17, where $\mathbf{Z} = \left[z^{-N_m}\right]_M$.

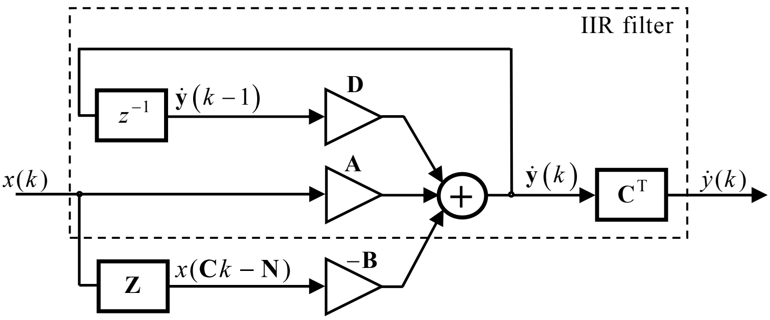


Fig. 17. FIR filter fast algorithm

The fast algorithm (14) expression form, using matrixes, is a compact way of algorithm expression, however, there is a system of M-number independent equations in case of practical realization.

$$\dot{y}_m(k) = a_m x(k) - b_m x(k - N_m) + d_m \dot{y}(k-1)\,, \tag{15}$$

where a_m, b_m, d_m - are complex coefficients, which are the m-th elements of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$ vectors.

The fast algorithm (14) or (15) can be directly realized by using DSPs, which include the instructions to multiplication with accumulation. At other cases, it is necessary to divide the algorithm (15) into two algorithms, which conform to real and imaginary components, i.e. two common filters will be realized (Mokeev, 2008b). Another method consists in algorithm forming, based on the operation fulfillment $y(k) = \mathrm{Re}(\dot{y}(k))$.

In the first case

$$y_{mc}(k) = a_{mc}x(k) - b_{mc}x(k-N) + d_{mc}y_{mc}(k-1) - d_{ms}y_{ms}(k-1)\,,$$

$$y_{ms}(k) = a_{ms}x(k) - b_{ms}x(k-N) + d_{mc}y_{ms}(k-1) + d_{ms}y_{mc}(k-1)\,,$$

where $\dot{y}_m(k) = y_{mc}(k) + jy_{ms}(k)$, $a_m = a_{mc} + ja_{ms}$, $b_m = b_{mc} + jb_{ms}$, $d_m = d_{mc} + jd_{ms}$.

The second method demands by one multiplication operation less

$$y_m(k) = c_{m0}x(k) + c_{m1}x(k-1) - c_{m2}x(k-N_m) - c_{m3}x(k-N_m-1) + h_{m1}y_m(k-1) - h_{m2}y_m(k-2)\,.$$

The fast algorithms synthesis for digital filters with integer coefficients, based on analogue filter-prototype descriptions, is considered in item 5.3.

5.2 Averaging FIR filter fast algorithms synthesis

One of the most extended problems of digital signal processing in measuring technology is connected to FIR filter use, realizing moving-average algorithm (Rabiner, 1975, Vanin, 1991). For reducing the computing expenditures, the digital filtering fast algorithms are applied at FIR filter implementation, including moving-average filters (Blahut, 1985, Nussbaumer, 1981, Yaroslavsky, 1984).

Averaging FIR filters are the special cases for FIR filters. Thus, the fast algorithm synthesis method, considered above, should be used for that kind of filter.

Let us contemplate the most elementary case – rectangular time window. The mathematical expression of analog filter-prototype will be

$$g(t) = k_1\left(1(t) - 1(t - T_1)\right),\ K(p) = \frac{k_1}{p}\left(1 - e^{-pT_1}\right),\ y(t) = k_1 \int_{t-T_1}^{t} x(\tau)d\tau\,,$$

T_1 - is averaging time (window length).

Using the transition methods from an analog filter-prototype to a digital filter, shown in the table 3 , in cases of first and second methods at $k_1 = 1/T$ the following known (Myasnikov, 2005) fast algorithm of moving-average will be obtained

$$y(k) = x(k) - x(k - N_1) + y(k-1)\,,$$

where $N_1 = T_1 / T$.

In case of bilinear transformation method application, there will be the following fast algorithm

$$y(k) = x(k) + x(k-1) - x(k-N_1) - x(k-N_1-1) + y(k-1) \,,$$

At usage of triangle time window, the following fast algorithm of averaging FIR filter realization will be obtained

$$y(k) = x(k) + x(k-1) - 2x(k-N_1-1) + x(k-2N_1-1) + 2y(k-1) - y(k-2) \,.$$

The considered moving-average realization algorithms involve recursive computations, as IIR filters do. However, the principal difference between them is a finite length of filter impulse function. This approach can be also applied to more complicated types of digital filters, including filters, which assure the moving-average computation in case of using different kinds of time windows (Mokeev, 2008a, 2008b, 2009c). The issues about averaging digital filter fast algorithms synthesis, based on given analog filter-prototype (13), considering the microprocessor finite digit capacity influence (Mokeev, 2008a), are investigated.

5.3 FIR filter fast algorithms synthesis, considering microprocessor finite digit capacity

The stability requirements for the discrete filter (5) at any value of FIR filter system function poles $K(z)$ are always ensured. The situation can be changed in case of filter coefficients quantization - at failed coefficient selection, instead of FIR filter IIR filter will be obtained. At negative real components of filter impulse function complex frequencies it is important to assure the filter impulse function level being out of its length is less than a value, specified before.

During the digital FIR filters designing, particular attention should be given to ensuring the impulse response finiteness and filter stability in case, that at least one complex frequency of filter impulse function has a positive real component, as an unstable filter can be obtained at filter coefficients quantization.

Let us consider an example of digital FIR filter synthesis for DSP with the support to fixed point data operations (four numbers to the left of the decimal point). In case of using the method of invariant impulse responses, based on the analogue filter-prototype (13) at $T = 500$ microseconds, the following fast algorithm will be obtained

$$\begin{bmatrix} \dot{y}_1(k) \\ \dot{y}_2(k) \end{bmatrix} = \begin{bmatrix} -0{,}0171 - j0{,}0364 \\ 0{,}0158 + j0{,}0105 \end{bmatrix} x(k) - \begin{bmatrix} 0{,}0049 + j0{,}0115 \\ -0{,}0045 - j0{,}0037 \end{bmatrix} x(k-102) + \begin{bmatrix} (0{,}9881 + j0{,}0308)\dot{y}_1(k-1) \\ (0{,}9841 + j0{,}0922)\dot{y}_2(k-1) \end{bmatrix}.$$

The fast algorithm efficiency is 17 times higher, than algorithm, based on discrete convolution realization with DSP support of complex multiplication/accumulation operations has, and 9 times higher in case of using the ordinary DSPs.

Fast algorithm synthesis for digital filters with integer coefficients is an ambiguous problem, which can be simpler solved by several iterations on the basis of the following expression

$$\dot{\mathbf{y}}_n(k) = \frac{\mathbf{A}_n x_n(k) - \mathbf{B}_n x(\mathbf{C}k - \mathbf{N}) + \mathbf{D}_n \dot{y}_n(k-1)}{m_3} \,,$$

where $\mathbf{A}_n = \mathrm{Int}(m_1\mathbf{A})$, $\mathbf{B}_n = \mathrm{Int}(m_2\mathbf{B})$, $\mathbf{D}_n = \mathrm{Int}(m_1\mathbf{D})$, Int - is an operator, taking an integral part of the number, $x_n(k)$ - is an input signal, considering amplitude quantization, m_1, m_2, m_3 - are scale integer coefficients.

To assure the finite duration of impulse response, the following conditions are required to be fulfilled

$$\mathrm{Int}\left(m_1\mathbf{A}^{\mathrm{T}}\mathbf{z}^{N}\right)-\mathrm{Int}\left(m_2\mathbf{B}^{\mathrm{T}}\mathbf{C}\right)\to 0\,.$$

The following averaging filter fast algorithm with the integer coefficients is obtained for the considered synthesis problem

$$\begin{bmatrix}\dot{y}_{n1}(k)\\ \dot{y}_{n2}(k)\end{bmatrix}=\begin{bmatrix}\dfrac{(-1711-j3642)x_n(k)-(489+j1146)x_n(k-102)+(9881+j308)\dot{y}_{n1}(k-1)}{10000}\\ \dfrac{(1577+j1053)x_n(k)-(-445-j371)x_n(k-102)+(9841+j922)\dot{y}_{n2}(k-1)}{10000}\end{bmatrix}.$$

The output signals for analog and digital signal processing system (fig. 11), using the averaging FIR filters, mentioned above (two filters for real and imaginary signal components $\dot{y}(t)$ or $\dot{y}(k)$ processing), for first harmonic module measuring $\left|\dot{X}_1(t)\right|$ and $\left|\dot{X}_1(k)\right|$ are shown on the fig 18. Digital and analog signal graphs are reduced to digital signal scale.

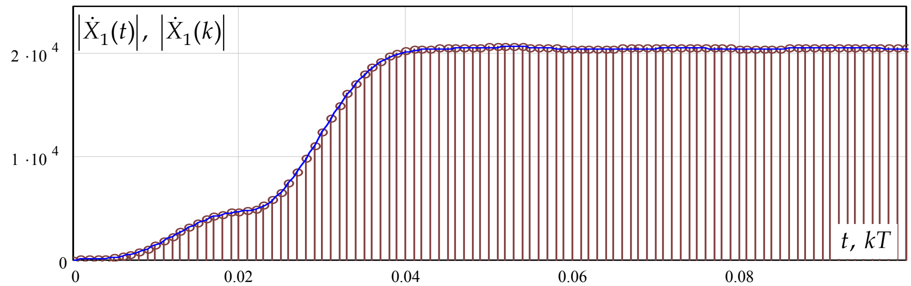


Fig. 18. Output signal

5.4 Fast algorithms synthesis of non-stationary FIR filter with using of the state space method

The expression for FIR filter (14) fast algorithm along with the mathematical description of analogue filter-prototype (7) can be interpreted as a definition, based on filter spectral representations in complex frequency coordinates, and as exposition on the basis of the state space method (Mokeev, 2008b). As is known, the advantage of the state space method consists in mathematical descriptions similarity of stationary and non-stationary systems. Thus, the expression for non-stationary filters can be obtained and interpreted by analogy on the basis of this approach. At that, the matrixes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ will be time dependent

$$\dot{\mathbf{y}}(k)=\mathbf{A}(k)x(k)-\mathbf{B}(k)x(\mathbf{C}k-\mathbf{N})+\mathbf{D}(k)\dot{\mathbf{y}}(k-1)\,. \qquad (16)$$

The algorithm for non-stationary filter with periodic coefficients, which is used for fast Fourier transform realization (Mokeev, 2008b), can be obtained on the basis of the expression above (16)

$$\dot{\mathbf{X}}(k)=\mathbf{W}(k)x(k)-\mathbf{W}(k-N)x(k-N)+\dot{\mathbf{X}}(k-1)\,, \qquad (17)$$

where $\mathbf{W}(k)=T\left[e^{-j\omega_m kT}\right]_M$, $\omega_m=m\omega_0$, $N=\dfrac{2\pi}{\omega_0 T}$, $\dot{\mathbf{X}}_m(k)$ - is spectral density of the signal

$x(t)$ on the basis of short-time Fourier transform application on the frequency $m\omega_0$, using rectangular time window.

Each component of the equation (17) is an analyzer of instantaneous signal spectrum on the specified frequency ω_m.

The fast algorithm of spectrum analyzer (17) has incontestable advantages over the FFT at $N > 5$ (Mokeev, 2008b). At that, it should be noted, that spectral density computation algorithm, as opposed to FFT, is not connected to the number of spectral density values and to uniform frequency scale.

The non-stationary filter algorithm with the periodic coefficients (17) is a special case of more general algorithm (16), which can be applied to describe more complicated types of filters, including adaptive digital filters.

5.5 Synthesis of spectrum analyzer fast algorithms

The spectrum analyzers, based on short-time Fourier transform, can be realized in different ways, including using the fast Fourier transform algorithms (Rabiner, 1975, Blahut, 1985, Nussbaumer, 1981).

The fast algorithms of mentioned spectrum analyzers can be also obtained on the basis of the approaches, considered in this chapter, including the non-stationary filter algorithm (17) with the periodic coefficients, which was contemplated above.

Another approach is based on subdividing the expression for the short-time Fourier transform on the specified frequency into two main operations: multiplication by complex exponent and further using the averaging filter. The issues of averaging FIR filter fast algorithms synthesis were considered in items 5.1 and 5.3.

The third approach is connected to using FIR filter fast algorithms with the orthogonal impulse functions (Mokeev, 2008b).

Let us consider the problems of fast spectrum analyzers synthesis in complex frequency coordinates. Two methods of fast spectrum analyzers realization on complex frequency coordinates, overcoming the difficulties of direct short-time Laplace transform implementation, are offered by the author in this paper (Mokeev, 2008b). The first method is based on using the FIR filter fast algorithms (4), as each finite component of filter with generalized impulse function makes spectrum analysis on the specified complex frequency.

The second method is connected to partitioning the expression for short-time Laplace transform on the given frequency into two basic operations: multiplication by complex exponent and further using the averaging filter with the transfer of exponential window to averaging filter (Mokeev, 2008b).

Considered approaches to FIR filter fast algorithms synthesis can be apply also for the case of wavelet transform fast algorithms, as is known, that wavelet transform is identical with the reconstructed FIR filter with the frequency responses, similar to band pass filter (Mokeev, 2008b).

6. Conclusion

It is shown in this chapter, that for many practical tasks it is reasonable to use the similar generalized mathematical models of analog and digital filter input signals and impulse functions in the form of a set of continuous/discrete semi-infinite or finite damped oscillatory components. To express signals and filters, it is sufficient to exercise the vectors of complex amplitudes and complex frequencies, and also time delay vectors.

For the signal and filter models, mentioned above, it is rational to use the spectral representations of the Laplace transform, in which the damped oscillatory component is a base transform function. Three new methods of analog and digital IIR and FIR filters analysis at semi-infinite and finite input signals were presented on the basis of the research into the spectral representations features of signal and filter frequency responses in complex frequency coordinates. The advantages of offered analysis methods consist in calculation simplicity, including solving problems of direct determination the performance of signal processing by frequency filters.

The application of spectral representations in complex frequency coordinates enables to combine the spectral approach and the state space method for frequency filter analysis and synthesis.

Spectral representations and linear system usage, based on Laplace transform, allow to ensure the effective solution of robust IIR and FIR filters synthesis problems. The filter synthesis problem instead of setting the requirements to separate areas of frequency response (pass band and rejection band) comes to dependence composition for filter transfer function on complex frequencies of input signal components. The synthesis is carried out with the growth of impulse function components number till the specified signal processing performance will be achieved.

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The new technology advances provide that a great number of system signals can be easily measured with a low cost. The main problem is that usually only a fraction of the signal is useful for different purposes, for example maintenance, DVD-recorders, computers, electric/electronic circuits, econometric, optimization, etc. Digital filters are the most versatile, practical and effective methods for extracting the information necessary from the signal. They can be dynamic, so they can be automatically or manually adjusted to the external and internal conditions. Presented in this book are the most advanced digital filters including different case studies and the most relevant literature.

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