PROPERTIES OF NEARLY-COMPACT SPACES

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ABSTRACT. A general product theorem for nearly-compact spaces and locally nearly-compact spaces is given along with other relating properties.

1. Introduction. A topological space \((X, T)\) is said to be nearly-compact [7] if every open cover of \(X\) has a finite subcollection, the interiors of the closures of which cover \(X\); a topological space \((X, T)\) is said to be locally nearly-compact [2] if each point has an open neighborhood whose closure is a nearly-compact subset of \(X\). It has been shown [7] that the product of two nearly-compact spaces is nearly-compact. The primary purpose of this paper is to prove a general product theorem for nearly-compact spaces. Using this result, a theorem pertaining to the product of locally nearly-compact spaces is obtained. Throughout, \(\text{cl}(A)\) will denote the closure of a set \(A\) and \(\text{Int}(A)\) will denote the interior of a set \(A\).

2. Product of nearly-compact spaces. In a topological space \((X, T)\) a set \(A\) is called regular-open if \(A = \text{Int}(\text{cl}(A))\) and regular-closed if \(A = \text{cl}(\text{Int}(A))\) [3, p. 92]. Since the intersection of two regular-open sets is regular-open, the regular-open sets form a base for a smaller semiregular topology \(T^*\) on \(X\), called the semiregularization of \(T\) [1, p. 138]. We call a closed (open) set \(A\) of a space \((X, T)\) star-closed (star-open) if and only if \(A\) is closed (open), respectively, in \((X, T^*)\). Thus it follows that each star-closed set is the intersection of a collection of regular-closed sets and each star-open set is the union of a collection of regular-open sets.

Remark 2.1. Let \(U\) be a regular-open set in the space \(S = \prod_{\alpha} Y_\alpha : Y_\alpha\) has topology \(T_\alpha\ \alpha \in \Delta\) \((S\) has the usual product topology denoted by \(T)\) containing the point \(\{y_0^\alpha\} \in \prod_{\alpha} Y_\alpha\). By noting the behavior of the closure and interior operators on basic open sets, it follows that there exists a finite collection of regular-open sets \(\{U_{\alpha_i}^m\}_{i=1}^n\) \((U_{\alpha_i} \text{ regular-open in } Y_{\alpha_i}\) and a

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regular-open set $G = \prod_{a \in \Delta} Y_a \times U_{a_1} \times U_{a_2} \times \cdots \times U_{a_n}$ containing
$\{y^0_a\}$ such that $\{y^0_a\} \in G \subset U$. Now if we let $S_* = \prod_{a \in \Delta} Y_a: Y_a$ has topology
$T_a$, $a \in \Delta$ and denote its topology by $T_0$. It follows from the preceding
discussion that the semiregular topology, $T_*$, associated with the space
$(S, T)$ is equal to the topology $T_0$.

Remark 2.2. A topological space $(X, T)$ is nearly-compact if and only
if each regular-open cover admits a finite subcover [7]. It was noted in [2,
Theorem 4.1] that a space $(X, T)$ is nearly-compact if and only if $(X, T_*)$
is compact. Using this result along with Remark 2.1, we next give a general
product theorem for nearly-compact spaces.

Theorem 2.1. Let $\{(Y_a, T_a): a \in \Delta\}$ be any family of spaces. Then
$\prod Y_a$ is nearly-compact if and only if each $Y_a$ is nearly-compact.

Proof. Assume that the $\prod Y_a$ is nearly-compact. Since each projection
$P_{\beta}: \prod Y_a \to Y_\beta$ is a continuous open surjection, [7, Theorem 3.1] shows that
each $Y_\beta$ is nearly-compact.

Conversely, assume that each $(Y_\alpha, T_\alpha)$ is nearly-compact and let $S = 
\prod Y_a: Y_a$ has topology $T_\alpha$ and $\alpha \in \Delta$ and $S_* = \prod Y_a: Y_a$ has topology
$T_a$, $a \in \Delta$. Now let $\{U_\beta: U_\beta$ regular-open in $S_*, \beta \in \Pi\}$ be a regular-
open cover of the space $S$. Using Remark 2.1 and Remark 2.2 it follows that
$\{U_\beta\}$ is a star-open cover in the compact space $S_*$. Consequently, $S$ is near-
ly-compact. This completes the proof.

Definition 2.1. A topological space $(X, T)$ is called locally nearly-com-
 pact if each point has an open neighborhood whose closure is a nearly-compact
subset of $X$ [2].

[2, Theorem 4.5] shows that if $(X, T)$ is a locally nearly-compact Haus-
dorff space, then $(X, T_\*)$ is locally compact. We next show that the converse
to [2, Theorem 4.5] is also true.

Lemma 2.1. Let $(X, T)$ be a topological space. Then $(X, T)$ is a local-
ly nearly-compact Hausdorff space if and only if $(X, T_\*)$ is a locally-compact
Hausdorff space.

Proof. Only the sufficiency requires proof. Assume that $(X, T_\*)$ is a
locally-compact Hausdorff space and let $x \in X$. Clearly $(X, T)$ is Hausdorff.
There exists a star-open set $V$ in $(X, T_\*)$ containing $x$ such that $cl_*(V)$
(closure in $(X, T_\*)$) is a compact subset in $(X, T_\*)$. Since $cl_*(V) = cl(V),
cl(V)$ is a nearly-compact subset in $(X, T)$. Therefore $(X, T)$ is locally
nearly-compact.
Theorem 2.2. Let \( \{ (Y_\alpha, T_\alpha) \colon \alpha \in \Delta \} \) be a family of Hausdorff spaces. Then \( \prod_\alpha Y_\alpha \) is locally nearly-compact if and only if all the \( Y_\alpha \) are locally nearly-compact and at most finitely many are not nearly-compact.

Proof. The proof is a standard, well-known argument after noting Remark 2.1 and Lemma 2.1.

3. General properties. A topological space \((X, T)\) is said to be completely Hausdorff [1, p. 146] if each pair of distinct points of \(X\) has disjoint closed neighborhoods. A Hausdorff space \((X, T)\) is said to be \(H\)-closed (called absolutely closed in [1]) if every open covering of \(X\) contains a finite subfamily whose closures cover \(X\) [6]. It follows from [7, Corollary 2.2] that a Hausdorff space \(X\) is nearly-compact if and only if it is an \(H\)-closed completely Hausdorff space.

Remark 3.1. We define a subset \(A \subset X\) to be an \(H\)-closed subset if for every collection \(\{ U_\alpha \colon U_\alpha \text{ open in } X, \alpha \in \Delta \}\) such that \(A \subset \bigcup_{\alpha} U_\alpha\), there exists a finite subcollection \(\{ U_{\alpha_i} \}_{i=1}^n\), such that \(A \subset \bigcup_{i=1}^n \text{cl}(U_{\alpha_i})\). Since points in a Hausdorff space \(X\) can be separated by disjoint regular-open sets [1, p. 138], it follows that \(H\)-closed subsets are star-closed in \(X\).

It is well known that the intersection of a descending family of continua in a compact Hausdorff space is a continuum [4, p. 43]. We next give an example to show that in general, a similar result need not hold in nearly-compact Hausdorff spaces.

Example 3.1. Let \(I^2 = [0, 1] \times [0, 1]\) have as a subbasis the usual open sets in \(I^2\) along with the complements of the set \(A \times I^1\), where \(A\) runs through the set of all subsets of \([0, 1]\). It follows from [1, Exercise 23b, p. 147] that \(I^2\) is a nearly-compact Hausdorff space, whose semiregular topology, \(T_*\), is the usual cartesian product topology of \(I^2\). For each \(n = 1, 2, 3, \ldots\), let \(F_n = \{(x, y) \in I^2 \colon y \leq 1/n\}\). Then \(\{ F_n \}_{n=1}^\infty\) is a descending family of nearly-compact connected subsets in \(I^2\) such that \(K = \bigcap_n F_n = [0, 1]\). It follows that \(K\), as a subspace of \(I^2\), is disconnected and not nearly-compact.

Remark 3.2. Even though the set \(K\) in Example 3.1 is not a nearly-compact subspace in \(I^2\), it is a nearly-compact subset since it is star-closed. Also we note that \(K\) is not a regular-closed set in \(I^2\) because \(\text{Int } K = \emptyset\). With this in consideration we give our next theorem.

Theorem 3.1. Let \((X, T)\) be a nearly-compact Hausdorff space and let \(\omega_\beta\) be an initial ordinal. Suppose that \(\{ F_\alpha \colon \alpha < \omega_\beta \}\) is a descending family \((\alpha < \gamma \text{ implies } F_\alpha \supset F_\gamma)\) of nonempty star-closed subsets in \(X\). Then
\[
(a) \quad K = \bigcap_{\alpha} F_\alpha
\]
will be a nonempty nearly-compact subset of \(X\).
(b) If each $F_\alpha$ is connected in $(X, T)$ and $K = \bigcap_\alpha F_\alpha$ is regular-closed, then $K$ will be connected in $(X, T)$.

**Proof.** First, we observe, as we have already noted, $(X, T)$ is a nearly-compact Hausdorff space if and only if $(X, T_*)$ is a compact Hausdorff space. Since each $F_\alpha$ is star-closed in $(X, T)$, it follows that each $F_\alpha$ is a compact subset in $(X, T_*)$. Therefore $\bigcap_\alpha F_\alpha = K$ is a nonempty compact subset in $(X, T_*)$ which implies that $K$ is a nonempty nearly-compact subset in $(X, T)$.

Now assume that each $F_\alpha$ is connected in $(X, T)$ and $\bigcap_\alpha F_\alpha = K$ is regular-closed in $(X, T)$. Clearly each $F_\alpha$ is connected in $(X, T_*)$. Therefore, [4, Lemma 2.8, p. 43] gives $K \subseteq \bigcap_\alpha F_\alpha$ compact and connected in $(X, T_*)$. Now suppose that $K$ is not connected in $(X, T)$. Then there exist two disjoint nonempty regular-closed sets $C_1$ and $C_2$ in the subspace $K$ such that $K = C_1 \cup C_2$. Since $K$ is regular-closed in $(X, T)$ and $C_i$ ($i = 1, 2$) is regular-closed in $K$, it follows that $C_i$ ($i = 1, 2$) is regular-closed in $(X, T)$ and consequently, closed in $(X, T_*)$. Therefore, $K$ is not connected in $(X, T_*)$ which is a contradiction. We conclude that $K$ is connected in $(X, T)$. This completes the proof.

**Remark 3.3.** A function is almost-continuous if the inverse images of regular-open sets are open and a function is almost-open if the images of regular-open sets are open [8].

We say a map $f : X \to Y$ is star-closed if the images of star-closed sets are closed.

**Theorem 3.2.** Let $f : (X, T_0) \to (Y, T)$ be an almost-continuous map of a nearly-compact space $X$ into a Hausdorff space $Y$. Then the images of star-closed subsets in $X$ will be $H$-closed subsets in $Y$. Moreover, $f$ will be a star-closed map.

**Proof.** Let $A$ be a star-closed subset in $X$. Since $X$ is nearly-compact, $A$ is an $H$-closed subset in $X$. Now since $f : (X, T_0) \to (Y, T)$ is almost-continuous if and only if $f : (X, T_0) \to (Y, T_*)$ is continuous, it follows that $f(H)$ is an $H$-closed subset in $(Y, T)$ and consequently, closed. Therefore, $f$ is a star-closed map. This completes the proof.

[5, Theorem 4] shows that when $f : X \to Y$ is an almost-continuous surjection of a connected space $X$ onto a space $Y$, then $Y$ will be connected. However, it was noted in [5] that almost-continuous maps do not preserve connected sets in general.

Our next theorem gives conditions when connected sets are preserved under almost-continuous maps.
Theorem 3.3. Let \( f: X \to Y \) be an almost-continuous almost-open map of a nearly-compact space \( X \) into a Hausdorff space \( Y \). Then the image of each regular-closed connected set in \( X \) will be a regular-closed connected set in \( Y \).

Proof. Let \( H \) be a connected regular-closed set in \( X \). We first show that \( f(H) = F \) is regular-closed in \( Y \). Theorem 3.2 assures us that \( F \) is closed. Suppose \( \text{cl}(\text{Int}(F)) \subset F \). Then there exists a \( y \in F \) such that \( y \notin \text{cl}(\text{Int}(F)) \). Since \( y \in f(H) = F \) there exists an \( x \in H \) such that \( f(x) = y \).

Now \( Y - \text{cl}(\text{Int}(F)) \) is a regular-open set containing \( f(x) \); therefore, there exists an open set \( W \) containing \( x \) such that \( f(W) \subset Y - \text{cl}(\text{Int}(F)) \). Hence, \( f(W) \cap \text{cl}(\text{Int}(F)) = \emptyset \). Since \( x \in H = \text{cl}(\text{Int}(H)) \), \( W \cap \text{Int}(H) \neq \emptyset \). Therefore, \( \emptyset \neq \{W \cap \text{Int}(H) \} \subset f(W) \cap \text{cl}(\text{Int}(F)) \subset f(W) \cap \text{Int}(H) \).

Thus \( f(W) \cap \text{cl}(\text{Int}(F)) \neq \emptyset \) which is a contradiction. We conclude that \( f(H) = \text{cl}(\text{Int}(f(H))) \) is a regular-closed set in \( Y \). Now assume that \( f(H) \) is not connected in \( Y \). Then there exist two nonempty, disjoint, regular-closed sets \( C_1 \) and \( C_2 \) in the subspace \( f(H) \) such that \( f(H) = C_1 \cup C_2 \). Since \( f(H) \) is regular-closed in \( Y \), it follows that \( C_i \) (\( i = 1, 2 \)) is regular-closed in \( Y \). Now the almost-continuity of \( f \) gives disjoint closed sets \( f^{-1}(C_1) \cap H \) and \( f^{-1}(C_2) \cap H \) in \( X \) such that \( H = ([f^{-1}(C_1)] \cap H) \cup ([f^{-1}(C_2)] \cap H) \). Therefore \( H \) is disconnected, which is a contradiction. Hence, we conclude that \( f(H) \) is a regular-closed connected subset in \( Y \).

Theorem 3.4. Let \( f: (X, T) \to (Y, T_0) \) be a continuous map of a nearly-compact space \( X \) into a regular space \( Y \). Then \( f(X) \) will be a compact subset in \( Y \).

Proof. Since \( Y \) is regular, \( f: (X, T) \to (Y, T_0) \) is continuous if and only if \( f: (X, T_*) \to (Y, T_0) \) is continuous \([6, \text{Theorem 2.15}]\). Now Remark 2.2 gives \( (X, T_*) \) compact. Consequently, it follows that \( f(X) \) is a compact subset in \( Y \). This completes the proof.

Our final theorem is an application of the notion of locally nearly-compactness.

Theorem 3.5. Let \( f: (X, T) \to (Y, T_0) \) be a continuous star-closed map of a locally nearly-compact Hausdorff space \( X \) into a regular space \( Y \). If \( f \) has nearly-compact point inverses, then \( f(X) \) will be a locally compact Hausdorff subspace of \( Y \).

Proof. We first observe that \( f: (X, T) \to (Y, T_0) \) is a star-closed continuous map if and only if \( f: (X, T_* \to (Y, T_0) \) is a closed continuous map.
Also we note that \( f^{-1}(y) \) is a nearly-compact subset in \((X, T)\) if and only if \( f^{-1}(y) \) is a compact subset in \((X, T_*).\) Consequently, it follows from the hypothesis that \( f: (X, T_*) \rightarrow (Y, T_0) \) is a proper map [1, p. 101]. Now by Lemma 2.1 we have that \((X, T_*)\) is locally compact. Therefore \( f: (X, T_*) \rightarrow (Y, T_0) \) is a proper mapping of a locally-compact Hausdorff space \((X, T_*).\) into a space \(Y.\) Hence, the result follows from [1, Corollary 2, p. 100] and the corollary following [1, Proposition 9, p. 105].

**Remark 3.4.** If the space \((Y, T_0)\) in Theorem 3.5 is given to be compact (resp. locally-compact), it does not imply the corresponding space \((X, T)\) is compact (resp. locally-compact). For let \(R\) be the reals, and \(T\) the topology on \(R\) having the open intervals and the set \(A = \{x \in R: x \text{ is rational and } 1/3 < x < 2/3\}\) as a subbasis. Give the set \(X = [0, 1]\) the subspace topology of \((R, T)\) and topologize the set \(Y = [0, 1]\) with the usual (Euclidean) subspace topology of \(R.\) It follows that the space \(X = [0, 1]\) is a nearly-compact (and consequently a locally nearly-compact) Hausdorff space which is not locally-compact. Let \(1: X \rightarrow Y\) be the identity map of \(X\) onto \(Y.\) The identity map \(1\) satisfies the hypothesis of Theorem 3.5, but the space \(X\) is not locally-compact.

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**REFERENCES**


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