

## CLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS

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Received 1 February 2004

Some classes of uniformly starlike and convex functions are introduced. The geometrical properties of these classes and their behavior under certain integral operators are investigated.

2000 Mathematics Subject Classification: 30C45, 30C50.

**1. Introduction.** Let  $A$  denote the class of functions of the form  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  which are analytic in the open unit disk  $U = \{z : |z| < 1\}$ . A function  $f$  in  $A$  is said to be starlike of order  $\beta$ ,  $0 \leq \beta < 1$ , written as  $f \in S^*(\beta)$ , if  $\operatorname{Re}[(zf'(z))/(f(z))] > \beta$ . A function  $f \in A$  is said to be convex of order  $\beta$ , or  $f \in K(\beta)$ , if and only if  $zf' \in S^*(\beta)$ .

Let  $SD(\alpha, \beta)$  be the family of functions  $f$  in  $A$  satisfying the inequality

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \quad z \in U, \quad \alpha \geq 0, \quad 0 \leq \beta < 1. \quad (1.1)$$

We note that for  $\alpha > 1$ , if  $f \in SD(\alpha, \beta)$ , then  $zf'(z)/f(z)$  lies in the region  $G \equiv G(\alpha, \beta) \equiv \{w : \operatorname{Re} w > \alpha|w-1| + \beta\}$ , that is, part of the complex plane which contains  $w = 1$  and is bounded by the ellipse  $(u - (\alpha^2 - \beta)/(\alpha^2 - 1))^2 + (\alpha^2/(\alpha^2 - 1))v^2 = \alpha^2(1 - \beta)^2/(\alpha^2 - 1)^2$  with vertices at the points  $((\alpha + \beta)/(\alpha + 1), 0)$ ,  $((\alpha - \beta)/(\alpha - 1), 0)$ ,  $((\alpha^2 - \beta)/(\alpha^2 - 1), (\beta - 1)/\sqrt{\alpha^2 - 1})$ , and  $((\alpha^2 - \beta)/(\alpha^2 - 1), (1 - \beta)/\sqrt{\alpha^2 - 1})$ . Since  $\beta < (\alpha + \beta)/(\alpha + 1) < 1 < (\alpha - \beta)/(\alpha - 1)$ , we have  $G \subset \{w : \operatorname{Re} w > \beta\}$  and so  $SD(\alpha, \beta) \subset S^*(\beta)$ . For  $\alpha = 1$  if  $f \in SD(\alpha, \beta)$ , then  $zf'(z)/f(z)$  belongs to the region which contains  $w = 2$  and is bounded by parabola  $u = (v^2 + 1 - \beta^2)/2(1 - \beta)$ .

Using the relation between convex and starlike functions, we define  $KD(\alpha, \beta)$  as the class of functions  $f \in A$  if and only if  $zf' \in SD(\alpha, \beta)$ . For  $\alpha = 1$  and  $\beta = 0$ , we obtain the class  $KD(1, 0)$  of uniformly convex functions, first defined by Goodman [1]. Rønning [3] investigated the class  $KD(1, \beta)$  of uniformly convex functions of order  $\beta$ . For the class  $KD(\alpha, 0)$  of  $\alpha$ -uniformly convex function, see [2]. In this note, we study the coefficient bounds and Hadamard product or convolution properties of the classes  $SD(\alpha, \beta)$  and  $KD(\alpha, \beta)$ . Using these results, we further show that the classes  $SD(\alpha, \beta)$  and  $KD(\alpha, \beta)$  are closed under certain integral operators.

**2. Main results.** First we give a sufficient coefficient bound for functions in  $SD(\alpha, \beta)$ .

**THEOREM 2.1.** *If  $\sum_{n=2}^{\infty} [n(1 + \alpha) - (\alpha + \beta)]|a_n| < 1 - \beta$ , then  $f \in SD(\alpha, \beta)$ .*

**PROOF.** By definition, it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - (1 + \beta) - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| \right| < \left| \frac{zf'(z)}{f(z)} + (1 - \beta) - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| \right|. \tag{2.1}$$

For the right-hand side and left-hand side of (2.1) we may, respectively, write

$$\begin{aligned} R &= \left| \frac{zf'(z)}{f(z)} + (1 - \beta) - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| \right| \\ &= \frac{1}{|f(z)|} |zf'(z) + (1 - \beta)f(z) - \alpha e^{i\theta} |zf'(z) - f(z)| | \\ &\geq \frac{1}{|f(z)|} \left[ (2 - \beta)|z| - \sum_{n=2}^{\infty} (n + 1 - \beta) |a_n| |z|^n - \alpha \sum_{n=2}^{\infty} (n - 1) |a_n| |z|^n \right] \\ &> \frac{|z|}{|f(z)|} \left[ 2 - \beta - \sum_{n=2}^{\infty} (n + 1 - \beta + n\alpha - \alpha) |a_n| \right], \end{aligned} \tag{2.2}$$

and similarly

$$L = \left| \frac{zf'(z)}{f(z)} - (1 + \beta) - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| \right| < \frac{|z|}{|f(z)|} \left[ \beta + \sum_{n=2}^{\infty} (n - 1 - \beta + n\alpha - \alpha) |a_n| \right]. \tag{2.3}$$

Now, the required condition (2.1) is satisfied, since

$$R - L > \frac{|z|}{|f(z)|} \left[ 2(1 - \beta) - 2 \sum_{n=2}^{\infty} [n(1 + \alpha) - (\alpha + \beta)] |a_n| \right] > 0. \tag{2.4}$$

The following two theorems follow from the above [Theorem 2.1](#) in conjunction with a convolution result of Ruscheweyh and Sheil-Small [5] and the already discussed relation between the classes  $SD(\alpha, \beta)$  and  $KD(\alpha, \beta)$ . □

**THEOREM 2.2.** *If  $\sum_{n=2}^{\infty} n[n(1 + \alpha) - (\alpha + \beta)]|a_n| < 1 - \beta$ , then  $f \in KD(\alpha, \beta)$ .*

**THEOREM 2.3.** *The classes  $SD(\alpha, \beta)$  and  $KD(\alpha, \beta)$  are closed under Hadamard product or convolution with convex functions in  $U$ .*

From [Theorem 2.3](#) and the fact that

$$F(z) = \frac{1 + \lambda}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt = f(z) * \sum_{n=1}^{\infty} \frac{1 + \lambda}{n + \lambda} z^n, \quad \text{Re } \lambda \geq 0, \tag{2.5}$$

we obtain the following corollary upon noting that  $\sum_{n=1}^{\infty} ((1 + \lambda)/(n + \lambda))z^n$  is convex in  $U$ .

**COROLLARY 2.4.** *If  $f$  is in  $SD(\alpha, \beta)$  or  $KD(\alpha, \beta)$ , so is  $F(z)$  given by (2.5).*

Similarly, the following corollary is obtained for

$$G(z) = \int_0^z \frac{f(t) - f(\mu t)}{t(1 - \mu)} dt = f(z) * \left( z + \sum_{n=2}^{\infty} \frac{1 - \mu^n}{n(1 - \mu)} z^n \right), \quad |\mu| \leq 1, \mu \neq 1. \tag{2.6}$$

**COROLLARY 2.5.** *If  $f$  is in  $SD(\alpha, \beta)$  or  $KD(\alpha, \beta)$ , so is  $G(z)$  given by (2.6).*

We observed that if  $\alpha > 1$  and if  $f \in SD(\alpha, \beta)$ , then  $(zf'(z)/f(z))_{z \in U} \subset E$ , where  $E$  is the region bounded by the ellipse  $(u - (\alpha^2 - \beta)/(\alpha^2 - 1))^2 + (\alpha^2/(\alpha^2 - 1))v^2 = \alpha^2(1 - \beta)^2/(\alpha^2 - 1)^2$  with the parametric form

$$w(t) = \frac{\alpha^2 - \beta}{\alpha^2 - 1} + \frac{\alpha(1 - \beta)}{\alpha^2 - 1} \cos t + \frac{i(1 - \beta)}{\sqrt{\alpha^2 - 1}} \sin t, \quad 0 \leq t < 2\pi. \tag{2.7}$$

Thus for  $\alpha > 1$  and  $z$  in the punctured unit disk  $U - \{0\}$ , we have  $f \in SD(\alpha, \beta)$  if and only if  $zf'(z)/f(z) \neq w(t)$  or  $zf'(z) - w(t)f(z) \neq 0$ . By Ruscheweyh derivatives (see [4]), we obtain  $f \in SD(\alpha, \beta)$ , if and only if  $f(z) * [z/(1 - z)^2 - w(t)z/(1 - z)] \neq 0$ ,  $z \in U - \{0\}$ . Consequently,  $f \in SD(\alpha, \beta)$ ,  $\alpha > 1$ , if and only if  $f(z) * h(z)/z \neq 0$ ,  $z \in U$  where  $h$  is given by the normalized function

$$h(z) = \frac{1}{1 - w(t)} \left[ \frac{z}{(1 - z)^2} - w(t) \frac{z}{1 - z} \right] \tag{2.8}$$

and  $w$  is given by (2.7). Conversely, if  $f(z) * h(z)/z \neq 0$ , then  $zf'(z)/f(z) \neq w(t)$ ,  $0 \leq t < 2\pi$ . Hence  $(zf'(z)/f(z))_{z \in U}$  lie completely inside  $E$  or its compliment  $E^c$ . Since  $(zf'(z)/f(z))_{z=0} = 1 \in E$ ,  $(zf'(z)/f(z))_{z \in U} \subset E$ , which implies that  $f \in SD(\alpha, \beta)$ . This proves the following theorem.

**THEOREM 2.6.** *The function  $f$  belongs to  $SD(\alpha, \beta)$ ,  $\alpha > 1$ , if and only if  $f(z) * h(z)/z \neq 0$ ,  $z \in U$  where  $h(z)$  is given by (2.8).*

**REFERENCES**

[1] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), no. 1, 87-92.  
 [2] S. Kanas and A. Wiśniowska, *Conic domains and starlike functions*, Rev. Roumaine Math. Pures Appl. **45** (2000), no. 4, 647-657.  
 [3] F. Rønning, *On starlike functions associated with parabolic regions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **45** (1991), 117-122.  
 [4] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109-115.  
 [5] St. Ruscheweyh and T. Sheil-Small, *Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv. **48** (1973), 119-135, Corrigendum in Comment. Math. Helv. **48** (1973), 194.

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