

L-fuzzy valued measure and integral

Vecislavs Ruza, Svetlana Asmuss

1 University of Latvia, Department of Mathematics
2 Institute of Mathematics and Computer Science of University of Latvia

Abstract

We continue to develop a construction of an L-fuzzy valued measure by extending a measure defined on a σ-algebra of crisp sets to an L-fuzzy valued measure defined on a T accumulative in the case when operations with L-sets and L-fuzzy numbers are defined by using the minimum triangular norm T. We introduce an L-fuzzy valued integral over an L-set with respect to an L-fuzzy valued measure, consider its properties and describe a method of L-fuzzy valued integration.

Keywords: L-set, L-fuzzy real number, L-fuzzy valued measure, L-fuzzy valued integral.

1. Introduction

One can find a lot of works regarding a fuzzy approach to measure and integral. The most important concepts and results concerning this topic are considered in [1], [2], [3]. Our interest is in developing a theory where not only sets are fuzzy, but also measure and integral take fuzzy real values. In the previous papers [4], [5] we suggested the construction that allows us to obtain an L-fuzzy valued measure defined on a T -algebra of L-sets by extension of a measure defined on a σ-algebra of crisp sets. We continue to develop the results obtained before and describe how an L-fuzzy valued measure defined on a T -algebra can be obtained for a given σ-algebra Φ ⊂ 2X and a finite measure μ : Φ → R+. On the next stage we introduce the concept of an L-fuzzy valued integral over a measurable L-set. Some properties of L-fuzzy valued integral are considered. We suppose that L is a complete, completely distributive lattice (see e.g. [6]) and operations with L-sets and L-fuzzy numbers are defined by using the minimum triangular norm T .

We give our preference to the fuzzy real numbers as they were first defined by B. Hutton [7] and then studied thoroughly in a series of papers (see e.g. [8], [9], [10]). The preference of using this approach for defining fuzzy real numbers is motivated by our intention to develop results on approximation from [11], [12]. For problems that can be solved only approximately the notion of the error of a method of approximation plays the fundamental role. In order to estimate the quality of approximation on an L-fuzzy set, we need an appropriate L-fuzzy analogue of a norm. Our intention is to use the L-fuzzy valued integral to define an L-fuzzy norm for investigation of the error of approximation on an L-set.

2. Preliminaries

2.1. L-sets

Given a (crisp) universe X and a complete, completely distributive lattice L(∧, ∨, 0L, 1L), an L-subset A of X (or, briefly, an L-set A) is a function A : X → L. The class of all L-subsets of X is denoted L X . The operations with L-sets A, B are defined by using the minimum triangular norm T M, its corresponding conorm S M and decreasing involution N:

\[ (A \land B)(x) = T M(A(x), B(x)), \]
\[ (A \lor B)(x) = S M(A(x), B(x)), \]
\[ A'(x) = N(A(x)). \]

A finite family of L-sets A 1, A 2, …, A n is said to be T M-disjoint (see e.g. [1]) iff for each k ∈ {1, …, n} we have \( \bigvee _{i=1,i\neq k}^{n} A_i \land A_k = 0 \). A countable family of L-sets is said to be T M-disjoint iff every finite subfamily of this family is T M-disjoint.

In order to consider an L-fuzzy valued T M-measure we consider classes of L-sets called T M-clans and T M-tribes (see e.g. [1]).

Definition 2.1. A subclass \( \mathcal{A} \subset L^X \) is called a T M-clan on X if the following properties are satisfied:

- \( \emptyset \in \mathcal{A} \);
- for all \( A \in \mathcal{A} \) we have \( A^c \in \mathcal{A} \);
- for all \( A, B \in \mathcal{A} \) we have \( A \land B \in \mathcal{A} \).

Definition 2.2. A subclass \( \Sigma \subset L^X \) is called a T M-tribe on X if the following properties are satisfied:

- \( \emptyset \in \Sigma \);
- for all \( A \in \Sigma \) we have \( A^c \in \Sigma \);
- for all sequences \((A_n)_{n \in \mathbb{N}} \subset \Sigma \) we have \( \bigwedge _{n=1}^\infty A_n \in \Sigma \).

2.2. L-fuzzy real numbers

For our purposes we use the L-fuzzy real numbers as they were first defined by B. Hutton [7].

Definition 2.3. An L-fuzzy real number is a function \( z : \mathbb{R} \to L \) such that

- \( z \) is non-increasing;
- \( \bigwedge _{t}^{\infty} z(t) = 0_L, \bigvee _{t}^{t} z(t) = 1_L; \)
- \( z \) is left semi-continuous, i.e. for all \( t_0 \in \mathbb{R} \) we have \( \bigwedge _{t<t_0}^{\infty} z(t) = z(t_0) \).
In the original papers of this subject (see [7], [8], [9]) \( L \)-fuzzy real numbers were defined not as order reversing functions, but as equivalence classes of such functions. However each class of equivalence has a unique left semi-continuous representative and therefore an \( L \)-fuzzy real number can be identified with this representative. A deep theoretical justification of viewing fuzzy numbers as distribution function was given by U. Höhle [10], who showed that such fuzzy real numbers can be obtained from the set of rational numbers \( \mathbb{Q} \) by means of Dedekind completion in the same way as real numbers \( \mathbb{R} \) are obtained from \( \mathbb{Q} \) if one applies the multiple-valued logic, instead of the binary logic which stands behind the Dedekind completion in the classic case.

The set of all \( L \)-fuzzy real numbers is called the \emph{the \( L \)-fuzzy real line} and it is denoted by \( \mathbb{R}(L) \). An \( L \)-fuzzy number \( z \) is called \emph{non-negative} if \( z(0) = 1_L \). We denote by \( \mathbb{R}_+(L) \) the set of all non-negative \( L \)-fuzzy real numbers.

Operations with \( L \)-fuzzy real numbers such as addition \( \oplus \) and multiplication by a real positive number \( a \) are defined as following:

\[
(z_1 \oplus z_2)(t) = \bigvee \left\{ z_1(t, \tau) \land z_2(t, \tau) \mid \tau \in \mathbb{R} \right\}, \quad (z)(t) = z(t, t).
\]

The supremum and the infimum of a set of non-negative \( L \)-fuzzy numbers \( F \subseteq \mathbb{R}_+(L) \) are defined by the formulas (see e.g. [11], [12]):

\[
(\inf F)(t) = \bigwedge \left\{ z(t) \mid z \in F, t \in \mathbb{R} \right\}
\]

\[
\sup F = \inf \left\{ z \mid z \in \mathbb{R}(L), z \geq z' \text{ for all } z' \in F \right\}.
\]

Taking into account that \( F \) is bounded from below it is easy to see that \( \inf F \) is an \( L \)-fuzzy real number. In case \( F \) is bounded from above (i.e. there exists \( z_0 \in \mathbb{R}(L) \) such that \( z \leq z_0 \) for all \( z \in F \) ), \( \sup F \) is an \( L \)-fuzzy real number, otherwise the condition

\[
\bigwedge_n \sup F(t) = 0_L
\]

does not necessarily hold.

Going forward we will need also the countable addition of non-negative fuzzy real numbers. Given a sequence of non-negative fuzzy real numbers \( (z_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_+(L) \) we consider the countable sum

\[
\bigoplus_{n=1}^{\infty} z_n = \sup \left\{ z_1 \oplus z_2 \oplus \cdots \oplus z_n \mid n \in \mathbb{N} \right\}.
\]

For \( a \in \mathbb{R}_+ \) and \( \alpha \in \mathbb{L} \) by \( z(a, \alpha) \) we denote a special type of non-negative \( L \)-fuzzy real numbers

\[
(z(a, \alpha))(t) = \begin{cases} 1, & t \leq 0, \\
\alpha, & 0 < t \leq a, \\
0, & t > a, \end{cases}
\]

that will play an important role in our work.

\section{Construction of \( L \)-fuzzy valued measure}

\subsection{Measurable \( L \)-sets}

For a given \( \sigma \)-algebra \( \Phi \subseteq 2^X \) and a finite measure \( \nu : \Phi \rightarrow \mathbb{R}_+ \) an \( L \)-fuzzy valued measure can be obtained by the following schema (see [4], [5]):

- For \( M \in \Phi, \alpha \in \mathbb{L} \) we define an \( L \)-fuzzy set

\[
(A(M, \alpha))(x) = \begin{cases} \alpha \in M, \\
0, & x \notin M. \end{cases}
\]

All these \( L \)-sets form a class of \( L \)-sets that we denote by \( \varnothing \):

\[\varnothing = \{A(M, \alpha)|M \in \Phi, \alpha \in \mathbb{L}\}.
\]

Note that the following properties hold for all \( L \)-sets \( A_1, A_2, \ldots, A_n \in \mathbb{L} \):

- \( \bigwedge_{i=1}^{n} A_i \in \mathbb{L} \);
- there exist such \( T \)-disjoint \( L \)-fuzzy sets \( B_1, B_2, \ldots, B_k \in \mathbb{L} \) that

\[
\bigvee_{i=1}^{n} A_i = \bigvee_{i=1}^{k} B_i.
\]

- Next we define an \( L \)-fuzzy valued function

\[
m : \varnothing \rightarrow \mathbb{R}_+(L)
\]

by the formula

\[
m(A(M, \alpha)) = \nu(A(M, \alpha)).
\]

Obviously,

- for all sets \( A_i = A(M_i, \alpha_i) \in \mathbb{L} \), \( i = 1, 2 \):

\[
m(A_1) \oplus m(A_2) = m(A_1 \land A_2) \oplus m(A_1 \lor A_2);
\]

- for all \( (M_n)_{n \in \mathbb{N}} \subseteq \Phi \) and \( M = \bigcup_{n \in \mathbb{N}} M_n \) we have

\[
\sup \{m(A(M_n, \alpha))| n \in \mathbb{N}\} = m(A(M, \alpha)).
\]
Now taking into account that consider a sequence \( T \)

\[ m(A, \alpha) = m(A \cup M_n, \alpha). \]

\[ \bigoplus_{n=1}^{\infty} m(A, \alpha) = m(A \bigcup_{n \in \mathbb{N}} M_n, \alpha). \]

• Now we extend \( m \) to the \( L \)-fuzzy valued function \( m^*: L^X \rightarrow \mathbb{R}_+ \) as following:

\[ m^*(E) = \text{Inf} \{ \bigoplus_{n=1}^{\infty} (m(E_n) \cap (E_n)_{n \in \mathbb{N}} \subseteq \varnothing : E \leq \bigcup_{n=1}^{\infty} E_n) \} \]

\( (m^* \) is an \( L \)-fuzzy valued analogue of an outer measure).

Let us note that

(i) for all \( E \in L^X \) there always exists such a sequence \( (E_n)_{n \in \mathbb{N}} \subseteq \varnothing \) that \( E \leq \bigcup_{n=1}^{\infty} E_n \);

(ii) \( m^* \) is bounded from above in the following sense:

\[ m^*(E) \leq z(\nu(X), 1_L) \text{ for all } E \in L^X; \]

(iii) for all \( E \in \varnothing \) we obtain \( m^*(E) = m(E) \);

(iv) for \( L \)-sets \( A, B \in L^X \) we have

\[ m^*(A) \oplus m^*(B) \geq m^*(A \cap B) \oplus m^*(A \cup B). \]

• Finally, we generalize to the fuzzy case the classical concept of \( m^* \)-measurability (in the sense of Caratheodory) and consider \( \Sigma \) - the class of all so called \( m^* \)-measurable \( L \)-sets .

**Definition 3.1.** A set \( E \in L^X \) is called a \( m^* \)-measurable if it satisfies the following conditions for all \( L \)-sets \( B \in L^X \):

\[ m^*(E) \oplus m^*(B) = m^*(E \cap B) \oplus m^*(E \cup B), \]

\[ m^*(E^c) \oplus m^*(B) = m^*(E^c \cap B) \oplus m^*(E^c \cup B). \]

Note that

(i) \( E^c \) is \( m^* \)-measurable for all \( m^* \)-measurable \( L \)-sets \( E; \)

(ii) all \( L \)-sets \( E \in \varnothing \) are \( m^* \)-measurable.

3.2. \( L \)-fuzzy valued measure of measurable \( L \)-sets

We consider \( \mu \) as the restriction of \( m^* \) to \( \Sigma \):

\[ \mu(E) = m^*(E) \text{ for all } E \in \Sigma. \]

**Theorem 3.2.** \( \mu \) is an \( L \)-fuzzy valued \( T_M \)-measure such that \( \mu/\varnothing = m. \)

As it was shown in [5] all \( m^* \)-measurable \( L \)-sets form a \( T_M \)-clan. To obtain that the class \( \Sigma \) is a \( T_M \)-tribe, we consider a sequence \( (E_n)_{n \in \mathbb{N}} \) of \( m^* \)-measurable \( L \)-sets. First we notice that

\[ m^*(\bigcup_{n=1}^{\infty} E_n) = \text{Sup} \{ m^*(\bigcup_{i=1}^{n} E_i) : n \in \mathbb{N} \}. \]

Now taking into account that \( \bigoplus_{i=1}^{n} E_i \) is \( m^* \)-measurable we obtain that for all \( L \)-sets \( B \in L^X \) and for all \( n \in \mathbb{N} \):

\[ m^*(\bigcup_{i=1}^{n} E_i) \oplus m^*(B) = \]

\[ m^*(\bigcup_{i=1}^{n} E_i) \oplus m^*(B) \leq \]

\[ m^*(\bigcup_{i=1}^{\infty} E_i) \oplus m^*(\bigcup_{i=1}^{\infty} E_i) \]

This means that for all \( n \in \mathbb{N} \)

\[ m^*(\bigcup_{i=1}^{n} E_i) \oplus m^*(B) \leq \]

\[ m^*(\bigcup_{i=1}^{\infty} E_i) \oplus m^*(\bigcup_{i=1}^{\infty} E_i) \]

and hence

\[ \text{Sup} \{ m^*(\bigcup_{i=1}^{n} E_i) : m \in \mathbb{N} \} \oplus m^*(B) \]

Finally we obtain

\[ m^*(\bigcup_{i=1}^{\infty} E_i) \oplus m^*(B) = \]

\[ m^*(\bigcup_{i=1}^{\infty} E_i) \oplus m^*(\bigcup_{i=1}^{\infty} E_i) \]

By analogy the result can be proved for \( \bigcap_{n=1}^{\infty} E_n \).

Thus by extension of a crisp measure \( \nu \) we obtain \( L \)-fuzzy valued measure

\[ \mu: \Sigma \rightarrow \mathbb{R}_+(L) \]

such that

(i) \( \mu/\varnothing = m; \)

(ii) \( \mu/\Phi = \nu. \)

The last equality means that for every \( M \in \Phi \) it holds

\[ \mu(A(M, 1_L)) = \nu(M, 1_L). \]

4. \( L \)-fuzzy valued integral

4.1. Definition of \( L \)-fuzzy valued integral

Our aim is to define an \( L \)-fuzzy valued integral

\[ \int_{E} f \, d\mu, \]

where \( E \in \Sigma \) and \( f : X \rightarrow \mathbb{R} \) is a non-negative measurable function with respect to \( \sigma \)-algebra \( \Phi. \)

By analogy with the classical case (see e.g. [13]) we define an \( L \)-fuzzy valued integral stepwise, first considering the case of simple non-negative measurable functions (for short SNMF):

\[ \int_{E} (\sum_{i=1}^{n} c_i \chi_{C_i}) \, d\mu = \sum_{i=1}^{n} (c_i \mu(C_i \cap E)), \]

whenever

- \( c_i \in \mathbb{R}_+, C_i \in \Phi \) for all \( i = 1, \ldots, n, \)
- \( \chi_{C_i} \) is the characteristic function of \( C_i, i = 1, \ldots, n, \)
\begin{itemize}
  \item \(C_1, \ldots, C_n\) are pairwise disjoint sets.
\end{itemize}

Then considering the case of non-negative measurable functions \(f\) (for short NMF):
\[
\int_E f \, d\mu = \text{Sup}\{\int_E g \, d\mu \mid g \leq f \text{ and } g \text{ is SNMF}\}.
\]

For \(\mathbb{I}_f = \int_E f \, d\mu\) due to properties of the supremum of a set of \(L\)-fuzzy numbers, we have
\begin{itemize}
  \item \(\mathbb{I}_f\) is non-increasing,
  \item \(\mathbb{I}_f(t) = \mathbb{I}_f(1)\),
  \item \(\mathbb{I}_f\) is left semi-continuous, i.e.
\end{itemize}

\[
\bigwedge_{t<0} \mathbb{I}_f(t) = \mathbb{I}_f(n).
\]

**Definition 4.1.** We say that a non-negative measurable function \(f\) is \(L\)-fuzzy integrable iff
\[
\bigwedge_{t} \mathbb{I}_f(t) = 0_L.
\]

### 4.2. Properties of \(L\)-fuzzy valued integral

For \(L\)-fuzzy integrable non-negative functions \(f_1, f_2, \ldots, f_n, \ldots\) and measurable \(L\)-sets \(E, E_1, E_2, \ldots, E_n, \ldots \in \Sigma\) the following properties of \(L\)-fuzzy valued integral are true.

1. \(\int_E d\mu = \mu(E)\)
2. \(r \in \mathbb{R}_+ \Rightarrow \int_E rf d\mu = r \int_E f d\mu\)
3. \(f_1 \leq f_2 \Rightarrow \int_E f_1 d\mu \leq \int_E f_2 d\mu\)
4. \(E_1 \leq E_2 \Rightarrow \int_E f d\mu \leq \int_{E_1} f d\mu \leq \int_{E_2} f d\mu\)
5. \((\int_E (f_1 + f_2) \, d\mu) = \int_E f_1 d\mu \oplus \int_E f_2 d\mu\)
6. \((E_1 \cap E_2 = \emptyset \Rightarrow \int_{E_1} f d\mu \oplus \int_{E_2} f d\mu = \int_{E_1 \cup E_2} f d\mu)\)
7. \(E_{n} \cap E_{n+1} \leq E_{n+1} \text{ and } \bigvee_{n \in \mathbb{N}} E_n = E \Rightarrow \int_E f \, d\mu = \text{Sup}\{\int_{E_n} f \, d\mu \mid n \in \mathbb{N}\}\)
8. \(\int_E f d\mu = \text{Sup}\{\int_{f_n} d\mu \mid n \in \mathbb{N}\}\)

### 5. Integration over a measurable fuzzy set

In this section we suggest a method of calculation of the fuzzy valued integral over a measurable fuzzy set \(E\) in the case when \(L = [0,1]\) and \(E\) is NMF (i.e. \(E\) is measurable with respect to \(\sigma\)-algebra \(\Phi\)).

The main idea of the method is based on the following reasoning. The fuzzy set we want to integrate over can be viewed as a non-negative function. Let us assume that this function is measurable with respect to \(\sigma\)-algebra \(\Phi\). It is known that every non-negative measurable function can be presented as a limit of a non-decreasing sequence of SNMF. Obviously, every fuzzy set that is SNMF can be presented as the union of \(T_M\)-disjoint fuzzy sets from the class \(\wp\). And the \(L\)-fuzzy valued integral over an element from the class \(\wp\) can be easily calculated.

This observation gives a reason for the following theorem.

**Theorem 5.1.** If \(E: X \to [0,1]\) is a measurable function with respect to \(\sigma\)-algebra \(\Phi\), then fuzzy set \(E\) is measurable with respect to \(T_M\)-tribe \(\Sigma\).

We describe the method gradually depending on the type of a fuzzy set \(E\): first considering the case when \(E\) is an element of the class \(\wp\), then extend it to the case when \(E\) is SNMF or a finite union of elements from the class \(\wp\) and, finally, the case when \(E\) is NMF.

#### 5.1. Integration over \(A(M, \alpha)\)

To show that for all \(A(M, \alpha) \in \wp\) it holds
\[
\int_{A(M, \alpha)} f d\mu = z\int_M f d\nu, \alpha),
\]
we use some special properties of the addition of fuzzy numbers \(z(\alpha, \beta)\) described in subsection 2.2.

- \(a_1, a_2 \in \mathbb{R}_+ \Rightarrow z(a_1, \alpha) \oplus z(a_2, \alpha) = z(a_1 + a_2, \alpha);\)
- \(c \in \mathbb{R}_+ \Rightarrow cz(a, \alpha) = z(ca, \alpha);\)
- \(a_i \in \mathbb{R}_+, i \in J \Rightarrow \text{Sup}\{z(a_i, \alpha) \mid i \in J\} = z(\text{Sup}\{a_i \mid i \in J\}, \alpha)\).

For \(f = \sum_{i=1}^n c_i \chi_{C_i}\) we get
\[
\int_{A(M, \alpha)} \sum_{i=1}^n c_i \chi_{C_i} \, d\mu = \bigoplus_{i=1}^n (c_i \mu(C_i \cap A(M, \alpha))) =
\]
\[
= \bigoplus_{i=1}^n (c_i z(M \cap C_i), \alpha) =
\]
\[
= z\sum_{i=1}^n c_i z(M \cap C_i), \alpha) = z\left(\int_M f d\nu, \alpha\right).
\]

In the case when \(f\) is NMF we have
\[
\int_{A(M, \alpha)} f d\mu = \text{Sup}\{\int_M g d\nu, \alpha) \mid g \leq f \text{ and } g \text{ is SNMF}\} =
\]
\[
= z(\text{Sup}\{\int_M g d\nu, \alpha) \mid g \leq f \text{ and } g \text{ is SNMF}\}, \alpha) =
\]
\[
= z(\int_M f d\nu, \alpha).
\]

#### 5.2. Integration over SNMF \(E\)

If \(E\) is SNMF then \(E(\mathbb{R}) = \{\alpha_1, \ldots, \alpha_n\}\). We assume that
\[
\alpha_1 > \alpha_2 > \ldots > \alpha_n
\]
and denote
\[
M_i = E^{-1}(\alpha_i), \ i = 1, \ldots, n.
\]
Then

- $i \neq j \Rightarrow M_i \cap M_j = \emptyset$;
- $\bigcup_{i=1}^n M_i = \mathbb{R}$;
- $E = \bigvee A(M_i, \alpha_i)$;
- $E^{\alpha_i} = \bigcup_{j=1}^i M_j$

where $E^{\alpha_i}$ as the $\alpha_i$-cut of fuzzy set $E$.

Taking into account the property of addition of fuzzy numbers:

$\bigoplus_{j=1}^n (a_i, \alpha_i)(t) =$

\[
\begin{cases}
1, t \leq 0, \\
\alpha_i, 0 < t \leq a_i, \\
\alpha_i+1, a_i + \ldots + a_i < t \leq a_i + \ldots + a_i+1, \\
\ldots \ldots, \\
0, t > a_i + \ldots + a_i,
\end{cases}
\]

we obtain

\[
\int_{E} f \, d\mu = \bigoplus_{i=1}^n \left( \int_{E^{\alpha_i}} f \, d\mu \right) =
\]

\[
\begin{cases}
1, t \leq \int_{M_i} f \, dv, \\
\alpha_i, \sum_{j=1}^{i-1} \int_{M_j} f \, dv < t \leq \sum_{j=1}^{i} \int_{M_j} f \, dv, \\
\ldots \ldots, \\
0, t > \sum_{j=1}^{n} \int_{M_j} f \, dv, \\
1, t \leq \int_{E^{\alpha_i}} f \, dv, \\
\alpha_i, \int_{E^{\alpha_i}} f \, dv < t \leq \int_{E^{\alpha_{i+1}}} f \, dv, \\
\ldots \ldots, \\
\alpha_{n}, \int_{E^{\alpha_{n-1}}} f \, dv < t \leq \int_{E^{\alpha_n}} f \, dv, \\
0, \text{otherwise}.
\end{cases}
\]

5.3. Integration over NMF $E$

As was already mentioned every NMF $E$ can be presented as the limit of a non-decreasing sequence of SNMF. To describe this sequence we use the same logic as in the previous subsection. Let us take a sequence $(E_n)_{n \in \mathbb{N}}$ such as:

- for all $n \in \mathbb{N}$: $E_n(\mathbb{R}) = \{ \alpha^n_1, \ldots, \alpha^n_{k_n} \}$;
- for all $n \in \mathbb{N}$: $\alpha^n_k > \alpha^n_{k-1}, i = 1, \ldots, k_n - 1$;
- $M^n_i = \{ x \in E | E(x) = \alpha^n_i \}$, $i = 1, \ldots, k_n$;
- $M^n_i = \{ x \in E | \alpha^n_i \leq E(x) < \alpha^n_{i-1} \}$, $i = 2, \ldots, k_n$;
- $E_n = \bigvee_{i=1}^{k_n} E(\alpha^n_i, M^n_i)$, $n \in \mathbb{N}$;
- $E = \bigvee_{n \in \mathbb{N}} E_n$.

Denoting $I = \int_{E} f \, d\mu$ and $I_n = \int_{E_n} f \, d\mu$ we get

\[
I = \text{Sup} \{ \int_{E_n} f \, d\mu | n \in \mathbb{N} \} = \text{Sup} \{ I_n | n \in \mathbb{N} \}.
\]

From the last equality we can get an approximate value of $I$ by fixing $n$. Obviously, the integral accuracy in this case will be dependent on $n$.

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