Thermodynamic Curvature of the BTZ Black Hole

Rong-Gen Cai[*] and Jin-Ho Cho[†]

Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea

Abstract

Some thermodynamic properties of the Bañados-Teitelboim-Zanelli (BTZ) black hole are studied to get the effective dimension of its corresponding statistical model. For this purpose, we make use of the geometrical approach to the thermodynamics: Considering the black hole as a thermodynamic system with two thermodynamic variables (the mass M and the angular momemtum J), we obtain two-dimensional Riemannian thermodynamic geometry described by positive definite Ruppeiner metric. From the thermodynamic curvature we find that the extremal limit is the critical point. The effective spatial dimension of the statistical system corresponding to the near-extremal BTZ black holes is one. Far from the extremal point, the effective dimension becomes less than one, which leads to one possible speculation on the underlying structure for the corresponding statistical model.

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[*]email: cai@atlantis.snu.ac.kr

[†]email: jhcho@galatica.snu.ac.kr

I. INTRODUCTION

Since the works of Bekenstein [1], Bardeen, Carter and Hawking [2], and Hawking [3], black holes have been known to have the ordinary thermodynamical properties. There are four laws of black hole mechanics analogous to the four laws of ordinary thermodynamics. (The surface gravity on the event horizon can be interpreted as the temperature of the black hole. One quarter of the event horizon area corresponds to the entropy of the black hole.) These correspondences proved not just to be an analogy but to be a realization of thermodynamics involving thermal radiation [3]. Now, it is widely believed that a black hole is a thermodynamic system.

The statistical interpretation of the black hole entropy has been one of the most fascinating subjects to understand the microstates behind its thermodynamics. So far, many progresses have been made for some specific models (extremal and near extremal black holes) in the string theory with the help of supersymmetry [4,5]. Another success was made for the BTZ black hole [6] making use of conformal symmetry on the boundary [7,8]. With the notion that many supergravity solutions of superstring theory can be related to the lower dimensional anti-de Sitter gravity solutions via U-duality [9], one can think these two approches might not be quite remote from each other. In fact recently, duality between string theory on various anti-de Sitter spacetimes and various conformal field theories was conjectured [10–12]. In this sense it becomes very important to find the conformal field theory corresponding to each black hole to account for its statistical origin of the entropy and the BTZ black hole will be the corner stone along this line.

Recently, it was reported for some near extremal cases including BTZ black hole, the relevant degrees of freedom concerned with the entropy live effectively on one spatial dimension [13,14]. This means that their corresponding conformal field theory must be $(1 + 1)$-dimensional. (So far, there have been many controversies on the exact place where the degrees of freedom reside, i.e., whether it is the interior of the black hole or else the horizon. According to the recent results above, it is likely that the exact place is concerned with just

one spatial dimension and that BTZ geometry partially constitutes the internal space for a class of black holes). In this paper, we ellaborate on this aspect for the BTZ black hole. For this purpose, we make use of the geometrical approch to the thermodynamical system, which is usually adopted in the ordinary thermodynamic fluctuation theory [15,16]. In the case at hand, the thermodynamic geometry is described in terms of two dimensional Ruppeiner metric (see [17] for the details on the thermodynamic geometry). We find that the thermodynamic curvature of the BTZ black holes is always positive and diverges strongly as the black hole approaches its extremal limit. This divergence is in agreement with the existence of the critical points, on which many authors have suggested in different contexts [18–23]. We find from the thermodynamic curvature that the effective spatial dimension of the statistical system corresponding to the BTZ black hole is 'one' in the extremal limit. However, this result does not continue to the region far from the limit, for which the effective dimension becomes less than one. This strange result suggests that the corresponding statistical model is composed of some extended objects rather than of point particle gas.

This paper is organized as follows: In section II, we give a brief review on the geometrical method on the thermodynamics. In section III, we apply this approach to the BTZ black hole and get the Ruppeiner metric and observe that the extremal black hole corresponds to the critical point. In section IV, the thermodynamic curvature and the effective spatial dimension of the corresponding statistical system is calculated. Section V concludes the paper with the discussion on the results.

II. GEOMETRICAL METHOD FOR THERMODYNAMICS

It was Weinhold [15] who first introduced the geometrical concept into the thermodynamics. He considered a sort of Riemannian metric, i.e., positive definite metric in the space of the thermodynamic parameters. The Weinhold metric $W_{\mu\nu}$ is defined as the second derivatives of the internal energy U with respect to the entropy and other extensive variables of a thermodynamic system $N^\mu \equiv (S, N^i)$,

$$(dN^\mu, dN^\nu) \equiv W_{\mu\nu} = \frac{\partial^2 U}{\partial N^\mu \partial N^\nu}. \tag{1}$$

In his picture, the second law of thermodynamics can be derived just from the Schwarz inequality [15].

$$(dN^\alpha, dN^\alpha)(dN^\beta, dN^\beta) - (dN^\alpha, dN^\beta)^2 \geq 0. \tag{2}$$

For example, the above inequality for the two vectors, the entropy S and its conjugate, the temperature, T, result in the stability condition $C_p \geq C_v$, where C_p and C_v are heat capacities at constant pressure and constant volume respectively. However, the geometry based on this metric was considered to have no physical meanings in the context of purely equilibrium thermodynamics [17].

In 1979, Ruppeiner [16] introduced a metric, which is defined as the second derivatives of the entropy with respect to the internal energy and other extensive variables of a thermodynamic system, $a^\mu \equiv (U, a^i)$,

$$S_{\mu\nu} = -\frac{\partial^2 S}{\partial a^\mu \partial a^\nu}. \tag{3}$$

Although the Weinhold metric and Ruppeiner metric are conformally related with the temperature T as the conformal factor,

$$W_{\mu\nu} dN^\mu dN^\nu = T S_{\mu\nu} da^\mu da^\nu, \tag{4}$$

the Ruppeiner geometry has its physical meaning in the equilibrium thermodynamic fluctuation theory. The components of the inverse Ruppeiner metric give us second moments of fluctuations. The advantage of this geometrical concept is that the fluctuation theory can be described in the covariant way so that for any set of thermodynamical variables, all the results can be obtained in the same way.

Since the proposal of Ruppeiner, many investigations on the geometric meanings of the Ruppeiner metric have been carried out in various thermodynamic systems, such as the ideal classical gas, ideal paramagnet, multicomponent ideal gas, ideal quantum gases, Takahashi gas, one-dimensional Ising model, van der Waals model and so on. In particular, the

Riemannian scalar curvature of the Ruppeiner metrics has been calculated for many thermodynamic systems. It turns out that the Riemannian scalar curvature $\Re$ of the Ruppeiner geometry is related to the correlation volume as,

$$\Re = \kappa_2 \xi^{\bar{d}}, \tag{5}$$

where κ_2 is a constant with absolute value of order unity, ξ is the correlation length, and $\bar{d}$ is the spatial dimension of the statistical system (not to be confused with the dimensionality of the Riemann geometry, which is the the number of thermodynamic quantities). For example, the Riemannian curvature is zero for single-component ideal gas, which implies that there is no interaction (including the statistical interaction) in the system. For ideal quantum Fermion gas, the thermodynamic curvature is always positive in the sign convention of [17] which we will adopt, while for the ideal quantum Boson gas, it is always negative and diverges strongly as the temperature approaches zero. This phenomenon is related to the Bose-Einstein condensation. Near the critical point, the Riemannian curvature always has an infinite divergence (for a detailed review see [17]) These properties are in accord with those of the correlation length. One important thing to note is that the relation (5) is valid even far from the critical point, although seemingly it looks like some kind of scaling behavior. Another important physical meaning of the scalar curvature is that it provides the lower bound on the volume for which the classical fluctuation theory is valid. If we take some sample volume of smaller size than this bound, we cannot apply the classical fluctuation theory.

III. RUPPEINER METRIC OF BTZ BLACK HOLE

A. BTZ Black Hole

The BTZ black hole is the solution of the (2+1)-dimensional Einstein gravity with a negative cosmological constant. The action describing the black hole is [6]

$$I = \frac{1}{2\pi} \int d^3x \sqrt{-g}(R + 2l^{-2}), \tag{6}$$

where R denotes the scalar curvature (we hope this not to be confused with the thermo-dynamic curvature $\Re$ discussed in the previous section) and l^{-2} represents the negative cosmological constant. The BTZ black hole metric is

$$ds^2 = -N(r)dt^2 + N^{-1}(r)dr^2 + r^2(N^\phi(r)dt + d\phi)^2, \tag{7}$$

where

$$N(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad \text{and} \quad N^\phi(r) = -\frac{J}{2r^2}, \tag{8}$$

M and J are two integration constants, which can be interpreted as the mass and angular momentum of the black holes, respectively. Throughout this paper, the units $8G = c = \hbar = k_B = 1$ will be used.

BTZ black hole (7) can be constructed by orbifolding the (2+1)-dimensional anti-de Sitter space with the spacelike Killing vector fields. Therefore the BTZ black hole is locally a constant curvature spacetime. Through the Einstein field equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = l^{-2}g_{\mu\nu}$, the scalar curvature of the BTZ black hole spacetime is

$$R = -6l^{-2}. \tag{9}$$

The BTZ black hole has two horizons

$$r_\pm^2 = \frac{1}{2}Ml^2(1 \pm \triangle), \quad \triangle = [1 - (J/Ml)^2]^{1/2}. \tag{10}$$

It is obvious that $J \leq Ml$ and $M \geq 0$ should be satisfied in order that the metric (7) to have the black hole structure. The Hawking temperature T of the hole is readily obtained

$$T = \frac{r_+^2 - r_-^2}{2\pi r_+ l^2} = \frac{M\triangle}{2\pi r_+}. \tag{11}$$

The Bekenstein-Hawking entropy is

$$S = 4\pi r_+ = 4\pi \left[\frac{M}{2}l^2(1 + \triangle)\right]^{1/2}. \tag{12}$$

These thermodynamic quantities obey the first law of thermodynamics

$$dM = TdS + \Omega_H dJ, \tag{13}$$

where $\Omega_H = J/2r_+^2$ is the angular velocity of the hole. Another feature of BTZ black holes is that its heat capacity is always positive, which is in contrast with the Schwarzschild case. According to the formula $C_J = (\partial M/\partial T)_J$, we have [23]

$$C_J = \frac{4\pi r_+ \triangle}{2 - \triangle}. \tag{14}$$

Because of $0 \leq \triangle \leq 1$, $C_J \geq 0$ always holds, which means the temperature to increase with the mass. Therefore, the BTZ black hole can be in stably thermal equilibrium with an arbitrary volume heat bath. When $\triangle = 1$, i.e. $J = 0$, we have $C_J = 4\pi l\sqrt{M}$. And when $\triangle = 0$, i.e. $J = Ml$, we have $C_J = 0$, corresponding to the extremal BTZ black holes. In that case, the two horizons of the hole coincide, and the Hawking temperature becomes zero.

Since the discovery of the BTZ black holes, many works have been done on the classical and quantum properties of BTZ black holes. The statistical explanation of entropy was first provided by Carlip [7] and the basic points are spelled out in the recent paper [8]. A nice review about the physics of BTZ black holes can be found in [24].

B. Thermodynamic Metric

Now we construct the thermodynamic geometry of BTZ black hole based on its thermodynamics. According to the definition (1), the Weinhold metric of BTZ black holes can be written down as

$$\begin{aligned} ds_{\rm W}^2 &\equiv \left(\frac{\partial^2 M}{\partial S^2}\right)_J dS^2 + \left(\frac{\partial^2 M}{\partial J^2}\right)_S dJ^2 \\ &= \frac{T}{C_J} dS^2 + I_S dJ^2, \end{aligned} \tag{15}$$

where mass M corresponds to the internal energy U while entropy S and angular momentum J are taken as the extensive variables N^μ. C_J is given by Eq. (14) and I_S is defined as

$$I_S \equiv \frac{1}{\Omega_H} \left(\frac{\partial \Omega_H}{\partial M} \right)_S = \frac{1}{M} + \frac{Ml^2 \triangle (1 + \triangle)}{J^2}. \tag{16}$$

It is easy to see that the Weinhold metric is regular even for the extremal BTZ black holes. Our interest is the Ruppeiner metric. Rewriting Eq. (13), we have

$$dS = \beta dM - \mu dJ, \tag{17}$$

where $\beta = 1/T$ and $\mu = \beta \Omega_H$. According to the definition (3), the Ruppeiner metric of the BTZ black holes in the $a = (M, J)$ coordinates is

$$ds_{\mathrm{R}}^2 \equiv - \left(\frac{\partial^2 S}{\partial M^2} \right)_J dM^2 - \left(\frac{\partial^2 S}{\partial J^2} \right)_M dJ^2$$
$$= \frac{1}{T^2 C_J} dM^2 + \frac{1}{T I_M} dJ^2, \tag{18}$$

where I_M is defined as

$$I_M \equiv \beta \left(\frac{\partial J}{\partial \mu} \right)_M = \left[\frac{1}{2r_+^2} + \frac{J^2}{8Mr_+^4 \triangle} + \frac{J^2}{2M^2 l^2 r_+^2 \triangle^2} \right]^{-1}. \tag{19}$$

Through a straightforward calculation, we can easily verify

$$ds_{\mathrm{R}}^2 = \beta ds_{\mathrm{W}}^2, \tag{20}$$

the conformal relation between the Ruppeiner and Weinhold metrics [25]. Because C_J and I_M are always positive, the line element ds_{R}^2 is positive definite, so does ds_{W}^2. This is quite different from the case of the dilaton black holes in N=2 supergravity theory [25] (in the paper some speculations on the relation between the Weinhold geometry and the moduli space geometry were given).

C. Extremal Black Hole as the Critical Point

In this section we see that the extremal black hole corresponds to the critical point. One characteristic feature of the critical point is the divergence of the correlation length. As is said in the previous section, the thermodynamic curvature is proportional to the correlation

volume. Therefore the divergence of the thermodynamic curvature at some point suggests that it is the critical point. Actually in the next section, we will see for our case this divergence occurs at the extreme black hole. However, in order to confirm exactly that the extreme black hole is the critical point, we have to show that at least one of the second derivatives of the thermodynamic potential (appropriately chosen for the system at hand) diverges in the extreme limit, i.e., as $\triangle \to 0$. In fact, the critical point can be defined in several ways, here we follows the definition in [26]: the point where some of the second derivatives of the thermodynamic potential diverge.

As for the thermodynamic potential, we choose the Helmoltz free energy $f = M - TS$. The Ruppeiner metric can be rewritten in the new coordinates (T, J).

$$
\begin{aligned}
ds_{\mathrm{R}}^2 &= \frac{1}{T}\left(-\frac{\partial^2 f}{\partial T^2} + \frac{\partial^2 f}{\partial J^2}\right) \\
&= \frac{C_J}{T^2} dT^2 + \frac{1}{TI_T} dJ^2,
\end{aligned}
\tag{21}
$$

where C_J is given in (14) and I_T is

$$
\begin{aligned}
I_T &\equiv \left(\frac{\partial J}{\partial \Omega_H}\right)_T \\
&= \left[\frac{1}{2r_+^2} + \frac{J^2}{4Mr_+^4 \triangle} + \left(\frac{Jl^2(1+\triangle)}{4r_+^4} + \frac{J^3}{4M^2 r_+^4 \triangle}\right)\mathcal{C}\right]^{-1},
\end{aligned}
\tag{22}
$$

with

$$
\mathcal{C} = \left[\frac{J}{4r_+^2} - \frac{J}{Ml^2 \triangle}\right]\left[\triangle + \frac{J^2}{M^2 l^2 \triangle} - \frac{Ml^2 \triangle(1+\triangle)}{4r_+^2} - \frac{J^2}{4Mr_+^2}\right]^{-1}.
$$

We can easily see that I_T vanishes in the extreme limit, in other words, the second derivative of the Helmoltz free energy with respect to the angular momentum J vanishes. This tells us that the extremal black hole is the critical point and its Hawking temperature $T_H = 0$ is the corresponding critical temperature. This zero temperature makes it hard to define the critical point in the usual way as the point where some of the fluctuations diverge. This is why we adopt the defintion given in [26]. In fact, as is mentioned before, the components of the Ruppeiner metric have the meaning of the second moments for the fluctuation. (This interpretation is not valid at the critical point unless we take the thermodynamic limit). The

metric (21) tells us in the thermodynamic limit that the following second moments vanish at the critical point.

$$\langle \delta T \delta T \rangle_J = T^2/C_J, \quad \langle \delta J \delta J \rangle_T = T I_T, \tag{23}$$

where the subscripts denote the variables which are kept fixed. These are quite different results from the ordinary statistical system with non-vanishing critical temperature. There, the fulctuaation diverges for the extensive quantities while it vanishes for the intensive quantities.

IV. THERMODYNAMIC CURVATURE

In the different coordinates, however, the scalar curvature invariants should keep unchanged for the same geometry. For a two-dimensional Riemannian geometry, the most important curvature invariant is the scalar curvature $\Re$. For this purpose, it is convenient to calculate it in the coordinates (M, J). In the following conventions,

$$\Re^\lambda_{\mu\nu\sigma} = \Gamma^\lambda_{\mu\nu,\sigma} - \Gamma^\lambda_{\mu\sigma,\nu} + \Gamma^\lambda_{\sigma\eta}\Gamma^\eta_{\mu\nu} - \Gamma^\lambda_{\nu\eta}\Gamma^\eta_{\mu\sigma}, \tag{24}$$

and

$$\Re_{\mu\nu} = \Re^\lambda_{\mu\lambda\nu}, \quad \Re = g^{\mu\nu}\Re_{\mu\nu}, \tag{25}$$

the Riemannian scalar curvature of the Ruppeiner metric (18) is

$$\Re = \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial M} \left(\frac{1}{\sqrt{g}} \frac{\partial g_{22}}{\partial M} \right) + \frac{\partial}{\partial J} \left(\frac{1}{\sqrt{g}} \frac{\partial g_{11}}{\partial J} \right) \right], \tag{26}$$

where

$$g_{11} = 1/(T^2 C_J), \quad g_{22} = 1/(T I_M). \tag{27}$$

Further expansion of expression for the thermodynamic curvature (26) is somewhat complicated. Here we do not present it explicitly. Instead we demonstrate the behavior of the scalar curvature numerically.

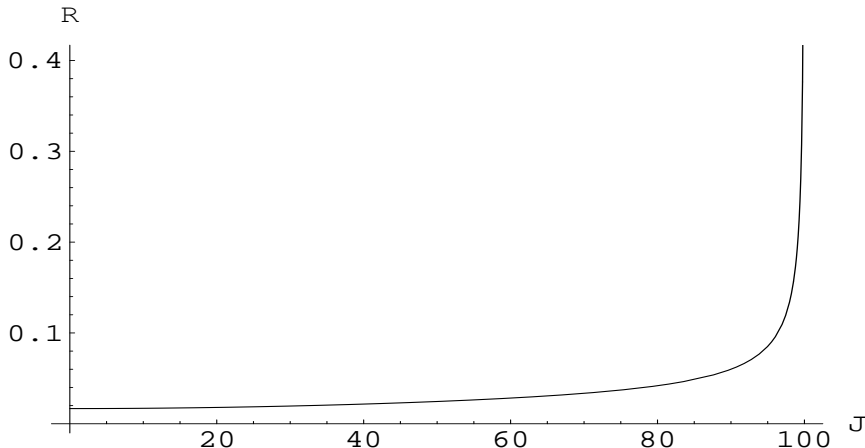FIG. 1. Thermodynamic curvature $\Re$ of the BTZ black hole thermodynamics versus the angular momentum J with $M = 1$ and $l = 100$

Fig. 1 shows the behaviour of the scalar curvature with respect to the angular momentum with the mass and the cosmological constant set to be $M = 1$ and $l = 100$. Since the radius l of the anti-de Stter space is concerned with the cosmological constant, it is reasonable to set it lager than any other scale that appear in the theory. The thermodynamic curvature becomes small as $J \to 0$ but not vanishes. It means that even for the spinless black hole, the statistical interaction is nontrivial in the corresponding statistical model. It should be noted that the situation becomes different if we initially work with the spinless black hole (setting $J = 0$). In that case, its thermodynamic geometry will be one dimensional and the scalar curvature will vanish. Therefore the above nonvanishing scalar curvature is a result coming from another dimension specified by J. When the extremal limit is approached, however, the curvature diverges strongly. More specifically, the inspection on the Eq. (26) reveals that the curvature diverges near the extremal limit like

$$\Re \sim \triangle^{-1}. \tag{28}$$

With the notion of the relation (5), we expect $\triangle$ to be related to the correlation length ξ. Lacking of a satisfactory quantum theory of gravity, unfortunately, we do not have knowledge about the exact correlation function for the gravitational interaction. However,

in Refs. [20,22,23] it was already argued that the inverse surface gravity $\kappa = 2\pi T$ of black holes may play the role of the correlation length

$$\xi = \frac{1}{2\kappa}. \tag{29}$$

Since $\kappa \sim \triangle$ near the extremal point, combining (28) and (5), we obtain the effective spatial dimension of the near-extremal BTZ black holes;

$$\bar{d} = 1. \tag{30}$$

In fact, [23] argued from the scaling laws, which hold near the critical point, that the effective spatial dimension of BTZ black holes is one. Here we have further verified this result via the Riemannian curvature of the BTZ black hole thermodynamics.

The divergence of the thermodynamic curvature, i.e., the correlation volume suggests that the extremal BTZ black hole is the critical point. This can be expected from several other aspects. For example, the extremal black hole corresponds to the BPS state in the supersymmetric extension. This means it preserves some of the full supersymmetries of the supergravity, which is not the case for the non-extremal black hole. Another fact is that the Hawking temperature vanishes for the extremal black hole, therfore no Hawking radiation happens. Moreover, beyond the extremality, the black hole becomes nakid. These discerning properties of extremal black hole support the view point that the extremal black hole is the critical point.

Recalling the behavior of the thermodynamic curvature of ideal Boson gas [17], we find that the behavior of the thermodynamic curvature for the BTZ black holes is similar to that of the Boson gas, but with an opposite sign. This signature difference looks strange, on which we will dicuss in the later section. Here it should also be mentioned that more recently Ghosh [14] has constructed a one-dimensional Boson gas model to provide the correct expression for entropies for extremal and near-extremal BTZ black holes. Therefore our result is consistent with the one-dimensional gas model of the near extremal BTZ black hole.

It is interesting to see what would be the effective dimension of the statistical system corresponding to the non-extremal BTZ black hole. One can extend the above procedure to get the effective dimension for arbitrary value of J, even far from the extremal point. The basic tool for this is the relation (5), which is valid even far from the critical point. That is to say, from

$$\Re = c \left(\frac{l\sqrt{1+\triangle}}{2\sqrt{2M\triangle}} \right)^{\bar{d}}, \tag{31}$$

we get

$$\bar{d} = \ln \frac{\Re}{c} \Big/ \ln \frac{l\sqrt{1+\triangle}}{2\sqrt{2M\triangle}}, \tag{32}$$

the coefficient c is determined by the condition that $\bar{d} = 1$ at the critical point, which is clear from the Eq. (28).
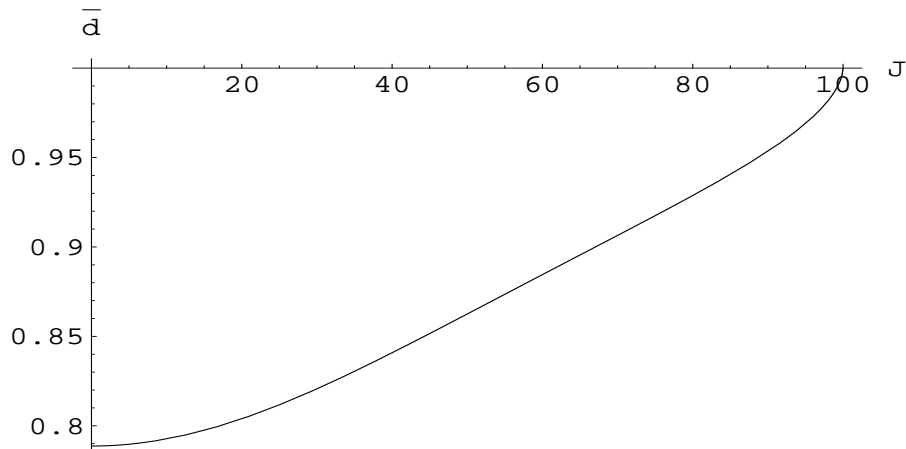

FIG. 2. The effective dimension $\bar{d}$ versus the angular momentum J with $M = 1$ and $l = 100$. The coefficient c is determined to be $3/(1250\pi)$.

Fig. 2 shows the result, where we set $l = 100$ and $M = 1$ (c is determined to be $3/(1250\pi)$). Remarkable thing is that the effective dimension is not constant and becomes less than one as J leaves the extremal point. This variation of the effective dimension means that for each value of the angular momentum J, the corresponding statistical model should be constructed differently. This is in accord with the recent viewpoints on the black hole. In fact, for the different types of black holes, their stringy counterparts are differently suggested

[27]. For the neutral black hole, the entropy can be matched with that of a single long string when its size becomes the string scale. As for the Ramond-Ramond charged black hole, its stringy counterpart is the open string gas living on the D-brane.

The fractal effective dimension looks absurd. The result becomes even worse when we choose the radius l of the anti-de Sitter space as the order of the black hole mass (the variation of thermodynamic curvature with respect to the angular momentum is insensitive in shape to the value of l as we see in Fig. 3). In this case the effective dimension sharply decreases to the negative value near the spinless black hole; see Fig. 4.
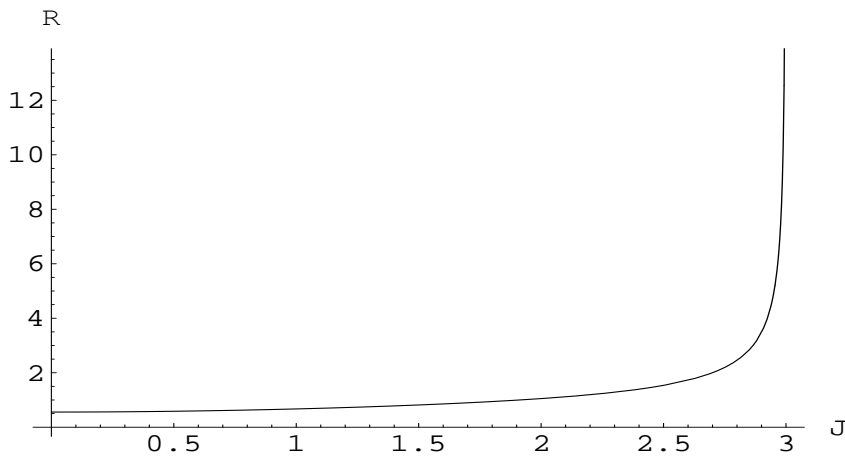

FIG. 3. Thermodynamic curvature $\Re$ versus the angular momentum J with $M = 1$ and $l = 3$
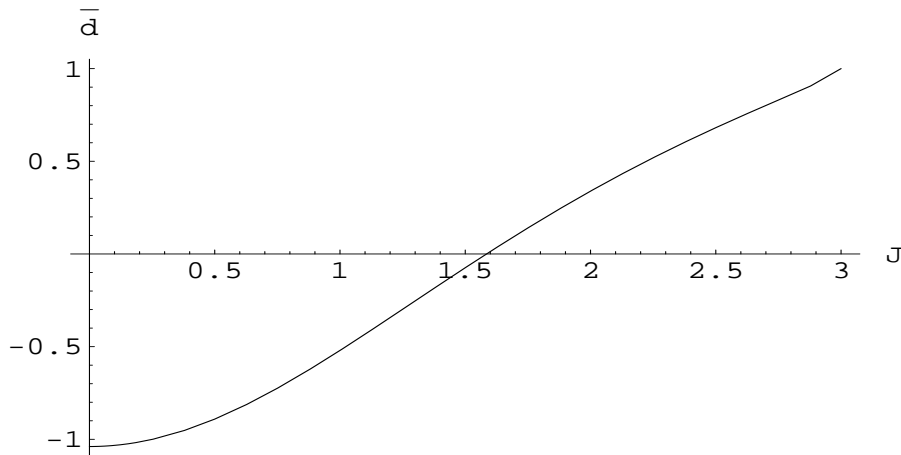

FIG. 4. The effective dimension $\bar{d}$ versus the angular momentum J with $M = 1$ and $l = 3$. c is determined to be $8/(3\pi)$.

Although it is very hard to find such an example in the ordinary thermodynamic system,

we can find one in the string gas (bootstrap) model. This model has been dsicussed exten-sively concerning about the hadron behaviour at high density and high temperature. For example, the mean energy of some string gas model behaves like $\langle E \rangle \sim (T_H - T)^{-1/2}$ near the Hagedorn temperature T_H [28]. If one try to interpret this result in terms of ordinary particle gas, which behaves like $\langle E \rangle \sim T^{\bar{d}+1}$ in $\bar{d}$ spatial dimension, he will get the negative effective dimension $\bar{d} = -3/2$. Of course, the string gas does not live in the negative spa-tial dimension. In this case, the negative effective dimension comes from the extrapolation of the interpretation of the particle system to the string gas. Therefore we have at least one possibity: The corresponding statistical model is not an ordinary sytem composed of particle-like gas. We think that some extended objects constitutes the system.

V. DISCUSSIONS

In the thermodynamics of BTZ black holes we have introduced the two-dimensional Ruppeiner metric in various coordinates. This metric is always positive definite. For general black holes, due to the fact that the heat capacity of some black holes may be negative, the Ruppeiner metric of black holes is not always positive definite. For example, for the Kerr black hole, the counterpart of the BTZ black hole in four dimensions, its heat capacity is positive for large angular momentum, and negative for small angular momentum, that is, the heat capacity changes its sign at some points in the parameter space. Therefore, the Ruppeiner metric for the Kerr black holes is not always positive definite. But near the critical points, the positive definiteness of the Ruppeiner metric is guaranteed.

Using the Ruppeiner metric we investigated the critical point for the BTZ black holes in the different thermodynamic coordinates. The divergence of the Riemannian curvature of the Ruppeiner metric suggests the existence of a critical point at the extremal limit of BTZ black holes. This is verified via the second derivative of the Helmoltz free energy with respect to the angular momentum.

Now we come back to the subtle point we mentioned above: the signature flip of the

curvature. As is said before, Ghosh recently found [14] the relation between the near extremal BTZ black hole and (1+1)-dimensional conformal field theory via the ideal bosonic gas model in one dimension. This strongly imples that the system is composed of bosons. However, the ideal quantum gas gives negative curvature for boson. This point is quite unclear to us but one definite thing is that our reslut reveals sharp divergence near the critical point, which is one characteristic of boson showing the condensation.

One way out is that the criterior based on the signature of the scalar curvature might not be valid for the sytem of the extended object. In fact, we can find one example in the Takahashi gas. This is the one dimensional system composed of rigid rods. In this model, the scalar curvature shows sharp peak at a point considered as the pseudo-phase transition point in the sense that it is still finite (See fig. 11 of the [17], with the notion that there is drawn the Gaussian curvature which has the opposite signature to that of Riemannian curvature). One can easily see that it shows negative Gaussian curvature, i.e., positive scalar curvature in the region of liquid phase (larger density region).

Another interesting thing is that the curve in this liquid phase region shows very similar shape to the our result. We think this is not a coincidence. If we grow the size of the rigid rod, the density becomes large and system will be in the liquid phase from some point. Further increase of the rod length terminates with zero scalar curvature at the point where the whole system is composed of one long rigid rod. Of course, this cannot be the exact statistical model corresponding to the BTZ black hole because the rigidity is too strong to give a nonvanishing curvature at the spinless black hole limit. Nevertherless the qualitative feature can be the same and one can view our result in the same way; Near the extremal point, the system is composed of very very short string gas. With the decrease of the angular momentum J, the size of the string gas becomes longer. Near the spinless black hole point, it becomes highly oscillatory single long string. In fact, this viewpoint has been discussed in many papers in different contexts but our results provide a strong support for the viewpoint.

ACKNOWLEDGEMENTS

REFERENCES

[1] J.D. Bekenstein, Phys. Rev. D 7 (1973) 949; 2333

[2] J.M. Bardeen, B. Carter and S.W. Hawking, Commun. Math. Phys. 31 (1973) 161

[3] S.W. Hawking, Nature 248 (1974) 30; Commun. Math. Phys. 43 (1975) 199

[4] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99

[5] G.T. Horowitz and A. Strominger, Phys. Rev. Lett. 77 2368 (1996)

[6] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849
    M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48 (1993) 1506

[7] S. Carlip, Phys. Rev. D 51 (1995) 632.

[8] A. Strominger, Black Hole Entropy from Near-Horizon Microstates, hep-th/9712251 (1997).

[9] S. Hyun, U-duality between Three and Higher Dimensional Black Hole, hep-th/9704005 (1997).

[10] J.M. Maldacena, The Large N Limit of Superconformal field theories and Supergravity, hep-th/9711200 (1997).

[11] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Gauge Theory Correlators from Non-Critical String Theory, hep-th/9802109 (1998).

[12] E. Witten, Anti De Sitter Space and Holography, hep-th/9802150 (1998).

[13] G.T. Horowitz, D.A. Lowe and J.M. Maldacena, Phys. Rev. Lett. 77 (1996) 430.

[14] A. Ghosh, Effectiveness of One-Dimensional Gas Models for Black Holes, hep-th/9801064 (1998).

[15] F. Weinhold, J. Chem. Phys. 63 (1975) 2479.

[16] G. Ruppeiner, Phys. Rev. A 20 (1979) 1608.

[17] G. Ruppeiner, Rev. Mod. Phys. 67 (1995) 605; 68 (1996) 313(E).

[18] A. Curir, Gen. Rel. Grav. 13 (1981) 417.

[19] D. Pavón and J.M. Rubí Phys. Rev. D 37 (1988) 2052;
D. Pavón, Phys. Rev. D 43 (1991) 2495.

[20] J. Traschen, Phys. Rev. D 50 (1994) 7144.

[21] O. Karabuki, Gen. Rel. Grav. 28 (1996) 843; Phys. Lett. A 217 (1996) 315.

[22] R.G. Cai, Phys. Rev. Lett. 78 (1997) 2531;
R.G. Cai and Y.S. Myung, Nucl. Phys. B495 (1997) 339.

[23] R.G. Cai, Z.J. Lu and Y.Z. Zhang, Phys. Rev. D 55 (1997) 853.

[24] S. Carlip, Class. Quantum Grav. 12 (1995) 2853.

[25] S. Ferrara, G.W. Gibbons and R. Kallosh, Nucl. Phys. B500 (1997) 75.

[26] J.M. Yeomans, Statistical Mechanics of Phase Transitions (Oxford University Press, Oxford, 1992).

[27] G.T. Horowitz and J. Polchinski, Phys. Rev. D 55 (1997) 2557.

[28] R.D. Carlitz, Phys. Rev. D 5 (1972) 3231;
S. Frautschi, Phys. Rev. D 3 (1971) 2821.