DAMPED VIBRATIONS PROBLEM OF BEAMS
FIXED ON THE ROTATIONAL DISK

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The problem considered in this paper is one of damped vibrations of the beam in transportation. The dynamical analysis of systems in motion is a very well-known issue, but many detailed cases have not been published yet. The considered case is applied with the double-sided fixed beams placed on a rotational disk. The disk treated as a rigid one is rotated with a constant angular velocity in the stationary reference frame. An application of vibrations of the beam is transformed from a local to global reference frame, damping forces and forces arising from the rotational motion of a local reference center are also taken into consideration. The beam is assumed as a homogenous one with symmetric cross-sections. This work is the dynamical analysis of such a type of vibrations, which is a mathematical one based on considering transient response and dynamical flexibility of this type of systems.

Keywords: Rotating beam; vibrations; dynamical flexibility; equations of motion.

1. Introduction

The beam is a fundamental construction element of numerous system types, machines and mechanisms. Beams are very often treated as the stationary vibrating systems without coupling between their local vibrations and the main motion. Such system types are considered in different aspects, for instance, in the kinematical and dynamical ones, in connection with the problem of controlling mechanical systems, etc. So far some solutions have been found by considering working motion (in the presented cases working motion means transportation) and local vibrations separately [Genta, 2005]. That assumption makes sense because the vibrations from elasticity of elements of the mechanical composition are much smaller than the main displacement and dislocation of this composition [Szefer, 2000, 2001]. Nowadays, there is an increased range of velocities and accelerations, achieved by using more efficient drives and in order to keep to a maximum power output of drives for the motion of mechanical systems, materials with lower mass density are used [Jamroziak & Bocian, 2008]. All those issues are brought about by creating new models of designing beam-like systems considered in this work.

The method used for the analysis of beam models in transportation is the dynamical flexibility method that is one of the very popular ways of analyzing the dynamics of systems. This method was used for the analysis of dynamical states of beam systems. The dynamical flexibility method gives an opportunity to specify the stability zones. These stability zones especially determine zones of minimal amplitude of vibrations, modes of vibrations and zeros of dynamical characteristics. A lot of publications focus on the subject area of vibrating
systems in transportation as distinguished from the ones concerning stationary systems. These aspects are the reasons for the widening of dynamical analysis.

The solution of the analyzed system was provided by an approximate method, the Galerkin’s method. Dynamical flexibilities of analyzed systems derived using the Galerkin’s method were compared with a dynamical flexibility of stationary systems derived from the exact method. The results confirm the high effectiveness of the Galerkin’s method. The characteristics obtained by this method overlap with the ones derived from the exact method both in the case of rotational systems and in the case of systems with zero-value angular velocity. Based on these results, the Galerkin’s method could be accepted as a sufficient method for the analysis of systems in motion.

The dynamic flexibility, as it is understood in this paper, is the amplitude of generalized displacement with a mathematical symbol in a direction of “i” generalized coordinate changed by a generalized harmonic force with an amplitude equal to one in the direction of “j” generalized coordinate. It can be expressed as follow:

\[ s_i = Y_{ij} s_j, \]  

(1)

Derivations of suitable dynamical flexibilities are applied to systems vibrating longitudinally and vibrating transversally in transportation. The solutions on the characteristics of amplitude in the function of frequency can be presented. There are many technical applications where the double-sided clamped beams in motion are implemented. Such system types can be put into use, for example, in turbines, pumps and rotors, etc. In many mechanisms and machines there are attached, mounted, elongated or fixed elements which can be treated as beam-like systems. An axis of such a type lies in radial direction. The paper concerns the known problem e.g. [Buchacz, 2003; Buchacz & Żółkiewski, 2004a, 2004b, 2005; Szefer, 2000, 2001; Żółkiewski, 2008, 2009] of vibrations of systems in transportation. Literature positions are applied in the analysis of longitudinal vibrations of rods [Buchacz, 2003; Buchacz & Żółkiewski, 2004a, 2004b] and transversal vibrations of beams [Buchacz & Żółkiewski, 2004a, 2004b, 2005; Żółkiewski, 2008]. As distinct from Jamrozik and Bocian’s paper [2008] that uses the so-called degenerated models allowing to describe phenomena in a more detailed way, this paper concerns interactions between the main motion and the local motion and in future will assess this level of interaction by estimating amplitudes of displacements by taking into consideration the interdependence of the main and local motion. There are many numerical applications dedicated to the dynamical analysis of rotational systems, for example, DynRot described by Genta [2005] or the Modyfit according to Żółkiewski [2008]. These applications provide the possibility of analyzing running systems or designing such types of systems, creating the modula-
tions of dynamical characteristics by changing working parameters or changing geometrical or material parameters. We can also estimate the critical speeds when the beam loses stability and prevent these disad
vantaged cases even at the level of designing or running technical systems.

2. Models of the Analyzed Beams

In this section, two different models of analyzed beams are described. Both models are homogeneous beams, the first double-sided clamped on the rotational disk and the second double-sided fixed on the rotational disk as well. The disk in both cases is assumed as a rigid one. The disk is rotated with an angular constant velocity \( \omega \) and the system is described in two reference frames. The local motion is transferred to the global reference frame. The beams have well-defined geometric parameters: symmetric cross-section, external dimensions

![Fig. 1. The model of analyzed system — clamped-clamped beam. The beam parameters used here are: \( \rho \) — mass-density, \( A \) — cross-section, \( l \) — length of beam, \( z \) — location of analyzed cross-section, \( M \) — mass of beam, \( \omega \) — angular velocity, \( Q \) — frequency, \( \mathbf{Q} \) — rotation matrix, \( \mathbf{s} \) — position vector, \( \mathbf{F} \) — harmonic force, \( E \) — Young modulus, \( I_z \) — geometric moment of inertia, \( \mathbf{w} \) — vector of displacement.](image-url)
Damped Vibrations of Rotated Beams

2.1. Equations of motion

The equations of motion can be derived using the d’Alembert’s principle and some examples of equations of motion are presented in [Buchacz, 2003] or in [Buchacz & Zolkiewski, 2004a, 2004b, 2005]. The principle is defined as:

$$\sum (F_i - M_i \cdot \ddot{r}_i) \delta r_i = 0,$$

where vector ‘r’ is a position vector of the analyzed section.

Generalized coordinates assumed as orthogonal projection on the coordinate axes of the global reference system can be written as:

$$q_1 = r_X,$$
$$q_2 = r_Y.$$  

after differentiating Eqs. (3) and (4) we can obtain the generalized velocities as follows:

$$\dot{q}_1 = \frac{dq_1}{dt} = \dot{r}_X = v_X,$$
$$\dot{q}_2 = \frac{dq_2}{dt} = \dot{r}_Y = v_Y.$$  

The transportation velocity treated as the angular velocity of a rotational disk is defined as follows:

$$\omega = [0 \ 0 \ \omega]^T,$$  

which provides the plane motion of the overall system.

A position vector assigned to the analyzed cross-section is described as:

$$\mathbf{s} = [s \ 0 \ 0]^T.$$  

2.2. The rotation matrix

The rotation matrix is used for a rigid rotation of the analyzed bar and is defined as follow:

$$Q = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

This classical method presents the orientation description of the system. The rotation angle is assumed as an angle between the x axis of the local reference frame and the X axis of the global reference frame.

2.3. Boundary problem

It is assumed that both ends of the beam are supported (Fig. 1). The beam is fixed on the rotational rigid disk. An analyzed section of the beam is loaded by a harmonic force with unitary amplitude on direction perpendicular to the center line of the beam. Forces on tips of the beam are assumed to equal zero and also displacements to equal zero, because the support of the system is double-sided. As a result, the boundary condition for the first beam (Fig. 1) should be written as follows:

$$w(0, t) = 0,$$
$$EI_z \frac{\partial^2 w(0, t)}{\partial x^2} = 0,$$
$$w(l, t) = 0,$$
$$EI_z \frac{\partial^2 w(l, t)}{\partial x^2} = 0,$$
$$\frac{\partial}{\partial x} \left[ EI_z \frac{\partial^2 w(s, t)}{\partial x^2} \right] = -2 \int_0^l F_0 \delta(x - l)e^{\Omega t}dx = -1e^{\Omega t}$$

in every time moment $t \geq 0$.
where

\[ k = \frac{2n + 1}{2} \pi, \]  

\[ n \] is a mode of vibrations.

Both ends of the second beam are clamped (Fig. 2). The beam is fixed on the rotational rigid disk. An analyzed section of the beam is loaded by a harmonic force with unitary amplitude on a direction perpendicular to the center line of the beam. The forces on tips of the beam are assumed to equal zero, because the system is double-sided clamped. Due to the above, boundary conditions for the second beam (Fig. 2) should be written as follows:

\[
\begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
\frac{\partial^2 w}{\partial \varphi^2} \\
\frac{\partial w}{\partial \varphi}
\end{bmatrix}
- \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega^2 \cdot s \\
\omega^2 \cdot \varphi \\
0
\end{bmatrix}
= 0
\]

\[
+ \frac{b}{M} \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega^2 \cdot s \\
\omega^2 \cdot \varphi \\
0
\end{bmatrix}
+ \frac{b}{M} \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
\frac{\partial w}{\partial \varphi} \\
\frac{\partial w}{\partial \varphi}
\end{bmatrix}
= 0
\]

\[
- 2 \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega^2 \cdot s \\
\omega^2 \cdot \varphi \\
0
\end{bmatrix}
+ \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega^2 \cdot s \\
\omega^2 \cdot \varphi \\
0
\end{bmatrix}
= 0
\]

\[
= -\frac{E \cdot I_y}{\rho \cdot A} \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
\frac{\partial^2 w}{\partial x^2} \\
\frac{\partial w}{\partial x}
\end{bmatrix}
\]

After solving the boundary problem, we can derive the eigenfunction of displacement of the following form:

\[ X(x) = \sin kx, \]  

where

\[ k = \frac{n}{\pi}. \]  

2.4. Mathematical model

After some calculations, the equations of motion in matrix form can be derived and can be written as [Zolkiewski, 2009]:

\[ \frac{\partial w(0, t)}{\partial x} = 0 \]

\[ w(l, t) = 0 \]

\[ \frac{\partial w}{\partial x} = 0 \]

\[ \frac{\partial}{\partial x} \left[ E I_{\rho} \frac{\partial w}{\partial x} \right] = -2 \int_0^l F_0 \delta(x - l) e^{i \Omega t} dx = -1 e^{i \Omega t} \]
for each point of range $D = \{(x, t), x \in (0, l), t \geq 0\}$.

Eq. (12) coincides with boundary conditions and initial conditions.

After assuming a constant angular velocity, an angular acceleration equals zero, then projecting the equations of motion Eq. (11) onto axes of the global reference frame, we get:

$$\frac{\partial^2 w_X}{\partial t^2} + \frac{E - I_z}{\rho \cdot A} \frac{\partial^3 w_X}{\partial x^3} + \omega^2 \cdot \left(\frac{\partial w_Y}{\partial t} - \omega \cdot (s \cdot \sin \varphi - w_X)\right) = 2 \cdot \omega \frac{\partial w_Y}{\partial t} - \omega^2 \cdot (s \cdot \sin \varphi - w_X)$$

which becomes:

$$\frac{\partial^2 w_Y}{\partial t^2} + \frac{E - I_z}{\rho \cdot A} \frac{\partial^3 w_Y}{\partial x^3} + \omega^2 \cdot \left(\frac{\partial w_X}{\partial t} - \omega \cdot (s \cdot \cos \varphi - w_Y)\right) = -\omega^2 \cdot (s \cdot \cos \varphi - w_Y) - 2 \cdot \omega \frac{\partial w_X}{\partial t},$$

where $b$ is a damping factor.

3. Dynamical Flexibility of the Beam

After orthogonalization of equations of motion Eqs. (12) and (14) by multiplication using the eigenfunction of displacement and after computing integrals from these equations in limits of integration from zero to the length of the beam, we can then obtain:

$$\int_0^l \frac{\partial^2 w_X}{\partial t^2} \cdot X(x) dx + \frac{E - I_z}{\rho \cdot A} \int_0^l \frac{\partial^3 w_X}{\partial x^3} \cdot X(x) dx$$

$$= -\omega^2 \int_0^l (s \cdot \cos \varphi - w_X) \cdot X(x) dx$$

$$+ 2 \cdot \omega \int_0^l \frac{\partial w_Y}{\partial t} \cdot X(x) dx$$

$$= -\frac{b}{M} \int_0^l \frac{\partial w_X}{\partial t} \cdot X(x) dx$$

$$+ \frac{b}{M} \cdot \omega \cdot \int_0^l (s \cdot \sin \varphi - w_Y) \cdot X(x) dx,$$

whereas a second equation will be:

$$\int_0^l \frac{\partial^2 w_Y}{\partial t^2} \cdot X(x) dx + \frac{E - I_z}{\rho \cdot A} \int_0^l \frac{\partial^3 w_Y}{\partial x^3} \cdot X(x) dx$$

$$= -\omega^2 \int_0^l (s \cdot \sin \varphi - w_Y) \cdot X(x) dx$$

$$+ 2 \cdot \omega \int_0^l \frac{\partial w_X}{\partial t} \cdot X(x) dx$$

$$= -\frac{b}{M} \int_0^l \frac{\partial w_Y}{\partial t} \cdot X(x) dx$$

$$+ \frac{b}{M} \cdot \omega \cdot \int_0^l (s \cdot \cos \varphi - w_X) \cdot X(x) dx.$$
where the norm equals:

\[ \gamma_\alpha^2 = \int_0^t X^2(x)dx. \]  

Assuming for simplification \( s = 0 \) and after taking into consideration the boundary conditions and after simple calculations, we get:

\[
\begin{align*}
\dot{a}^2 \cdot k^4 \cdot A_X \cdot e^{\Omega t} - \Omega^2 \cdot A_X \cdot e^{\Omega t} = & \\
\omega^2 \cdot A_X \cdot e^{\Omega t} + 2 \cdot j \cdot \omega \cdot A_Y \cdot e^{\Omega t} - & \\
& - j \cdot \frac{b}{M} \cdot \Omega \cdot A_X \cdot e^{\Omega t} + \frac{b}{M} \cdot \omega \cdot A_Y \cdot e^{\Omega t} \\
& + \frac{F_0 \cdot X(l)}{\rho \cdot A} \cdot \gamma_\alpha^2 \cdot \left[ \left( j \cdot \frac{b}{M} \cdot \Omega + a^2 \cdot k^4 - \omega^2 - \Omega^2 \right)^2 + \left( 2 \cdot j \cdot \omega \cdot \Omega + \frac{b}{M} \cdot \omega \right)^2 \right] \\
& + \sum_{n=1}^{\infty} \frac{F_0 \cdot X(l) \cdot X(x)}{\rho \cdot A} \cdot \gamma_\alpha^2 \cdot \left[ \left( j \cdot \frac{b}{M} \cdot \Omega + a^2 \cdot k^4 - \omega^2 - \Omega^2 \right)^2 + \left( 2 \cdot j \cdot \omega \cdot \Omega + \frac{b}{M} \cdot \omega \right)^2 \right]
\end{align*}
\]

and

\[
\begin{align*}
\dot{w}_X = & \\
& - \frac{b}{M} \cdot \int_0^t (s \cdot \sin \varphi - w_Y) \cdot X(x)dx - 2 \cdot \omega \cdot \int_0^t \frac{\partial w_X}{\partial t} \cdot X(x)dx \\
& - \frac{b}{M} \cdot \int_0^t \frac{\partial w_Y}{\partial t} \cdot X(x)dx + \frac{b}{M} \cdot \omega \cdot \int_0^t (s \cdot \cos \varphi - w_X) \cdot X(x)dx,
\end{align*}
\]

where

\[
\begin{align*}
\omega &= \sqrt{\left( \frac{b}{M} \cdot \Omega + a^2 \cdot k^4 - \omega^2 - \Omega^2 \right)^2 + \left( 2 \cdot j \cdot \omega \cdot \Omega + \frac{b}{M} \cdot \omega \right)^2} \\
\end{align*}
\]

We can finally obtain the system’s displacements as follows:

\[
\begin{align*}
\omega &= \sqrt{\left( \frac{b}{M} \cdot \Omega + a^2 \cdot k^4 - \omega^2 - \Omega^2 \right)^2 + \left( 2 \cdot j \cdot \omega \cdot \Omega + \frac{b}{M} \cdot \omega \right)^2} \\
\end{align*}
\]

and

\[
\begin{align*}
\omega_Y &= \sqrt{\left( \frac{b}{M} \cdot \Omega + a^2 \cdot k^4 - \omega^2 - \Omega^2 \right)^2 + \left( 2 \cdot j \cdot \omega \cdot \Omega + \frac{b}{M} \cdot \omega \right)^2} \\
\end{align*}
\]

where

\[
\begin{align*}
\omega &= \sqrt{\left( \frac{b}{M} \cdot \Omega + a^2 \cdot k^4 - \omega^2 - \Omega^2 \right)^2 + \left( 2 \cdot j \cdot \omega \cdot \Omega + \frac{b}{M} \cdot \omega \right)^2} \\
\end{align*}
\]
Up to the definition the mathematical form of the modulus of dynamical flexibility of the considered systems can be obtained as:

$$Y = \sqrt{\frac{X(x)Y(l)}{\rho_s A n \left( Y_1 + Y_2 \right)^2} \left( Y_1 Y_3 + Y_2 Y_4 \right)^2 + (Y_1^2 - Y_2^2 + Y_3^2 - Y_4^2)Y_4 - Y_2 Y_3},$$

(29)

where the individual elements of the dynamical flexibility are as follows:

$$Y_1 = (a^2 k^4 - \omega^2 - \Omega^2)^2 - \left( \frac{b}{M} \Omega \right)^2$$

$$+ \left( \frac{b}{M} \omega \right) - 4\omega^2 \Omega^2,$$

(30)

$$Y_2 = \frac{b}{M} (a^2 k^4 - \omega^2 - \Omega^2)\Omega + 4 \frac{b}{M} \omega^2 \Omega,$$

(31)

$$Y_3 = a^2 k^4 - \omega^2 - \Omega^2,$$

(32)

$$Y_4 = \frac{b}{M} \Omega,$$

(33)

Fig. 3. The sample juxtaposition of characteristics of damped and nondamped systems.

Fig. 4. The sample juxtaposition of characteristics of stationary (pink line) and rotating systems.
Numerical examples in the form of dynamical characteristics have been presented. In Figs. 3–5, are presented samples of dynamical flexibilities in chart form. Figure 3 presents the sample juxtaposition of characteristics of damped and nondamped systems, and Fig. 4 presents the sample juxtaposition of characteristics of the stationary system and the rotating system without damping. In Fig. 5, the characteristics of the systems are presented with large damping of vibrations. The charts were generated using the Modyfit application [Zółkiewski, 2008].

4. Conclusions
This paper considers the vibration problem of beams fixed on a rotational rigid disk. The beam was located onto the rotational disk that rotates with a constant angular velocity. The beam moves in terms of the plane motion and the model presented here makes possible to consider local and global vibrations, taking into consideration the transportation effect. The clamped–clamped beam and the supported–supported one, both on a rotational disk, were analyzed in this work. The systems can be put into practice in many types of turbines, pumps or rotors, especially the high-speed ones. This analysis can be also used in the analysis of complex systems where one of the components of such a complex system is an analyzed beam. In mathematical models, the internal damping was taken into consideration. The damping forces were assumed as function of velocities of displacements, the energy of damping was transferred from rotation to vibrations. The damping forces were assumed as rotating ones, which means that damping forces were rotated with the beam and the energy transfer was provided in the direction of vibrations. This type of damping can also be used for active damping and the model will be proper in such situations as well. Future works will be dedicated to the dynamical analysis of complex systems and the systems considering nonlinearities.

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