The algebraic Bethe ansatz for open vertex models

Guang-Liang Li[a] , **Kang-Jie Shi**[b]

[a] *Department of Applied Physics, Xi'an Jiaotong University, Xi'an, 710049, P.R.China*

[b] *Institute of Modern Physics, Northwest University, Xi'an, 710069, P.R.China*

E-mail: leegl@mail.xjtu.edu.cn

March 2, 2008

ABSTRACT: We present a unified algebraic Bethe ansatz for open vertex models associated with the non-exceptional $A^{(2)}_{2n}, A^{(2)}_{2n-1}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$ Lie algebras. By using the method, we solve these models with the trivial K matrix and find that our results agree with that obtained by analytical Bethe ansatz.

PACS number: 05.50.+q,75.10.Hk,75.10.Jm
KEYWORDS: algebraic Bethe ansatz; open boundary conditions

Contents

1 Introduction

The algebraic Bethe ansatz(ABA)[1]-[3] is proved to be a more powerful mathematical method in constructing and solving integrable models. One important feature of this method is that it permits us to present an algebraic formulation of the Bethe states. The construction of exact eigenvectors, besides being an interesting problem on its own, is certainly an important step in the program of solving integrable systems. This step, however, depends much on our ability to disentangle the Yang-Baxter algebra in terms appropriate commutation relations. The simplest structure of commutation rules has been discovered in the context of six-vertex model[1],[3] and its multistate generalization[4],[5].

However, there are many models such as B_n, C_n, D_n vertex models etc. whose eigenvectors are rather complicated to be constructed. Fortunately, Martins and Ramos successfully generalized the ABA and threw light on how to construct such general muti-particle eigenvectors. They presented a unified algebraic Bethe ansatz for a large family of vertex models with periodic boundary and solved them by the ABA [6]-[9].

In the framework of Martins's work, recently, the method has been applied to some nineteen-vertex like models and Hubbard-like models models with open boundary conditions. In refs.[10]-[15], the eigenvectors are explicitly constructed and the algebraic Bethe ansatz is demonstrated. The results show that the ABA could be suit for some systems with higher rank algebra symmetry.

There are many models having been solved by the so called analytical Bethe ansatz. This method was firstly proposed by Reshetikhin for close chains[16], then generalized by Nepomechie and his collaborators into the quantum-algebra-invariant open chains [17]. Later, a set of open vertex models associated with non-exceptional Lie algebras such as $A^{(2)}_{2n}$, $A^{(2)}_{2n-1}$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$ vertex models[18] were solved by the analytical Bethe ansatz.

Compared with the ABA, the results obtained by the analytical Bethe ansatz are far from rigorous. So it is well worth reconsidering these models by the ABA.

In this paper we will consider the $A^{(2)}_{2n}$, $A^{(2)}_{2n-1}$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$ vertex models with the trivial K matrix. These models with period boundary conditions have been solved by Reshetikhin in terms of analytical Bethe ansatz and by Lima-santos with the help of the ABA[19], respectively. Their quantum-algebra-invariant open chains (corresponding to the trivial diagonal reflecting K matrix) have already been considered by Nepomechie et al.[18]. Here, we will present a unified ABA for these models. We find that the fundamental commutation relations rules have a common form in terms of the corresponding Boltzmann weights. As a consequence, the derivations of the eigenvectors, the eigenvalues and the associated Bethe ansatz equations also have a quite general character for these vertex models. Our results agree with that obtained by the analytical Bethe ansatz.

We organize our paper as following. In section 2 we demonstrate the ABA for the open $A^{(2)}_{2n}$, $A^{(2)}_{2n-1}$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$ vertex models and present the results for these model with the trivial K matrix. A brief summary and discussion about our results are included in section 3. Some coefficients and detail derivations are given as the Appendices.

2 The algebraic Bethe ansatz

2.1 The R matrices and K matrices

The R matrices for the $A^{(2)}_{2n}, A^{(2)}_{2n-1}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$ models[20],[21] can be expressed as [18]

$$\begin{aligned} R^{(n)}(u) \;&=\; a_n(u)\sum_{i\neq\bar{i}} E_{ii}\otimes E_{ii} + b_n(u)\sum_{i\neq j,\bar{j}} E_{ii}\otimes E_{jj} + \Big(\sum_{i<\bar{i}} c_n(u,i) + \sum_{i>\bar{i}} \bar{c}_n(u,i)\Big) E_{i\bar{i}}\otimes E_{\bar{i}i} \\ &+\Big(\sum_{i<j,j\neq\bar{i}} d_n(u,i,j) + \sum_{i>j,j\neq\bar{i}} \bar{d}_n(u,i,j)\Big) E_{ij}\otimes E_{\bar{i}\bar{j}} + e_n(u)\sum_{i\neq\bar{i}} E_{ii}\otimes E_{\bar{i}\bar{i}} \\ &+\delta_{i\bar{i}} f_n(u) E_{ii}\otimes E_{\bar{i}\bar{i}} + \Big(g_n(u)\sum_{i<j,j\neq\bar{i}} + \bar{g}_n(u)\sum_{i>j,j\neq\bar{i}}\Big) E_{ij}\otimes E_{ji}. \qquad (1) \end{aligned}$$

The summations run over 1 to q, $i+\bar{i}=q+1$, $q=2n+1$ for $A^{(2)}_{2n}, B^{(1)}_n$ and $q=2n$ for $A^{(2)}_{2n-1}, C^{(1)}_n, D^{(1)}_n$, $(E_{ij})_{kl}=\delta_{ik}\delta_{jl}$. The R matrices satisfy the following properties

$$\begin{aligned} regularity \;&:\; R^{(n)}_{12}(0) = \rho_n(0)^{\frac{1}{2}}\mathcal{P}_{12}, \\ unitarity \;&:\; R^{(n)}_{12}(u)R^{(n)}_{21}(-u) = \rho_n(u), \\ PT-symmetry \;&:\; \mathcal{P}_{12}R^{(n)}_{12}(u)\mathcal{P}_{12} = [R^{(n)}_{12}]^{t_1t_2}(u), \\ crossing-unitarity \;&:\; M^{(n)}_1 R^{(n)}_{12}(u)^{t_2} {M^{(n)}_1}^{-1} R^{(n)}_{12}(-u-2\vartheta_n)^{t_1} = \rho_n(u+\vartheta_n). \end{aligned}$$

Where $\rho_n(u)=a_n(u)a_n(-u)$, $\mathcal{P}=\mathcal{P}^{ik}_{jl}E_{ij}\otimes E_{kl}$, $\mathcal{P}^{ik}_{jl}=\delta_{il}\delta_{jk}$, $\vartheta_n=-\sqrt{-1}\pi-2\kappa\eta$ for $A^{(2)}_{2n}, A^{(2)}_{2n-1}$ and $\vartheta_n=-2\kappa\eta$ for $B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$, $\kappa=2n+1, 2n, 2n-1, 2n+2, 2n-2$ for $A^{(2)}_{2n}, A^{(2)}_{2n-1}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$, respectively. t_i denotes the transposition in i-th space,

$M_1^{(n)} = M^{(n)} \otimes 1$ and

$$M_{ij}^{(n)} = \begin{cases} \delta_{ij} e^{4(n+1-\tilde{i})\eta} & \text{for} A_{2n}^{(2)}, \;\; B_n^{(1)} \\ \delta_{ij} e^{4(n+1/2-\tilde{i})\eta} & \text{for} A_{2n-1}^{(2)}, \;\; C_n^{(1)}, \;\; D_n^{(1)} \end{cases} \tag{2}$$

with

$$\tilde{i} = \begin{cases} i+\frac{1}{2}, & 1 \leq i < \frac{q+1}{2} \\ i, & i = \frac{q+1}{2} \\ i-\frac{1}{2}, & \frac{q+1}{2} < i \leq q, \qquad \text{for } A_{2n}^{(2)}, B_n^{(1)}, D_n^{(1)} \end{cases} \tag{3}$$

$$\tilde{i} = \begin{cases} i-\frac{1}{2}, & 1 \leq i \leq n \\ i+\frac{1}{2}, & n+1 \leq i \leq 2n. \qquad \text{for } A_{2n-1}^{(2)}, C_n^{(1)} \end{cases} \tag{4}$$

The R matrices also satisfy the Yang-Baxter equation(YBE)[22]

$$R_{12}^{(n)}(u-v)R_{13}^{(n)}(u)R_{23}^{(n)}(v) = R_{23}^{(n)}(v)R_{13}^{(n)}(u)R_{12}^{(n)}(u-v), \tag{5}$$

$R_{12}^{(n)}(u) = R^{(n)}(u) \otimes 1, R_{23}^{(n)}(u) = 1 \otimes R^{(n)}(u)$ etc., $R_{21}^{(n)} = \mathcal{P}_{12} R_{12}^{(n)} \mathcal{P}_{12}$. For a $N \times N$ square lattice, if we can find the K matrices satisfying the so called reflection equations

$$R_{12}^{(n)}(u-v) \overset{1}{K}_-(u) R_{21}^{(n)}(u+v) \overset{2}{K}_-(v) = \overset{2}{K}_-(v) R_{12}^{(n)}(u+v) \overset{1}{K}_-(u) R_{21}^{(n)}(u-v), \tag{6}$$

$$\begin{aligned} & R_{12}^{(n)}(-u+v) \overset{1}{K}{}_+^{t_1}(u) \overset{1}{M}{}^{-1} R_{21}^{(n)}(-u-v-2\xi_n) \overset{1}{M} \overset{2}{K}{}_+^{t_2}(v) \\ = & \overset{2}{K}{}_+^{t_2}(v) \overset{1}{M} R_{12}^{(n)}(-u-v-2\xi_n) \overset{1}{M}{}^{-1} \overset{1}{K}{}_+^{t_1}(u) R_{21}^{(n)}(-u+v), \end{aligned} \tag{7}$$

where $\overset{1}{K}_\pm(u) = K_\pm(u) \otimes 1, \overset{2}{K}_\pm(u) = 1 \otimes K_\pm(u)$, then the transfer matrix defined as

$$t(u) = \text{tr} K_+(u) U(u) \tag{8}$$

constitutes an one-parameter commutative family, i.e. $[t(u), t(v)] = 0$. Here

$$U(u) = T(u) K_-(u) T^{-1}(-u), \tag{9}$$

$$T(u) = R_{01}^{(n)}(u) R_{02}^{(n)} \cdots R_{0N}^{(n)}(u). \tag{10}$$

The corresponding integrable open chain Hamiltonian takes the form

$$H = \sum_{k=1}^{N-1} H_{k,k+1} + \frac{1}{2} \overset{1}{K'}_-(0) + \frac{\text{tr} \overset{0}{K}_+(0) H_{N,0}}{\text{tr} K_+(0)}, \tag{11}$$

with $H_{k,k+1} = \mathcal{P}_{k,k+1} R'_{kk+1}(u)|_{u=0}$.

From Eq.(6) and Eq.(7), we can see that, given a solution $K_-(u)$ of Eq.(6), the matrix

$$K_+(u) = K_-^t(-u-\vartheta) M^{(n)} \tag{12}$$

satisfies Eq.(7). The general solution to Eq(6) of $A_{2n}^{(2)}, A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ models have been obtained in ref [23], here we will consider the trivial diagonal ones. For Eq.(6), there are three kinds of diagonal K matrices, the trivial K matrix, K matrices without arbitrary parameter and K matrices with one arbitrary parameter. Here we denote them as $K_-^{(1)}(n)$, $K_-^{(2)}(u,n,p_-)$ and $K_-^{(3)}(u,n,\zeta_-)$, respectively. Correspondingly, by Eq.(12), we can obtain three kinds of diagonal K matrices $K_+^{(1)}(n)$, $K_+^{(2)}(u,n,p_+)$ and $K_+^{(3)}(u,n,\zeta_+)$ for Eq.(7). Where $p_\pm$ are integer number, $\zeta_\pm$ are free parameters.

2.2 The vacuum state and commutation relations

Firstly, we write the double-monodromy matrix (9) as

$$U(u)=\begin{pmatrix} A(u) & B_1(u) & B_2(u) & \cdots & B_{2n-1} & F(u) \\ D_1(u) & A_{11}(u) & A_{12}(u) & \cdots & A_{1q-2}(u) & E_1(u) \\ D_2(u) & A_{21}(u) & A_{22}(u) & \cdots & A_{2q-2}(u) & E_2(u) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{q-2}(u) & A_{q-21}(u) & A_{q-22}(u) & \cdots & A_{q-2q-2}(u) & E_{q-2}(u) \\ G(u) & C_1(u) & C_2(u) & \cdots & C_{q-2}(u) & A_2(u) \end{pmatrix}. \tag{13}$$

With the help of Eqs.(5,6), we can prove that $U(u)$ in Eq.(13) satisfy the following equation

$$R_{12}^{(n)}(u-v)\overset{1}{U}(u)R_{21}^{(n)}(u+v)\overset{2}{U}(v)=\overset{2}{U}(v)R_{12}^{(n)}(u+v)\overset{1}{U}(u)R_{21}^{(n)}(u-v). \tag{14}$$

Now we Introduce the vacuum state,

$$|0\rangle=\prod^{\otimes N}(1,0,\cdots,0)^t, \tag{15}$$

where $(1,0,\cdots,0)$ is a $1\times q$ matrix, t denotes the transposition.

Applying the double-row monodromy matrix eq.(13) on the vacuum state eq.(15), we can find

$$\begin{aligned} &D_a(u)|0\rangle=0, \quad C_a(u)|0\rangle=0, \quad G(u)|0\rangle=0, \\ &B_a(u)|0\rangle\neq0, \quad E_a(u)|0\rangle\neq0, \quad F(u)|0\rangle\neq0, \\ &A_{aa}(u)|0\rangle\neq0, \quad A_{ab}(u)|0\rangle=0 \quad (a\neq b), \\ &A(u)|0\rangle\neq0, \quad A_2(u)|0\rangle\neq0. \quad (a=1,2,\cdots,q-2) \end{aligned} \tag{16}$$

From Eq.(16), we can see that D_a, C_a and B_a, E_a, F play the role of annihilation operators and creation operators on the vacuum state, respectively. The A, A_{aa}, A_2 are diagonal operators on the vacuum state. Considering the definition of $U(u)$ Eq.(9), we have

$$A(u)|0\rangle = T(u)_{11}K_-(u)_1T^{-1}(-u)_{11}|0\rangle, \tag{17}$$

$$\begin{aligned} A_{aa}(u)|0\rangle &= T(u)_{a+11}K_-(u)_1T^{-1}(-u)_{1a+1}|0\rangle \\ &\quad +T(u)_{a+1a+1}K_-(u)_{a+1}T^{-1}(-u)_{a+1a+1}|0\rangle, \end{aligned} \tag{18}$$

$$A_2(u)|0\rangle = \sum_{i=1}^{q}T(u)_{qi}K_-(u)_iT^{-1}(-u)_{iq}|0\rangle. \tag{19}$$

In above equations, the first term of eq.(18) and the previous $q-1$ terms of eq.(19) can not be calculated directly but it can be worked out by using the following method. Taking $v=-u$ in the Yang-Baxter equation, we can get

$$T_2^{-1}(-u)R_{12}(2u)T_1(u)=T_1(u)R_{12}(2u)T_2^{-1}(-u). \tag{20}$$

Taking special indices in this relation and applying both sides of this relation to the vacuum state, we find:

$$\begin{aligned}
T(u)_{a+11}T^{-1}(-u)_{1a+1}|0\rangle &= \tilde{f}_1(u)\left(T^{-1}(-u)_{11}T(u)_{11}-T(u)_{a+1a+1}T^{-1}(-u)_{a+1a+1}\right)|0\rangle,\\
T(u)_{qi}T^{-1}(-u)_{iq}|0\rangle &= M_{ii}^{(n)}\tilde{f}_2(u)\left(T^{-1}(-u)_{ii}T(u)_{ii}-T(u)_{qq}T^{-1}(-u)_{qq}\right)|0\rangle, (i\neq 1,q)\\
T(u)_{q1}T^{-1}(-u)_{1q}|0\rangle &= \Bigg(\tilde{f}_3(u)T^{-1}(-u)_{11}T(u)_{11}-\tilde{f}_1(u)\tilde{f}_2(u)\sum_{i=2}^{q-1}M_{ii}^{(n)}T(u)_{ii}T^{-1}(-u)_{ii}\\
&\quad +(\tilde{f}_1(u)\tilde{f}_2(u)\sum_{i=2}^{q-1}M_{ii}^{(n)}-\tilde{f}_3(u))T^{-1}(-u)_{qq}T(u)_{qq}\Bigg)|0\rangle,
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\tilde{f}_1(u) &= \frac{\bar{g}_n(2u)}{a_n(2u)}, \qquad \tilde{f}_3(u)=\frac{\bar{c}_n(2u,q)}{a_n(2u)},\\
\tilde{f}_2(u) &= \frac{\bar{c}_n(2u,q)g_n(2u)-\bar{g}_n(2u)a_n(2u)}{g_n(2u)\bar{g}_n(2u)\sum_{i=2}^{q-1}M_{ii}^{(n)}-a_n(2u)\sum_{i=2}^{q-1}M_{ii}^{(n)}R^{(n)}(2u)_{ji}^{ij}}
\end{aligned} \tag{22}$$

with arbitrary $j\in[2,q-1]$. Simple calculations show that

$$\tilde{f}_2(u)=\begin{cases} -\dfrac{e^{u-4\eta}\sinh(2\eta)}{\sinh(u-2(\kappa-1)\eta)}, & \text{for } A_{2n}^{(2)}, C_n^{(1)}\\ -\dfrac{e^{u}\sinh(2\eta)}{\sinh(u-2(\kappa-1)\eta)}. & \text{for } A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)} \end{cases} \tag{23}$$

Introducing two new operators

$$\tilde{A}_{ab}(u) = A_{ab}(u)-\tilde{f}_1(u)A(u)\delta_{ab}, \tag{24}$$

$$\tilde{A}_2(u) = A_2(u)-\tilde{f}_3(u)A(u)-\tilde{f}_2(u)\sum_{a=1}^{q-2}M_{aa}^{(n-1)}\tilde{A}_{aa}(u), \tag{25}$$

here we should notice that $M_{aa}^{(n-1)}=M_{a+1a+1}^{(n)}$, then we have

$$A(u)|0\rangle = K_-(u)_1[a_n(u)]^{2N}\rho_n(u)^{-N}|0\rangle=\omega_1(u)|0\rangle, \tag{26}$$

$$\tilde{A}_{aa}(u)|0\rangle = (K_-(u)_{a+1} - \tilde{f}_1(u)K_-(u)_1)[b_n(u)]^{2N}\rho_n(u)^{-N}|0\rangle = k^-(u)_a\omega(u)|0\rangle, \quad (27)$$

$$\tilde{A}_2(u)|0\rangle = \Big\{K_-(u)_q - \tilde{f}_2(u)\sum_{a=1}^{q-2} M_{aa}^{(n-1)}\Big(K_-(u)_{a+1} - \tilde{f}_1(u)K_-(u)_1\Big) - \tilde{f}_3(u)K_-(u)_1\Big\}[e_n(u)]^{2N}\rho_n(u)^{-N}|0\rangle = \omega_q(u)|0\rangle. \quad (28)$$

In terms of new operators, the transfer matrix (8) can be rewritten as

$$t(u) = w_1(u)A(u) + \sum_{a=1}^{q-2} w(u)k_a^+(u)\tilde{A}_{aa}(u) + w_q(u)\tilde{A}_2(u) \quad (29)$$

with

$$w_1(u) = K_+(u)_1 + \tilde{f}_3(u)K_+(u)_q + \tilde{f}_1(u)\sum_{a=1}^{q-2} K_+(u)_{a+1},$$
$$w(u)k_a^+(u) = K_+(u)_{a+1} + M_{qq}^{(n)}\tilde{f}_2(u)K_+(u)_q, \quad w_q(u) = K_+(u)_q. \quad (30)$$

Where $k^{\mp}(u) = K_{\mp}^{(1)}(n-1)$, $K_{\mp}^{(2)}(\tilde{u}, n-1, p_{\mp}-1)$ or $K_{\mp}^{(3)}(\tilde{u}, n-1, \tilde{\zeta}_{\mp})$ depend on the choice of boundary, $\tilde{u} = u - 2\eta, \tilde{\zeta}_{\mp} = \zeta \pm \eta$.

In order to construct the general m-particle state, we need to find the commutation relations between the creation, diagonal and annihilation fields. Rewriting the equation (14) in component form

$$R(u_-)_{c_1c_2}^{a_1a_2}U(u)_{d_1}^{c_1}R(u_+)_{d_2b_1}^{c_2d_1}U(v)_{b_2}^{d_2}$$
$$= U(v)_{c_2}^{a_2}R(u_+)_{c_1d_2}^{a_1c_2}U(u)_{d_1}^{c_1}R(u_-)_{b_2b_1}^{d_2d_1}, \quad (31)$$

where the repeated indices sum over 1 to q, $u_- = u - v$, $u_+ = u + v$. In the following formulaes, the repeated indices will mean summing over 1 to $q-2$ unless there are some special notifications. Taking some components of Eq.(31), we can obtain the following fundamental commutation relations for those models,

$$B_a(u)B_b(v) + \delta_{\bar{a}b}g_1(u,v,a)F(u)A(v) + g_2(u,v,a)F(u)\tilde{A}_{\bar{a}b}(v)$$
$$= \hat{r}(u_-)_{ba}^{dc}[B_d(v)B_c(u) + \delta_{\bar{d}c}g_1(v,u,d)F(v)A(u) + g_2(v,u,d)F(v)\tilde{A}_{\bar{d}c}(u)], \quad (32)$$

$$A(u)B_a(v) = a_1^1(u,v)B_a(v)A(u) + a_2^1(u,v)B_a(u)A(v) + a_3^1(u,v)B_d(u)\tilde{A}_{da}(v) + a_4^1(u,v,\bar{a})F(u)D_{\bar{a}}(v) + a_5^1(u,v)F(u)C_a(v) + a_6^1(u,v,\bar{a})F(v)D_{\bar{a}}(u), \quad (33)$$

$$\tilde{A}_{ab}(u)B_c(v) = \tilde{r}(u_+)_{dg}^{ae}\bar{r}(u_-)_{cb}^{gf}B_e(v)\tilde{A}_{df}(u) + R_1^A(u,v)_{cb}^{af}B_f(u)A(v) + R_2^A(u,v)_{db}^{af}B_f(u)\tilde{A}_{dc}(v) + \delta_{\bar{b}c}R_3^A(u,v,\bar{b})E_a(u)A(v) + R_4^A(u,v,\bar{b})E_a(u)\tilde{A}_{\bar{b}c}(v) + R_5^A(u,v)_{cb}^{af}F(u)D_{\bar{f}}(v) + \delta_{ab}R_6^A(u,v)F(u)C_c(v) + R_7^A(u,v)_{cb}^{af}F(v)D_{\bar{f}}(u)$$

$$+R_8^A(u,v)_{cb}^{af}F(v)C_f(u), \quad (34)$$

$$\begin{aligned}\tilde{A}_2(u)B_a(v) &= a_1^3(u,v)B_a(v)\tilde{A}_2(u)+a_2^3(u,v)B_a(u)A(v)+a_3^3(u,v)B_d(u)\tilde{A}_{da}(v)\\ &+a_4^3(u,v,a)E_{\bar{a}}(u)A(v)+a_5^3(u,v,\bar{d})E_d(u)\tilde{A}_{\bar{d}a}(v)+a_6^3(u,v,\bar{a})F(u)D_{\bar{a}}(v)\\ &+a_7^3(u,v)F(u)C_a(v)+a_8^3(u,v,\bar{a})F(v)D_{\bar{a}}(u)+a_9^3(u,v)F(v)C_a(u), \quad (35)\end{aligned}$$

$$\begin{aligned}A(u)F(v) &= b_1^1(u,v)F(v)A(u)+b_2^1(u,v)F(u)A(v)+b_3^1(u,v,d)F(u)\tilde{A}_{dd}(v)\\ &+b_4^1(u,v)F(u)\tilde{A}_2(v)+b_5^1(u,v,d)B_{\bar{d}}(u)B_d(v)+b_6^1(u,v)B_d(u)E_d(v), \quad (36)\end{aligned}$$

$$\begin{aligned}\tilde{A}_{ab}(u)F(v) &= b_1^2(u,v)F(v)\tilde{A}_{ab}(u)+\delta_{ab}b_2^2(u,v)F(u)A(v)+R_1^F(u,v)_{b\bar{a}}^{dc}F(u)\tilde{A}_{\bar{d}c}(v)\\ &+\delta_{ab}b_3^2(u,v)F(u)\tilde{A}_2(v)+R_2^F(u,v)_{b\bar{a}}^{dc}B_d(u)B_c(v)+R_3^F(u,v)_{db}^{ac}B_c(u)E_d(v)\\ &+b_4^2(u,v)E_a(u)B_b(v)+b_5^2(u,v,\bar{b})E_a(u)E_{\bar{b}}(v), \quad (37)\end{aligned}$$

$$\begin{aligned}\tilde{A}_2(u)F(v) &= b_1^3(u,v)F(v)\tilde{A}_2(u)+b_2^3(u,v)F(u)A(v)+b_3^3(u,v,d)F(u)\tilde{A}_{dd}(v)\\ &+b_4^3(u,v)F(u)\tilde{A}_2(v)+b_5^3(u,v,\bar{d})B_{\bar{d}}(u)B_d(v)+b_6^3(u,v)B_d(u)E_d(v)\\ &+b_7^3(u,v)E_d(u)B_d(v)+b_8^3(u,v,d)E_d(u)E_{\bar{d}}(v). \quad (38)\end{aligned}$$

$$\begin{aligned}D_a(u)B_b(v) &= R_1^D(u,v)_{db}^{ac}B_c(v)D_d(u)+c_1^1(u,v,\bar{a})B_{\bar{a}}(v)C_b(u)+\delta_{ab}c_2^1(u,v)F(v)G(u)\\ &+c_3^1(u,v,\bar{a})B_{\bar{a}}(u)C_b(v)+c_4^1(u,v)E_a(u)C_b(v)+\delta_{ab}c_5^1(u,v)A(v)A(u)\\ &+c_6^1(u,v)A(v)\tilde{A}_{ab}(u)+\delta_{ab}c_7^1(u,v)A(u)A(v)+c_8^1(u,v)A(u)\tilde{A}_{ab}(v)\\ &+c_9^1(u,v)\tilde{A}_{ab}(u)A(v)+c_{10}^1(u,v)\tilde{A}_{ad}(u)\tilde{A}_{db}(v), \quad (39)\end{aligned}$$

$$\begin{aligned}C_a(u)B_b(v) &= R_1^C(u,v)_{ba}^{dc}B_d(v)C_c(u)+R_2^C(u,v)_{db}^{\bar{a}c}B_c(v)D_d(u)+\delta_{a\bar{b}}c_1^2(u,v,\bar{a})F(v)G(u)\\ &+c_2^2(u,v)B_a(u)C_b(v)+c_3^2(u,v,\bar{a})E_{\bar{a}}(u)C_b(v)+\delta_{a\bar{b}}c_4^2(u,v,\bar{a})A(v)A(u)\\ &+R_3^C(u,v)_{ba}^{dc}A(v)\tilde{A}_{\bar{d}c}(u)+\delta_{a\bar{b}}c_5^2(u,v,\bar{a})A(v)\tilde{A}_2(u)\\ &+\delta_{a\bar{b}}c_6^2(u,v,\bar{a})A(u)A(v)+c_7^2(u,v,\bar{a})A(u)\tilde{A}_{\bar{a}b}(v)\\ &+R_4^C(u,v)_{ba}^{dc}\tilde{A}_{\bar{d}c}(u)A(v)+R_5^C(u,v)_{ea}^{dc}\tilde{A}_{\bar{d}c}(u)\tilde{A}_{eb}(v)\\ &+\delta_{a\bar{b}}c_8^2(u,v,\bar{a})\tilde{A}_2(u)A(v)+c_9^2(u,v,\bar{a})\tilde{A}_2(u)\tilde{A}_{\bar{a}b}(v), \quad (40)\end{aligned}$$

$$\begin{aligned}B_a(u)E_b(v) &= R_1^{be}(u,v)_{bd}^{ca}E_c(v)B_d(u)+R_2^{be}(u,v)_{\bar{b}a}^{dc}B_d(v)B_c(u)+\delta_{ab}e_1^1(u,v,a)F(v)A(u)\\ &+R_3^{be}(u,v)_{\bar{b}a}^{dc}F(v)\tilde{A}_{\bar{d}c}(u)+\delta_{ab}e_2^1(u,v,a)F(u)A(v)+R_4^{be}(u,v)_{bd}^{ca}F(u)\tilde{A}_{cd}(v)\\ &+\delta_{ab}e_3^1(u,v,a)F(u)\tilde{A}_2(v), \quad (41)\end{aligned}$$

where the coefficients $R_i^\alpha(u,v)_{cd}^{ab}$ are zeroes except for $a=b=c=d, a=c\neq b=d, a=d\neq b=c$ and $a+b=c+d=q-1$, $(\alpha = A, D, etc., i=1,2,etc.)$,

$$g_1(u,v,a) = -\frac{d_n(u_-,1,\bar{a}+1)b_n(2v)}{e_n(u_-)a_n(2v)}, \quad g_2(u,v,a) = \frac{d_n(u_+,1,\bar{a}+1)}{b_n(u_+)} \qquad a+\bar{a}=q-1.$$

The $\hat{r}(u), \tilde{r}(u)$ and $\bar{r}(u)$ are given by

$$\hat{r}(u) = \frac{1}{e_n(u)}\frac{b_n(u)}{a_n(u)}R^{(n-1)}(u), \quad \tilde{r}(u) = \frac{1}{a_n(u)}R^{(n-1)}(u-4\eta), \quad \bar{r}(u) = \frac{1}{e_n(u)}R^{(n-1)}(u),$$

respectively. The other coefficients are omitted here for their long and tedious expressions.

2.3 The m-particle state

Inferred from the commutation relation Eq.(32), we can construct the general m-particle state for these models as follow. Let

$$\begin{aligned}
\Phi_m^{b_1\cdots b_m}(v_1,\cdots,v_m) &= B_{b_1}(v_1)\Phi_{m-1}^{b_2\cdots b_m}(v_2,\cdots,v_m)\\
&+F(v_1)\sum_{i=2}^{m}\Phi_{m-2}^{d_3\cdots d_m}(v_2,\cdots,\check{v}_i,\cdots,v_m)S_{b_2\cdots b_m}^{d_2\cdots d_m}(v_i;\{\check{v}_1,\check{v}_i\})\\
&\times\Lambda_1^{m-2}(v_i;\{\check{v}_1,\check{v}_i\})g_1(v_1,v_i,b_1)A(v_i)\delta_{\bar{b}_1 d_2}\\
&+F(v_1)\sum_{i=2}^{m}\Phi_{m-2}^{d_3\cdots d_m}(v_2,\cdots,\check{v}_i,\cdots,v_m)[\tilde{T}^{m-2}(v_i;\{\check{v}_1,\check{v}_i\})_{c_3\cdots c_m}^{d_3\cdots d_m}]_{\bar{b}_1 c_2}\\
&\times S_{b_2\cdots b_m}^{c_2\cdots c_m}(v_i;\{\check{v}_1,\check{v}_i\})g_2(v_1,v_i,b_1),
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
S_{b_1\cdots b_m}^{d_1\cdots d_m}(v_i;\{\check{v}_i\}) &= \hat{r}_{c_2 b_1}^{d_1 d_2}(v_1-v_i)\hat{r}_{c_3 b_2}^{c_2 d_3}(v_2-v_i)\cdots\hat{r}_{b_i b_{i-1}}^{c_{i-1} d_i}(v_{i-1}-v_i)\prod_{j=i+1}^{m}\delta_{d_j b_j}\\
[\tilde{T}^m(u;\{v_m\})_{c_1\cdots c_m}^{d_1\cdots d_m}]_{ab} &= \tilde{r}_{h_1 g_1}^{a d_1}(u+v_1)\tilde{r}_{h_2 g_2}^{h_1 d_2}(u+v_2)\cdots\tilde{r}_{h_m g_m}^{h_{m-1} d_m}(u+v_m)\tilde{A}_{h_m f_m}(u)\\
&\quad\bar{r}_{c_m f_{m-1}}^{g_m f_m}(u-v_m)\bar{r}_{c_{m-1} f_{m-2}}^{g_{m-1} f_{m-1}}(u-v_{m-1})\cdots\bar{r}_{c_1 b}^{g_1 f_1}(u-v_1)
\end{aligned} \tag{43}$$

with $S_{b_1\cdots b_m}^{d_1\cdots d_m}(v_1;v_2,\cdots,v_m) = \prod_{i=1}^m \delta_{d_i b_i}$, $[\tilde{T}^0(u)]_{ab} = \tilde{A}_{ab}(u)$, $\Lambda_l^m(u;v_1,v_2,\cdots,v_m) = \prod_{i=1}^m a_1^l(u,v_i)$, $(l=1,3)$, $\Phi_0 = 1, \Phi_1^{b_1}(v_1) = B_{b_1}(v_1)$. The $\check{v}_i$ means missing of v_i in the sequence.

Then the general m-particle state is defined by

$$|\Upsilon_m(v_1,\cdots,v_m)\rangle = \Phi_m^{b_1\cdots b_m}(v_1,\cdots,v_m)F^{b_1\cdots b_m}|0\rangle, \tag{44}$$

which satisfy the $n-1$ exchange conditions

$$\begin{aligned}
&\Phi_m^{b_1\cdots b_i b_{i+1}\cdots b_m}(v_1,\cdots,v_i,v_{i+1},\cdots,v_m)F^{b_1\cdots b_m}|0\rangle =\\
&\quad\Phi_m^{b_1\cdots a_i a_{i+1}\cdots b_m}(v_1,\cdots,v_{i+1},v_i,\cdots,v_m)\hat{r}_{b_i b_{i+1}}^{a_{i+1} a_i}(v_i-v_{i+1})F^{b_1\cdots b_m}|0\rangle\ .
\end{aligned} \tag{45}$$

It is easy to verify Eq.(45) excepting $i=1$ by the Yang-Baxter equation. The proof for the case $i=1$ becomes very involved but we can prove it by mathematical induction method.

2.4 The eigenvalue and Bethe equations

We can apply the operators x ($x = A, \tilde{A}_{aa}, \tilde{A}_2$) on the eigenstate ansatz and obtain (see Appendix B)

$$\begin{aligned}
x(u)|\Upsilon_m(v_1,\cdots,v_m)\rangle &= |\tilde{\Psi}_x(u,\{v_m\})\rangle \\
&+\sum_{i=1}^{m} h_1^x(u,v_i,d)|\Psi^{(1)}_{m-1}(u,v_i;\{v_m\})_{dd}\rangle \\
&+\sum_{i=1}^{m} h_2^x(u,v_i,d)|\tilde{\Psi}^{(2)}_{m-1}(u,v_i;\{v_m\})_{dd}\rangle \\
&+\sum_{i=1}^{m} h_3^x(u,v_i,\bar{\alpha}_x)|\Psi^{(3)}_{m-1}(u,v_i;\{v_m\})_{\alpha_x\alpha_x}\rangle \\
&+\sum_{i=1}^{m} h_4^x(u,v_i,\bar{\alpha}_x)|\tilde{\Psi}^{(4)}_{m-1}(u,v_i;\{v_m\})_{\alpha_x\alpha_x}\rangle \\
&+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \tilde{H}^x_{1,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(5)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle \\
&+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \tilde{H}^x_{2,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(6)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle \\
&+\sum_{i=1}^{n-1}\sum_{j=i+1}^{m} \tilde{H}^x_{3,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(7)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle \\
&+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \tilde{H}^x_{4,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(8)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle,
\end{aligned} \tag{46}$$

where the expression of $|\tilde{\Psi}\rangle$'s and coefficients $\tilde{H}^x_{j,d_1}$ ($j = 1, 2, 3, 4$) are given in Appendix B. Using Eq.(29), we then get

$$\begin{aligned}
t(u)|\Upsilon_m(v_1,\cdots,v_m)\rangle &= w_1(u)\omega_1(u)\Lambda_1^m(u;v_1,\cdots,v_m)|\Upsilon_m(v_1,\cdots,v_m)\rangle \\
&+w(u)\omega(u)\Lambda_2^m(u;v_1,\cdots,v_m)\Phi_m^{d_1\cdots d_m}(v_1,\cdots,v_m)\tau_1(\tilde{u};\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}F^{b_1\cdots b_m}|0\rangle \\
&+w_q(u)\omega_q(u)\Lambda_3^m(u;v_1,\cdots,v_m)|\Upsilon_m(v_1,\cdots,v_m)\rangle + u.t.,
\end{aligned} \tag{47}$$

where $u.t.$ denotes the unwanted terms,

$$\begin{aligned}
&\tau_1(\tilde{u};\{\tilde{v}_m\})^{d_1\cdots d_m}_{c_1\cdots c_m} = \\
&\quad k^+(u)_a L(\tilde{u},\tilde{v}_1)^{ad_1}_{h_1g_1} L(\tilde{u},\tilde{v}_2)^{h_1d_2}_{h_2g_2}\cdots L(\tilde{u},\tilde{v}_m)^{h_{m-1}d_m}_{h_mg_m} k^-(u)_{h_m} \\
&\quad \times L^{-1}(-\tilde{u},\tilde{v}_m)^{h_mg_m}_{f_{m-1}c_m} L^{-1}(-\tilde{u},\tilde{v}_{m-1})^{f_{m-1}g_{m-1}}_{f_{m-2}c_{m-1}}\cdots L^{-1}(-\tilde{u},\tilde{v}_1)^{f_1g_1}_{ac_1}.
\end{aligned} \tag{48}$$

with $\tilde{v}_i = v_i - 2\eta$, and

$$\begin{aligned}
L(\tilde{u},\tilde{v})^{ab}_{cd} &= R^{(n-1)}(\tilde{u}+\tilde{v})^{ab}_{cd}, \\
L^{-1}(-\tilde{u},\tilde{v})^{ab}_{cd} &= \frac{R^{(n-1)}(\tilde{u}-\tilde{v})^{ba}_{dc}}{\rho_{n-1}(\tilde{u}-\tilde{v})}.
\end{aligned} \tag{49}$$

Thus, we get the conclusion that$|\Upsilon_m(v_1,\cdots,v_m)\rangle$ is the eigenstate of $t(u)$, i.e.

$$
\begin{aligned}
t(u)|\Upsilon_m(v_1,\cdots,v_m)\rangle = &\{w_1(u)\omega_1(u)\Lambda_1^m(u;v_1,\cdots,v_m)\\
&+w(u)\omega(u)\Lambda_2^m(u;v_1,\cdots,v_m)\Gamma_1(\tilde{u};\{\tilde{v}_m\};\{v_{m_1}^{(1)}\})\\
&+w_q(u)\omega_q(u)\Lambda_3^m(u;v_1,\cdots,v_m)\}|\Upsilon_m(v_1,\cdots,v_m)\rangle\\
&=\Gamma(u;\{v_m\})|\Upsilon_m(v_1,\cdots,v_m)\rangle,
\end{aligned}
\tag{50}
$$

if the parameters satisfy

$$\tau_1(\tilde{u};\{\tilde{v}_m\})F^{b_1\cdots b_m}=\Gamma_1(\tilde{u};\{\tilde{v}_m\};\{v_{m_1}^{(1)}\})F^{b_1\cdots b_m}\ , \tag{51}$$

$$\Gamma_1(\tilde{v}_i;\{\tilde{v}_m\};\{v_{m_1}^{(1)}\})=-\rho_{n-1}^{-\frac{1}{2}}(0)\frac{\omega_1(v_i)\Lambda_1^{m-1}(v_i;\{\check{v}_i\})}{\omega(v_i)\Lambda_2^{m-1}(v_i;\{\check{v}_i\})}\beta_1(v_i).\ \ (i=1,\cdots,m) \tag{52}$$

All unwanted terms cancel out by the following three kinds of identities

$$\beta_1(v_i)=T^{(d_1)}(v_i)\frac{w_1(u)a_2^1(u,v_i)+\sum_{d=1}^{q-2}w_{d+1}(u)R_1^A(u,v_i)_{d_1d}^{dd_1}+w_q(u)a_2^3(u,v_i)}{w_1(u)a_3^1(u,v_i)+\sum_{d=1}^{q-2}w_{d+1}(u)R_2^A(u,v_i)_{d_1d}^{dd_1}+w_q(u)a_3^3(u,v_i)}, \tag{53}$$

$$\beta_1(v_i)=T^{(d_1)}(v_i)\frac{w_{\bar{d}_1+1}(u)R_3^A(u,v_i,d_1)+w_q(u)a_4^3(u,v_i,d_1)}{w_{\bar{d}_1+1}(u)R_4^A(u,v_i,d_1)+w_{2n+1}(u)a_5^3(u,v_i,d_1)}, \tag{54}$$

$$
\begin{aligned}
&\sum_{l=1}^{q}w_l(u)\tilde{H}_{1,d_1}^{x_l}(u,v_i,v_j)-[\sum_{l=1}^{q}w_l(u)\tilde{H}_{2,d_1}^{x_l}(u,v_i,v_j)]\beta_1(v_i)\\
&-[\sum_{l=1}^{q}w_l(u)\tilde{H}_{3,d_1}^{x_l}(u,v_i,v_j)]\beta_1(v_j)+[\sum_{l=1}^{q}w_l(u)\tilde{H}_{4,d_1}^{x_l}(u,v_i,v_j)]\beta_1(v_i)\beta_1(v_j)=0
\end{aligned}
\tag{55}
$$

with $x_1=A, x_{l+1}=\tilde{A}_{ll}, x_q=\tilde{A}_2$, $d_1=1,2,\cdots,q-2$ and do not sum in Eqs.(53,54,55).

From Eqs.(48), (50) and (51) , we can see that the diagonalization of $\tau(u)$ is reduced to finding the eigenvalue of $\tau_1(\tilde{u};\{\tilde{v}_m\})$ which is just the transfer matrix of the same open model with its R matrix given by $R^{(n-1)}(u)$. Repeating the procedure j times, we can obtain the eigenvalue $\Gamma_j(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\};\{v_{m_j}^{(j)}\})$ with its transfer matrix constructed by $R^{(n-j)}(u)$,

$$
\begin{aligned}
&\Gamma_j(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\};\{v_{m_j}^{(j)}\})\\
&\quad=w_1^{(j)}(u^{(j)})\omega_1^{(j)}(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\})\Lambda_1^{m_j}(u^{(j)};\{v_{m_j}^{(j)}\})\\
&\quad+w^{(j)}(u^{(j)})\omega^{(j)}(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\})\Lambda_2^{m_j}(u^{(j)};\{v_{m_j}^{(j)}\})\Gamma_{j+1}(u^{(j+1)};\{\tilde{v}_{m_j}^{(j)}\};\{v_{m_{j+1}}^{(j+1)}\})\\
&\quad+w_{q-2j}^{(j)}(u^{(j)})\omega_{q-2j}^{(j)}(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\})\Lambda_3^{m_j}(u^{(j)};\{v_{m_j}^{(j)}\})
\end{aligned}
\tag{56}
$$

with $u^{(j)}=u-2j\eta$, $\tilde{v}_k^{(j)}=v_k^{(j)}-2\eta$, $\{v_{m_j}^{(j)}\}=\{v_1^{(j)},\cdots,v_{m_j}^{(j)}\}$, $\{v_{m_0}^{(0)}\}=\{v_m\}$, $\{v_{m_{-1}}^{(-1)}\}=\{\tilde{v}_{m_{-1}}^{(-1)}\}=\{0\}$, $m_{-1}=N, m_0=m$,

$$
\begin{aligned}
&\omega_1^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) = \bar{\omega}_1^{(j)}(u^{(j)})\xi_1^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}),\\
&\omega^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) = \bar{\omega}^{(j)}(u^{(j)})\xi_2^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}),\\
&\omega_{q-2j}^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) = \bar{\omega}_{q-2j}^{(j)}(u^{(j)})\xi_3^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\})
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
\xi_1^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) &= \prod_{i=1}^{m_{j-1}} \frac{a_{n-j}(u^{(j)}+\tilde{v}_i^{(j-1)})a_{n-j}(u^{(j)}-\tilde{v}_i^{(j-1)})}{a_{n-j}(u^{(j)}-\tilde{v}_i^{(j-1)})a_{n-j}(\tilde{v}_i^{(j-1)}-u^{(j)})},\\
\xi_2^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) &= \prod_{i=1}^{m_{j-1}} \frac{b_{n-j}(u^{(j)}+\tilde{v}_i^{(j-1)})b_{n-j}(u^{(j)}-\tilde{v}_i^{(j-1)})}{a_{n-j}(u^{(j)}-\tilde{v}_i^{(j-1)})a_{n-j}(\tilde{v}_i^{(j-1)}-u^{(j)})},\\
\xi_3^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) &= \prod_{i=1}^{m_{j-1}} \frac{e_{n-j}(u^{(j)}+\tilde{v}_i^{(j-1)})e_{n-j}(u^{(j)}-\tilde{v}_i^{(j-1)})}{a_{n-j}(u^{(j)}-\tilde{v}_i^{(j-1)})a_{n-j}(\tilde{v}_i^{(j-1)}-u^{(j)})}.
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
\Lambda_1^{m_j}(u^{(j)};\{v^{(j)}_{m_j}\}) &= \prod_{i=1}^{m_j} \frac{a_{n-j}(-\tilde{u}^{(j)}+\tilde{v}_i^{(j)})a_{n-j}(-\tilde{u}^{(j)}-\tilde{v}_i^{(j)})}{b_{n-j}(-\tilde{u}^{(j)}+\tilde{v}_i^{(j)})b_{n-j}(-\tilde{u}^{(j)}-\tilde{v}_i^{(j)})}\\
\Lambda_2^{m_j}(u^{(j)};\{v^{(j)}_{m_j}\}) &= \prod_{i=1}^{m_j} \frac{a_{n-j-1}(-\tilde{u}^{(j)}+\tilde{v}_i^{(j)})a_{n-j-1}(\tilde{u}^{(j)}-\tilde{v}_i^{(j)})}{e_{n-j}(\tilde{u}^{(j)}-\tilde{v}_i^{(j)})e_{n-j}(\tilde{u}^{(j)}+\tilde{v}_i^{(j)})}\\
\Lambda_3^{m_j}(u^{(j)};\{v^{(j)}_{m_j}\}) &= \prod_{i=1}^{m_j} \frac{b_{n-j}(\tilde{u}^{(j)}-\tilde{v}_i^{(j)})b_{n-j}(\tilde{u}^{(j)}+\tilde{v}_i^{(j)})}{e_{n-j}(\tilde{u}^{(j)}-\tilde{v}_i^{(j)})e_{n-j}(\tilde{u}^{(j)}+\tilde{v}_i^{(j)})}.
\end{aligned}
$$

The coefficients w's, $\bar{\omega}$'s vary with the corresponding transfer matrix. The Bethe anzatz equations are

$$
\begin{aligned}
&\Gamma_{j+1}(\tilde{v}_i^{(j)};\{\tilde{v}^{(j)}_{m_j}\};\{v^{(j+1)}_{m_{j+1}}\})\\
&\quad = -\rho_{n-j-1}^{-\frac{1}{2}}(0)\frac{\omega_1^{(j)}(v_i^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\})\Lambda_1^{m_j-1}(v_i^{(j)};\{\check{v}_i^{(j)}\})}{\omega^{(j)}(v_i^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\})\Lambda_2^{m_j-1}(v_i^{(j)};\{\check{v}_i^{(j)}\})}\beta_{j+1}(v_i^{(j)}).\ \ (i=1,\cdots,m_j)
\end{aligned}
\tag{59}
$$

When $j=n-1$, our problem become to obtain the eigenvalues of the nineteen-vertex models $A_2^{(2)}$, $B_1^{(1)}$ for $A_{2n}^{(2)}$, $B_n^{(1)}$ and six-vertex models $A_1^{(2)}, C_1^{(1)}$ for $A_{2n-1}^{(2)}, C_n^{(1)}$, respectively. For $D_n^{(1)}$ when $j=n-2$, we will arrive at $D_2^{(1)}$ model whose eigenvalues are given in terms of the product of the eigenvalues of two $A_1^{(1)}$ vertex models[19]. The six-vertex and nineteen-vertex models have been well solved[13],[15]. Thus, we can get the the whole eigenvalues and the Bethe ansatz equations of transfer matrix for the $A_{2n}^{(2)}, A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ vertex model with open boundary conditions.

2.5 The quantum-algebra-invariant cases

Here we will consider the trivial K matrices for $A_{2n}^{(2)}, A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ models (quantum-algebra-invariant case). The K matrices take the forms

$$
K_- = 1, \quad K_+ = M. \tag{60}
$$

We begin with $A_{2n}^{(2)}, B_n^{(1)}$ models. After a not very long derivation, we can rewrite Eq.(50) in detail, which is

$$\begin{aligned}\Gamma(u) = \Gamma_0(u) &= w_1^{(0)}(u^{(0)})\bar{\omega}_1^{(0)}(u^{(0)})\xi_1^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{A}^{(m_0)}(u)\\ &+w_q^{(0)}(u^{(0)})\bar{\omega}_q^{(0)}(u^{(0)})\xi_3^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{C}^{(m_0)}(u)\\ &+\sum_{j=0}^{n-2}\mu_j(u^{(j)})\nu_j(u^{(j)})w_1^{(j+1)}(u^{(j+1)})\bar{\omega}_1^{(j+1)}(u^{(j+1)})\xi_2^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{B}^{(m_j,m_{j+1})}(u)\\ &+\sum_{j=0}^{n-2}\mu_j(u^{(j)})\nu_j(u^{(j)})w_{q-2(j+1)}^{(j+1)}(u^{(j+1)})\bar{\omega}_{q-2(j+1)}^{(j+1)}(u^{(j+1)})\xi_2^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\bar{\mathcal{B}}^{(m_j,m_{j+1})}(u)\\ &+\mu_{n-1}(u^{(n-1)})\nu_{n-1}(u^{(n-1)})\xi_2^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{B}^{(m_{n-1})}(u), \end{aligned} \tag{61}$$

where the identity $\Lambda_2^{m_j}(u^{(j)};\{v_{m_j}^{(j)}\})\xi_2^{(j+1)}(u^{(j+1)};\{\tilde{v}_{m_j}^{(j)}\}) = 1$ is used,

$$\mu_j(u^{(j)}) = \prod_{i=0}^{j}\bar{\omega}^{(i)}(u^{(i)}), \quad \nu_j(u^{(j)}) = \prod_{i=0}^{j}w^{(i)}(u^{(i)}), \tag{62}$$

$$\mathcal{A}^{(m_0)}(u) = \prod_{k=1}^{m_0}\frac{a_n(-\tilde{u}^{(0)}+\tilde{v}_k^{(0)})a_n(-\tilde{u}^{(0)}-\tilde{v}_k^{(0)})}{b_n(-\tilde{u}^{(0)}+\tilde{v}_k^{(0)})b_n(-\tilde{u}^{(0)}-\tilde{v}_k^{(0)})}, \tag{63}$$

$$\mathcal{C}^{(m_0)}(u) = \prod_{k=1}^{m_0}\frac{b_n(\tilde{u}^{(0)}-\tilde{v}_k^{(0)})b_n(\tilde{u}^{(0)}+\tilde{v}_k^{(0)})}{e_n(\tilde{u}^{(0)}-\tilde{v}_k^{(0)})e_n(\tilde{u}^{(0)}+\tilde{v}_k^{(0)})}, \tag{64}$$

$$\begin{aligned}\mathcal{B}^{(m_j,m_{j+1})}(u) = &\prod_{k=1}^{m_j}\frac{a_{n-j-1}(\tilde{u}^{(j)}+\tilde{v}_k^{(j)})a_{n-j-1}(\tilde{u}^{(j)}-\tilde{v}_k^{(j)})}{e_{n-j}(\tilde{u}^{(j)}+\tilde{v}_k^{(j)})e_{n-j}(\tilde{u}^{(j)}-\tilde{v}_k^{(j)})}\\ &\times\prod_{l=1}^{m_{j+1}}\frac{a_{n-j-1}(-\tilde{u}^{(j+1)}+\tilde{v}_l^{(j+1)})a_{n-j-1}(-\tilde{u}^{(j+1)}-\tilde{v}_l^{(j+1)})}{b_{n-j-1}(-\tilde{u}^{(j+1)}+\tilde{v}_l^{(j+1)})b_{n-j-1}(-\tilde{u}^{(j+1)}-\tilde{v}_l^{(j+1)})},\end{aligned} \tag{65}$$

$$\begin{aligned}\bar{\mathcal{B}}^{(m_j,m_{j+1})}(u) = &\prod_{k=1}^{m_j}\frac{e_{n-j-1}(\tilde{u}^{(j)}+\tilde{v}_k^{(j)})e_{n-j-1}(\tilde{u}^{(j)}-\tilde{v}_k^{(j)})}{e_{n-j}(\tilde{u}^{(j)}+\tilde{v}_k^{(j)})e_{n-j}(\tilde{u}^{(j)}-\tilde{v}_k^{(j)})}\\ &\times\prod_{l=1}^{m_{j+1}}\frac{b_{n-j-1}(\tilde{u}^{(j+1)}-\tilde{v}_l^{(j+1)})b_{n-j-1}(\tilde{u}^{(j+1)}+\tilde{v}_l^{(j+1)})}{e_{n-j-1}(\tilde{u}^{(j+1)}-\tilde{v}_l^{(j+1)})e_{n-j-1}(\tilde{u}^{(j+1)}+\tilde{v}_l^{(j+1)})},\\ &\qquad (j=0,1,2,\cdots,n-2)\end{aligned} \tag{66}$$

$$\mathcal{B}^{(m_{n-1})}(u) = \prod_{k=1}^{m_{n-1}}\frac{f_0(\tilde{u}^{(n-1)}+\tilde{v}_k^{(n-1)})f_0(\tilde{u}^{(n-1)}-\tilde{v}_k^{(n-1)})}{e_1(\tilde{u}^{(n-1)}+\tilde{v}_k^{(n-1)})e_1(\tilde{u}^{(n-1)}-\tilde{v}_k^{(n-1)})},$$

where $f_0(u)$ is just the matric elements $f_n(u)$ when $n=0$. The more explicit expression of Bethe equations Eq.(59) are

$$\prod_{k=1}^{m_{j-1}} \frac{a_{n-j}(\tilde{v}_i^{(j)}+\tilde{v}_k^{(j-1)})a_{n-j}(\tilde{v}_i^{(j)}-\tilde{v}_k^{(j-1)})}{b_{n-j}(\tilde{v}_i^{(j)}+\tilde{v}_k^{(j-1)})b_{n-j}(\tilde{v}_i^{(j)}-\tilde{v}_k^{(j-1)})}$$
$$\times \prod_{l=1}^{m_{j+1}} \frac{b_{n-j-1}(-\tilde{v}_i^{(j)}+\tilde{v}_l^{(j+1)})b_{n-j-1}(-\tilde{v}_i^{(j)}-\tilde{v}_l^{(j+1)})}{a_{n-j-1}(-\tilde{v}_i^{(j)}+\tilde{v}_l^{(j+1)})a_{n-j-1}(-\tilde{v}_i^{(j)}-\tilde{v}_l^{(j+1)})}$$
$$\times \prod_{s=1\neq i}^{m_j} \frac{a_{n-j}(-\tilde{v}_i^{(j)}+\tilde{v}_s^{(j)})a_{n-j}(-\tilde{v}_i^{(j)}-\tilde{v}_s^{(j)})}{b_{n-j}(-\tilde{v}_i^{(j)}+\tilde{v}_s^{(j)})b_{n-j}(-\tilde{v}_i^{(j)}-\tilde{v}_s^{(j)})} \frac{e_{n-j}(\tilde{v}_i^{(j)}+\tilde{v}_s^{(j)})e_{n-j}(\tilde{v}_i^{(j)}-\tilde{v}_s^{(j)})}{a_{n-j-1}(\tilde{v}_i^{(j)}+\tilde{v}_s^{(j)})a_{n-j-1}(\tilde{v}_i^{(j)}-\tilde{v}_s^{(j)})}$$
$$= -\frac{w_1^{(j+1)}(\tilde{v}_i^{(j)})a_{n-j-1}(2\tilde{v}_i^{(j)}))}{\beta_{j+1}(v_i^{(j)})} \frac{\bar{\omega}^{(j)}(v_i^{(j)})\bar{\omega}_1^{(j+1)}(\tilde{v}_i^{(j)})}{\bar{\omega}_1^{(j)}(v_i^{(j)})}, \quad (i=1,\cdots,m_j; j\neq n-1) \tag{67}$$

$$\prod_{k=1}^{m_{n-2}} \frac{a_1(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-2)})a_1(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-2)})}{b_1(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-2)})b_1(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-2)})}$$
$$\times \prod_{l=1\neq i}^{m_{n-1}} \frac{a_1(-\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})a_1(-\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})}{b_1(-\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})b_1(-\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})} \frac{e_1(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})e_1(\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})}{f_0(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})f_0(\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})}$$
$$= -\frac{\bar{\omega}^{(n-1)}(v_i^{(n-1)})}{\bar{\omega}_1^{(n-1)}(v_i^{(n-1)})\beta_n(v_i^{(n-1)})}, \quad (i=1,\cdots,m_{n-1}) \tag{68}$$

From Eqs.(26, 27, 28, 30) we can obtain

$$\bar{\omega}_1^{(j)}(u^{(j)})=1, \quad \bar{\omega}^{(j)}(u^{(j)}) = \frac{e^{2\eta}\sinh(u^{(j)})}{\sinh(u^{(j)}-2\eta)},$$
$$\bar{\omega}_{2(n-j)+1}^{(j)}(u^{(j)}) = \frac{e^{2(2(n-j)-1)\eta}\sinh(u^{(j)})\cosh(u^{(j)}-(2(n-j)+3)\eta)}{\sinh(u^{(j)}-4(n-j)\eta)\cosh(u^{(j)}-(2(n-j)+1)\eta)}. \tag{69}$$

$$w_1^{(j)}(u^{(j)}) = \frac{\sinh(u^{(j)}-2(2(n-j)+1)\eta)\cosh(u^{(j)}-(2(n-j)-1)\eta)}{\sinh(u^{(j)}-2\eta)\cosh(u^{(j)}-(2(n-j)+1)\eta)},$$
$$w^{(j)}(u^{(j)}) = \frac{e^{-2\eta}\sinh(u^{(j)}-2(2(n-j)+1)\eta)}{\sinh(u^{(j)}-4(n-j)\eta)}, \quad w_{2(n-j)+1}^{(j)}(u^{(j)}) = e^{-2(2(n-j)-1)\eta}, \tag{70}$$

$$\beta_{j+1}(v_i^{(j)}) = \begin{cases} -\dfrac{2e^{2\eta}\sinh(v_i^{(j)})\sinh(v_i^{(j)}-4(n-j)\eta)\cosh(v_i^{(j)}-(2n-2j-1)\eta)}{\sinh(v_i^{(j)}-2\eta)}, & j\leq n-2 \\ -\dfrac{e^{2\eta}\sinh(v_i^{(n-1)})}{\sinh(v_i^{(n-1)}-2\eta)} & j=n-1 \end{cases} \tag{71}$$

and

$$\bar{\omega}_1^{(j)}(u^{(j)}) = 1, \quad \bar{\omega}^{(j)}(u^{(j)}) = \frac{e^{2\eta}\sinh(u^{(j)})}{\sinh(u^{(j)}-2\eta)},$$
$$\bar{\omega}_{2(n-j)+1}^{(j)}(u^{(j)}) = \frac{e^{2(2(n-j)-1)\eta}\sinh(u^{(j)})\sinh(u^{(j)}-(2(n-j)-3)\eta)}{\sinh(u^{(j)}-4(n-j-1)\eta)\sinh(u^{(j)}-(2(n-j)-1)\eta)}. \quad (72)$$

$$w_1^{(j)}(u^{(j)}) = \frac{\sinh(u^{(j)}-2(2(n-j)+1)\eta)\sinh(u^{(j)}-(2(n-j)-1)\eta)}{\sinh(u^{(j)}-2\eta)\sinh(u^{(j)}-(2(n-j)-1)\eta)},$$
$$w^{(j)}(u^{(j)}) = \frac{e^{-2\eta}\sinh(u^{(j)}-2(2(n-j)-1)\eta)}{\sinh(u^{(j)}-4(n-j-1)\eta)}, \quad w_{2(n-j)+1}^{(j)}(u^{(j)}) = e^{-2(2(n-j)-1)\eta}, \quad (73)$$

$$\beta_{j+1}(v_i^{(j)}) = \begin{cases} -\dfrac{2e^{2\eta}\sinh(v_i^{(j)})\sinh(v_i^{(j)}-4(n-j-1)\eta)\sinh(v_i^{(j)}-(2n-2j+1)\eta)}{\sinh(v_i^{(j)}-2\eta)}, & j \le n-2 \\ -\dfrac{e^{2\eta}\sinh(v_i^{(n-1)})}{\sinh(v_i^{(n-1)}-2\eta)} & j = n-1 \end{cases} \quad (74)$$

for $A_{2n}^{(2)}$ and $B_n^{(1)}$, respectively. The Eq.(50) for $A_{2n-1}^{(2)}, C_n^{(1)}$ and $D_n^{(1)}$ can be written as

$$\begin{aligned}\Gamma(u) = \Gamma_0(u) &= w_1^{(0)}(u^{(0)})\bar{\omega}_1^{(0)}(u^{(0)})\xi_1^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{A}^{(m_0)}(u) \\ &+ w_q^{(0)}(u^{(0)})\bar{\omega}_q^{(0)}(u^{(0)})\xi_3^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{C}^{(m_0)}(u) \\ &+ \sum_{j=0}^{n-2}\mu_j(u^{(j)})\nu_j(u^{(j)})w_1^{(j+1)}(u^{(j+1)})\bar{\omega}_1^{(j+1)}(u^{(j+1)})\xi_2^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{B}^{(m_j,m_{j+1})}(u) \\ &+ \sum_{j=0}^{n-2}\mu_j(u^{(j)})\nu_j(u^{(j)})w_{q-2(j+1)}^{(j+1)}(u^{(j+1)})\bar{\omega}_{q-2(j+1)}^{(j+1)}(u^{(j+1)})\xi_2^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\bar{\mathcal{B}}^{(m_j,m_{j+1})}(u)\end{aligned} \quad (75)$$

and

$$\begin{aligned}\Gamma(u) = \Gamma_0(u) &= w_1^{(0)}(u^{(0)})\bar{\omega}_1^{(0)}(u^{(0)})\xi_1^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{A}^{(m_0)}(u) \\ &+ w_q^{(0)}(u^{(0)})\bar{\omega}_q^{(0)}(u^{(0)})\xi_3^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{C}^{(m_0)}(u) \\ &+ \sum_{j=0}^{n-3}\mu_j(u^{(j)})\nu_j(u^{(j)})w_1^{(j+1)}(u^{(j+1)})\bar{\omega}_1^{(j+1)}(u^{(j+1)})\xi_2^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{B}^{(m_j,m_{j+1})}(u) \\ &+ \sum_{j=0}^{n-3}\mu_j(u^{(j)})\nu_j(u^{(j)})w_{q-2(j+1)}^{(j+1)}(u^{(j+1)})\bar{\omega}_{q-2(j+1)}^{(j+1)}(u^{(j+1)})\xi_2^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\bar{\mathcal{B}}^{(m_j,m_{j+1})}(u) \\ &+ \mu_{n-3}(u^{(n-3)})\nu_{n-3}(u^{(n-3)})w_1^{(n-1)}(u^{(n-2)})\bar{\omega}_1^{(n-1)}(u^{(n-2)})\xi_2^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\mathcal{B}^{(m_{n-2},m_{n-1})}(u) \\ &+ \mu_{n-3}(u^{(n-3)})\nu_{n-3}(u^{(n-3)})w_2^{(n-1)}(u^{(n-2)})\bar{\omega}_2^{(n-1)}(u^{(n-2)})\xi_2^{(0)}(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\})\bar{\mathcal{B}}^{(m_{n-2},m_{n-1})}(u),\end{aligned} \quad (76)$$

respectively. Here Eqs.(65,66,67) hold until $j=n-3$ for $A^{(2)}_{2n-1}, C^{(1)}_n$ and $j=n-4$ for $D^{(1)}_n$. The rest $\mathcal{B}$'s and $\bar{\mathcal{B}}$'s are

$$\mathcal{B}^{(m_{n-2},m_{n-1})}(u)=\prod_{k=1}^{m_{n-2}}\frac{a_1(\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)a_1(\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)}{e_2(\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)e_2(\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)} \times\prod_{l=1}^{m_{n-1}}\frac{a_1(-\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l)a_1(-\tilde{u}^{(n-1)}-\tilde{v}^{(n-1)}_l)}{e_1(-\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l)e_1(-\tilde{u}^{(n-1)}-\tilde{v}^{(n-1)}_l)},$$

$$\bar{\mathcal{B}}^{(m_{n-2},m_{n-1})}(u)=\prod_{k=1}^{m_{n-2}}\frac{e_1(\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)e_1(\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)}{e_2(\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)e_2(\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)} \times\prod_{l=1}^{m_{n-1}}\frac{a_1(\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l)a_1(\tilde{u}^{(n-1)}-\tilde{v}^{(n-1)}_l)}{e_1(\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l)e_1(\tilde{u}^{(n-1)}-\tilde{v}^{(n-1)}_l)}.$$

for $A^{(2)}_{2n-1}$,

$$\mathcal{B}^{(m_{n-2},m_{n-1})}(u)=\prod_{k=1}^{m_{n-2}}\frac{a_1(\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)a_1(\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)}{e_2(\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)e_2(\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)} \times\prod_{l=1}^{m_{n-1}}\frac{a(-\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l)b(\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l+4\eta)}{b(-\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l)a(\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l+4\eta)},$$

$$\bar{\mathcal{B}}^{(m_{n-2},m_{n-1})}(u)=\prod_{k=1}^{m_{n-2}}\frac{e_1(\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)e_1(\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)}{e_2(\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)e_2(\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)} \times\prod_{l=1}^{m_{n-1}}\frac{a(\tilde{u}^{(n-1)}-\tilde{v}^{(n-1)}_l)a(\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l-4\eta)}{b(\tilde{u}^{(n-1)}-\tilde{v}^{(n-1)}_l)a(\tilde{u}^{(n-1)}+\tilde{v}^{(n-1)}_l+4\eta)}.$$

for $C^{(1)}_n$ and

$$\mathcal{B}^{(m_{n-3},m_{n-2},m_{n-1})}(u)=\prod_{s=1}^{m_{n-3}}\frac{a_2(\tilde{u}^{(n-3)}+\tilde{v}^{(n-3)}_s)a_2(\tilde{u}^{(n-3)}-\tilde{v}^{(n-3)}_s)}{e_3(\tilde{u}^{(n-3)}+\tilde{v}^{(n-3)}_s)e_3(\tilde{u}^{(n-3)}-\tilde{v}^{(n-3)}_s)} \times\prod_{k=1}^{m_{n-2}}\frac{a(-\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)a(-\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)}{b(-\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)b(-\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)} \times\prod_{l=1}^{m_{n-1}}\frac{a(-\tilde{u}^{(n-2)}+\tilde{v}^{(n-1)}_l)a(-\tilde{u}^{(n-2)}-\tilde{v}^{(n-1)}_l)}{b(-\tilde{u}^{(n-2)}+\tilde{v}^{(n-1)}_l)b(-\tilde{u}^{(n-2)}-\tilde{v}^{(n-1)}_l)},$$

$$\mathcal{B}^{(m_{n-2},m_{n-1})}(u)=\prod_{k=1}^{m_{n-2}}\frac{a(-\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)a(-\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)}{b(-\tilde{u}^{(n-2)}+\tilde{v}^{(n-2)}_k)b(-\tilde{u}^{(n-2)}-\tilde{v}^{(n-2)}_k)} \times\prod_{l=1}^{m_{n-1}}\frac{a(\tilde{u}^{(n-2)}-\tilde{v}^{(n-1)}_l)a(\tilde{u}^{(n-2)}+\tilde{v}^{(n-1)}_l)}{b(\tilde{u}^{(n-2)}-\tilde{v}^{(n-1)}_l)b(\tilde{u}^{(n-2)}+\tilde{v}^{(n-1)}_l)},$$

$$\bar{\mathcal{B}}^{(m_{n-3},m_{n-2},m_{n-1})}(u) = \prod_{s=1}^{m_{n-3}} \frac{e_2(\tilde{u}^{(n-3)}+\tilde{v}_s^{(n-3)})e_2(\tilde{u}^{(n-3)}-\tilde{v}_s^{(n-3)})}{e_3(\tilde{u}^{(n-3)}+\tilde{v}_s^{(n-3)})e_3(\tilde{u}^{(n-3)}-\tilde{v}_s^{(n-3)})}$$
$$\times \prod_{k=1}^{m_{n-2}} \frac{a(\tilde{u}^{(n-2)}+\tilde{v}_k^{(n-2)})a(\tilde{u}^{(n-2)}-\tilde{v}_k^{(n-2)})}{b(\tilde{u}^{(n-2)}+\tilde{v}_k^{(n-2)})b(\tilde{u}^{(n-2)}-\tilde{v}_k^{(n-2)})}$$
$$\times \prod_{l=1}^{m_{n-1}} \frac{a(\tilde{u}^{(n-2)}+\tilde{v}_l^{(n-1)})a(\tilde{u}^{(n-2)}-\tilde{v}_l^{(n-1)})}{b(\tilde{u}^{(n-2)}+\tilde{v}_l^{(n-1)})b(\tilde{u}^{(n-2)}-\tilde{v}_l^{(n-1)})},$$

$$\bar{\mathcal{B}}^{(m_{n-2},m_{n-1})}(u) = \prod_{k=1}^{m_{n-2}} \frac{a(\tilde{u}^{(n-2)}+\tilde{v}_k^{(n-2)})a(\tilde{u}^{(n-2)}-\tilde{v}_k^{(n-2)})}{b(\tilde{u}^{(n-2)}+\tilde{v}_k^{(n-2)})b(\tilde{u}^{(n-2)}-\tilde{v}_k^{(n-2)})}$$
$$\times \prod_{l=1}^{m_{n-1}} \frac{a(-\tilde{u}^{(n-2)}-\tilde{v}_l^{(n-1)})a(-\tilde{u}^{(n-2)}+\tilde{v}_l^{(n-1)})}{b(-\tilde{u}^{(n-2)}-\tilde{v}_l^{(n-1)})b(-\tilde{u}^{(n-2)}+\tilde{v}_l^{(n-1)})}, \tag{77}$$

for $D_n^{(1)}$, where $a(u)=\sinh(\frac{u}{2}-4\eta), b(u)=\sinh(\frac{u}{2})$ for $C_n^{(1)}$ and $a(u)=\sinh(\frac{u}{2}-2\eta), b(u)=\sinh(\frac{u}{2})$ for $D_n^{(1)}$. The other Bethe equations are

$$\prod_{k=1}^{m_{n-3}} \frac{a_2(\tilde{v}_i^{(n-2)}+\tilde{v}_k^{(n-3)})a_2(\tilde{v}_i^{(n-2)}-\tilde{v}_k^{(n-3)})}{b_2(\tilde{v}_i^{(n-2)}+\tilde{v}_k^{(n-3)})b_2(\tilde{v}_i^{(n-2)}-\tilde{v}_k^{(n-3)})}$$
$$\times \prod_{l=1}^{m_{n-1}} \frac{e_1(-\tilde{v}_i^{(n-2)}+\tilde{v}_l^{(n-1)})e_1(-\tilde{v}_i^{(n-2)}-\tilde{v}_l^{(n-1)})}{a_1(-\tilde{v}_i^{(n-2)}+\tilde{v}_l^{(n-1)})a_1(-\tilde{v}_i^{(n-2)}-\tilde{v}_l^{(n-1)})}$$
$$\times \prod_{s=1\neq i}^{m_{n-2}} \frac{a_2(-\tilde{v}_i^{(n-2)}+\tilde{v}_s^{(n-2)})a_2(-\tilde{v}_i^{(n-2)}-\tilde{v}_s^{(n-2)})}{b_2(-\tilde{v}_i^{(n-2)}+\tilde{v}_s^{(n-2)})b_2(-\tilde{v}_i^{(n-2)}-\tilde{v}_s^{(n-2)})} \frac{e_2(\tilde{v}_i^{(n-2)}+\tilde{v}_s^{(n-2)})e_2(\tilde{v}_i^{(n-2)}-\tilde{v}_s^{(n-2)})}{a_1(\tilde{v}_i^{(n-2)}+\tilde{v}_s^{(n-2)})a_1(\tilde{v}_i^{(n-2)}-\tilde{v}_s^{(n-2)})}$$
$$= -\frac{w_1^{(n-1)}(\tilde{v}_i^{(n-2)})a_1(2\tilde{v}_i^{(n-2)}))}{\beta_{n-1}(v_i^{(n-2)})} \frac{\bar{\omega}^{(n-2)}(v_i^{(n-2)})\bar{\omega}_1^{(n-1)}(\tilde{v}_i^{(n-2)})}{\bar{\omega}_1^{(n-2)}(v_i^{(n-2)})}, \quad (i=1,\cdots,m_{n-2}) \tag{78}$$

$$\prod_{k=1}^{m_{n-2}} \frac{a_1(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-2)})a_1(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-2)})}{e_1(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-2)})e_1(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-2)})}$$
$$\times \prod_{l=1\neq i}^{m_{n-1}} \frac{a_1(-\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})a_1(-\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})}{a_1(\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})a_1(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})}$$
$$= -\frac{\bar{\omega}_2^{(n-1)}(v_i^{(n-1)})}{\bar{\omega}_1^{(n-1)}(v_i^{(n-1)})\beta_n(v_i^{(n)})} \quad (i=1,\cdots,m_{n-1}) \tag{79}$$

for $A_{2n-1}^{(2)}$,

$$\prod_{k=1}^{m_{n-3}} \frac{a_2(\tilde{v}_i^{(n-2)}+\tilde{v}_k^{(n-3)})a_2(\tilde{v}_i^{(n-2)}-\tilde{v}_k^{(n-3)})}{b_2(\tilde{v}_i^{(n-2)}+\tilde{v}_k^{(n-3)})b_2(\tilde{v}_i^{(n-2)}-\tilde{v}_k^{(n-3)})}$$

$$\times \prod_{l=1}^{m_{n-1}} \frac{b(-\tilde{v}_i^{(n-2)}+\tilde{v}_l^{(n-1)})a(\tilde{v}_i^{(n-2)}+\tilde{v}_l^{(n-1)}+4\eta)}{a(-\tilde{v}_i^{(n-2)}+\tilde{v}_l^{(n-1)})b(\tilde{v}_i^{(n-2)}+\tilde{v}_l^{(n-1)}+4\eta)}$$

$$\times \prod_{s=1\neq i}^{m_{n-2}} \frac{a_2(-\tilde{v}_i^{(n-2)}+\tilde{v}_s^{(n-2)})a_2(-\tilde{v}_i^{(n-2)}-\tilde{v}_s^{(n-2)})}{b_2(-\tilde{v}_i^{(n-2)}+\tilde{v}_s^{(n-2)})b_2(-\tilde{v}_i^{(n-2)}-\tilde{v}_s^{(n-2)})} \frac{e_2(\tilde{v}_i^{(n-2)}+\tilde{v}_s^{(n-2)})e_2(\tilde{v}_i^{(n-2)}-\tilde{v}_s^{(n-2)})}{a_1(\tilde{v}_i^{(n-2)}+\tilde{v}_s^{(n-2)})a_1(\tilde{v}_i^{(n-2)}-\tilde{v}_s^{(n-2)})}$$

$$= -\frac{w_1^{(n-1)}(\tilde{v}_i^{(n-2)})a_1(2\tilde{v}_i^{(n-2)}))}{\beta_{n-1}(v_i^{(n-2)})} \frac{\bar{\omega}^{(n-2)}(v_i^{(n-2)})\bar{\omega}_1^{(n-1)}(\tilde{v}_i^{(n-2)})}{\bar{\omega}_1^{(n-2)}(v_i^{(n-2)})}, \quad (i=1,\cdots,m_{n-2}) \qquad (80)$$

$$\prod_{k=1}^{m_{n-2}} \frac{a_1(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-2)})a_1(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-2)})}{e_1(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-2)})e_1(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-2)})}$$

$$\times \prod_{l=1\neq i}^{m_{n-1}} \frac{a(-\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})b(-\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)}-4\eta)}{a(\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})a(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)}-4\eta)}$$

$$= -\frac{\bar{\omega}_2^{(n-1)}(v_i^{(n-1)})}{\bar{\omega}_1^{(n-1)}(v_i^{(n-1)})\beta_n(v_i^{(n)})} \quad (i=1,\cdots,m_{n-1}) \qquad (81)$$

for $C_n^{(1)}$ and for $D_n^{(1)}$,

$$\prod_{k=1}^{m_{n-4}} \frac{a_3(\tilde{v}_i^{(n-3)}+\tilde{v}_k^{(n-4)})a_3(\tilde{v}_i^{(n-3)}-\tilde{v}_k^{(n-4)})}{b_3(\tilde{v}_i^{(n-3)}+\tilde{v}_k^{(n-4)})b_3(\tilde{v}_i^{(n-3)}-\tilde{v}_k^{(n-4)})}$$

$$\times \prod_{l=1}^{m_{n-2}} \frac{b(-\tilde{v}_i^{(n-3)}+\tilde{v}_l^{(n-2)})b(-\tilde{v}_i^{(n-3)}-\tilde{v}_l^{(n-2)})}{a(-\tilde{v}_i^{(n-3)}+\tilde{v}_l^{(n-2)})a(-\tilde{v}_i^{(n-3)}-\tilde{v}_l^{(n-2)})}$$

$$\times \prod_{t=1}^{m_{n-1}} \frac{b(-\tilde{v}_i^{(n-3)}+\tilde{v}_t^{(n-1)})b(-\tilde{v}_i^{(n-3)}-\tilde{v}_t^{(n-1)})}{a(-\tilde{v}_i^{(n-3)}+\tilde{v}_t^{(n-1)})a(-\tilde{v}_i^{(n-3)}-\tilde{v}_t^{(n-1)})}$$

$$\times \prod_{s=1\neq i}^{m_{n-3}} \frac{a_3(-\tilde{v}_i^{(n-3)}+\tilde{v}_s^{(n-3)})a_3(-\tilde{v}_i^{(n-3)}-\tilde{v}_s^{(n-3)})}{b_3(-\tilde{v}_i^{(n-3)}+\tilde{v}_s^{(n-3)})b_3(-\tilde{v}_i^{(n-3)}-\tilde{v}_s^{(n-3)})} \frac{e_3(\tilde{v}_i^{(n-3)}+\tilde{v}_s^{(n-3)})e_3(\tilde{v}_i^{(n-3)}-\tilde{v}_s^{(n-3)})}{a_2(\tilde{v}_i^{(n-3)}+\tilde{v}_s^{(n-3)})a_2(\tilde{v}_i^{(n-3)}-\tilde{v}_s^{(n-3)})}$$

$$= -\frac{[W_1(\tilde{v}_i^{(n-3)})]^2 a_2(2\tilde{v}_i^{(n-3)}))}{\beta_{n-2}(v_i^{(n-3)})} \frac{\bar{\omega}^{(n-3)}(v_i^{(n-3)})[\Omega_1(\tilde{v}_i^{(n-3)})]^2}{\bar{\omega}_1^{(n-3)}(v_i^{(n-3)})}, \quad (i=1,\cdots,m_{n-3}) \qquad (82)$$

$$\prod_{k=1}^{m_{n-3}} \frac{a(\tilde{v}_i^{(n-2)}+\tilde{v}_k^{(n-3)})a(\tilde{v}_i^{(n-2)}-\tilde{v}_k^{(n-3)})}{b(\tilde{v}_i^{(n-2)}+\tilde{v}_k^{(n-3)})b(\tilde{v}_i^{(n-2)}-\tilde{v}_k^{(n-3)})}$$

$$\times \prod_{l=1\neq i}^{m_{n-2}} \frac{a(-\tilde{v}_i^{(n-2)}+\tilde{v}_l^{(n-2)})a(-\tilde{v}_i^{(n-2)}-\tilde{v}_l^{(n-2)})}{a(\tilde{v}_i^{(n-2)}-\tilde{v}_l^{(n-2)})a(\tilde{v}_i^{(n-2)}+\tilde{v}_l^{(n-2)})}$$

$$= -\frac{\Omega_2(v_i^{(n-2)})}{\Omega_1(v_i^{(n-2)})\beta_{n-1}(v_i^{(n-2)})}. \quad (i=1,\cdots,m_{n-2}) \qquad (83)$$

$$\prod_{k=1}^{m_{n-3}} \frac{a(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-3)})a(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-3)})}{b(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-3)})b(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-3)})}$$

$$\times \prod_{l=1\neq i}^{m_{n-1}} \frac{a(-\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})a(-\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})}{a(\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})a(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})}$$

$$= -\frac{\Omega_2(v_i^{(n-1)})}{\Omega_1(v_i^{(n-1)})\beta_{n-1}(v_i^{(n-1)})}. \ (i=1,\cdots,m_{n-1}) \quad (84)$$

The coefficients are

$$\bar{\omega}_1^{(j)}(u^{(j)}) = 1, \quad \bar{\omega}^{(j)}(u^{(j)}) = \frac{e^{2\eta}\sinh(u^{(j)})}{\sinh(u^{(j)}-2\eta)},$$
$$\bar{\omega}_{2(n-j)}^{(j)}(u^{(j)}) = \frac{e^{2(2(n-j))\eta}\sinh(u^{(j)})\cosh(u^{(j)}-2((n-j)-1)\eta)}{\sinh(u^{(j)}-2(2(n-j)-1)\eta)\cosh(u^{(j)}-2(n-j)\eta)}. \quad (85)$$

$$w_1^{(j)}(u^{(j)}) = \frac{\sinh(u^{(j)}-2(2(n-j))\eta)\cosh(u^{(j)}-2((n-j)+1)\eta)}{\sinh(u^{(j)}-2\eta)\cosh(u^{(j)}-2(n-j)\eta)},$$
$$w^{(j)}(u^{(j)}) = \frac{e^{-2\eta}\sinh(u^{(j)}-2(2(n-j))\eta)}{\sinh(u^{(j)}-2(2(n-j)-1)\eta)}, \quad w_{2(n-j)}^{(j)}(u^{(j)}) = e^{-2(2(n-j))\eta}, \quad (86)$$

$$\beta_{j+1}(v_i^{(j)}) = \begin{cases} -\dfrac{2e^{2\eta}\sinh(v_i^{(j)})\sinh(v_i^{(j)}-2(2(n-j)-1)\eta)\cosh(v_i^{(j)}-2((n-j)+1)\eta)}{\sinh(v_i^{(j)}-2\eta)}, & j \leq n-2 \\ -\dfrac{e^{4\eta}\sinh(2v_i^{(n-1)})}{\sinh(2v_i^{(n-1)}-4\eta)}. & j = n-1 \end{cases} \quad (87)$$

for $A_{2n-1}^{(2)}$,

$$\bar{\omega}_1^{(j)}(u^{(j)}) = 1, \quad \bar{\omega}^{(j)}(u^{(j)}) = \frac{e^{2\eta}\sinh(u^{(j)})}{\sinh(u^{(j)}-2\eta)},$$
$$\bar{\omega}_{2(n-j)}^{(j)}(u^{(j)}) = \frac{e^{2(2(n-j))\eta}\sinh(u^{(j)})\sinh(u^{(j)}-2((n-j)+2)\eta)}{\sinh(u^{(j)}-2(2(n-j)+1)\eta)\sinh(u^{(j)}-2((n-j)+1)\eta)}. \quad (88)$$

$$w_1^{(j)}(u^{(j)}) = \frac{\sinh(u^{(j)}-2(2(n-j)+2)\eta)\sinh(u^{(j)}-2(n-j)\eta)}{\sinh(u^{(j)}-2\eta)\sinh(u^{(j)}-2((n-j)+1)\eta)},$$
$$w^{(j)}(u^{(j)}) = \frac{e^{-2\eta}\sinh(u^{(j)}-2(2(n-j)+2)\eta)}{\sinh(u^{(j)}-2(2(n-j)+1)\eta)}, \quad w_{2(n-j)}^{(j)}(u^{(j)}) = e^{-2(2(n-j))\eta}, \quad (89)$$

$$\beta_{j+1}(v_i^{(j)}) = \begin{cases} -\dfrac{2e^{2\eta}\sinh(v_i^{(j)})\sinh(v_i^{(j)}-2(2(n-j)+1)\eta)\sinh(v_i^{(j)}-2(n-j)\eta)}{\sinh(v_i^{(j)}-2\eta)}, & j \leq n-2 \\ -\dfrac{e^{4\eta}\sinh(v_i^{(n-1)})}{\sinh(v_i^{(n-1)}-4\eta)}. & j = n-1 \end{cases} \quad (90)$$

for $C_n^{(1)}$ and

$$\bar{\omega}_1^{(j)}(u^{(j)}) = 1, \quad \bar{\omega}^{(j)}(u^{(j)}) = \frac{e^{2\eta}\sinh(u^{(j)})}{\sinh(u^{(j)} - 2\eta)},$$
$$\bar{\omega}_{2(n-j)}^{(j)}(u^{(j)}) = \frac{e^{2(2(n-j)-2)\eta}\sinh(u^{(j)})\sinh(u^{(j)} - 2((n-j)-2)\eta)}{\sinh(u^{(j)} - 2(2(n-j)-3)\eta)\sinh(u^{(j)} - 2((n-j)-1)\eta)}$$
$$(j \leq n-3) \quad (91)$$

$$w_1^{(j)}(u^{(j)}) = \frac{\sinh(u^{(j)} - 2(2(n-j)-2)\eta)\sinh(u^{(j)} - 2(n-j)\eta)}{\sinh(u^{(j)} - 2\eta)\sinh(u^{(j)} - 2((n-j)-3)\eta)},$$
$$w^{(j)}(u^{(j)}) = \frac{e^{-2\eta}\sinh(u^{(j)} - 2(2(n-j)-2)\eta)}{\sinh(u^{(j)} - 2(2(n-j)-3)\eta)}, \quad w_{2(n-j)}^{(j)}(u^{(j)}) = e^{-2(2(n-j)-2)\eta},$$
$$(j \leq n-3) \quad (92)$$

$$w_1^{(n-2)}(u^{(n-2)})\bar{\omega}_1^{(n-2)}(u^{(n-2)}) = 2[W_1(u^{(n-2)})]^2[\Omega_1(u^{(n-2)})]^2,$$
$$w_1^{(n-1)}(u^{(n-2)})\bar{\omega}_1^{(n-1)}(u^{(n-2)}) = 2W_1(u^{(n-2)})W_2(u^{(n-2)})\Omega_1(u^{(n-2)})\Omega_2(u^{(n-2)}),$$
$$w_4^{(n-2)}(u^{(n-2)})\bar{\omega}_4^{(n-2)}(u^{(n-2)}) = 2[W_2(u^{(n-2)})]^2[\Omega_2(u^{(n-2)})]^2,$$
$$w_2^{(n-1)}(u^{(n-2)})\bar{\omega}_2^{(n-1)}(u^{(n-2)}) = 2W_1(u^{(n-2)})W_2(u^{(n-2)})\Omega_1(u^{(n-2)})\Omega_2(u^{(n-2)})$$

with

$$\Omega_1(u) = 1, \qquad \Omega_2(u) = \frac{e^{2\eta}\sinh(u)}{\sinh(u-2\eta)},$$
$$W_1(u) = \frac{\sinh(u-4\eta)}{\sinh(u-2\eta)}, \qquad W_2(u) = e^{-2\eta}. \quad (93)$$

$$\beta_{j+1}(v_i^{(j)}) = \begin{cases} -\dfrac{2e^{2\eta}\sinh(v_i^{(j)})\sinh(v_i^{(j)} - 2(2(n-j)-3)\eta)\sinh(v_i^{(j)} - 2(n-j)\eta)}{\sinh(v_i^{(j)} - 2\eta)}, & j \leq n-3 \\ -\dfrac{e^{2\eta}\sinh(v_i^{(n-j)})}{\sinh(v_i^{(n-j)} - 2\eta)}. & j = n-2, n-1 \end{cases}$$
$$(94)$$

for $D_n^{(1)}$. Up to now, we have gotten the whole eigenvalues and the Bethe equations of transfer matrix for the $A_{2n}^{(2)}, A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ vertex model with trivial K matrix. We have checked that our results can agree with that obtained by the analytical Bethe ansatz[18].

3 Conclusions

In the framework of algebraic Bethe ansatz, we solve the $A_{2n}^{(2)}, A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ vertex model with trivial K matrix and find that our results agree with that obtained by analytical Bethe ansatz method [18]. We can further consider these models (except for $A_{2n}^{(2)}$) with the non-trivial K matrices.

Here some open vertex models, such as $D_n^{(2)}$, are not considered due to their complicated R matrices. For these models, we need to generalize the ABA further. Additionally, for the ABA has been generalized to the spin chain with non-diagonal reflecting matrices[24]-[27], it is interesting problem on how to apply the method to other models with higher rank algebras.

Acknowledgements

This work is supported by Program for NCET under Grant No. NCET-04-0929 and NNSFC under Grant No.10575080.

A R matric elements

The R matric elements are

$$\begin{aligned}
a_n(u) &= 2\sinh(\frac{u}{2}-2\eta)\sinh(\frac{u}{2}-\kappa\eta),\\
b_n(u) &= 2\sinh(\frac{u}{2})\sinh(\frac{u}{2}-\kappa\eta),\\
c_n(u,i) &= d_n(u,i,\bar{i})-2e^{-\frac{u}{2}}\sinh(2\eta)\sinh(\frac{u}{2}-\kappa\eta),\\
\bar{c}_n(u,i) &= \bar{d}_n(u,i,\bar{i})-2e^{\frac{u}{2}}\sinh(2\eta)\sinh(\frac{u}{2}-\kappa\eta),\\
d_n(u,i,j) &= 2\epsilon_i\epsilon_j e^{-\frac{u}{2}+(\kappa+2(\tilde{i}-\tilde{j}))\eta}\sinh(2\eta)\sinh(\frac{u}{2}),\\
\bar{d}_n(u,i,j) &= 2\epsilon_i\epsilon_j e^{\frac{u}{2}+(2(\tilde{i}-\tilde{j})-\kappa)\eta}\sinh(2\eta)\sinh(\frac{u}{2}),\\
e_n(u) &= 2\sinh(\frac{u}{2})\sinh(\frac{u}{2}-(\kappa-2)\eta),\\
f_n(u) &= b_n(u)+2\sinh(2\eta)\sinh(\kappa\eta),\\
g_n(u) &= -2e^{-\frac{u}{2}}\sinh(2\eta)\sinh(\frac{u}{2}-\kappa\eta),\\
\bar{g}_n(u) &= -2e^{\frac{u}{2}}\sinh(2\eta)\sinh(\frac{u}{2}-\kappa\eta)
\end{aligned} \tag{A.1}$$

for $B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ and

$$a_n(u) = 2\sinh(\frac{u}{2}-2\eta)\cosh(\frac{u}{2}-\kappa\eta),$$

$$
\begin{aligned}
b_n(u) &= 2\sinh(\frac{u}{2})\cosh(\frac{u}{2}-\kappa\eta),\\
c_n(u,i) &= d_n(u,i,\bar{i}) - 2e^{-\frac{u}{2}}\sinh(2\eta)\cosh(\frac{u}{2}-\kappa\eta),\\
\bar{c}_n(u,i) &= \bar{d}_n(u,i,\bar{i}) - 2e^{\frac{u}{2}}\sinh(2\eta)\cosh(\frac{u}{2}-\kappa\eta),\\
d_n(u,i,j) &= -2\epsilon_i\epsilon_j e^{-\frac{u}{2}+(\kappa+2(\tilde{i}-\tilde{j}))\eta}\sinh(2\eta)\sinh(\frac{u}{2}),\\
\bar{d}_n(u,i,j) &= 2\epsilon_i\epsilon_j e^{\frac{u}{2}+(2(\tilde{i}-\tilde{j})-\kappa)\eta}\sinh(2\eta)\sinh(\frac{u}{2}),\\
e_n(u) &= 2\sinh(\frac{u}{2})\cosh(\frac{u}{2}-(\kappa-2)\eta),\\
f_n(u) &= b_n(u) - 2\sinh(2\eta)\cosh(\kappa\eta),\\
g_n(u) &= -2e^{-\frac{u}{2}}\sinh(2\eta)\cosh(\frac{u}{2}-\kappa\eta),\\
\bar{g}_n(u) &= -2e^{\frac{u}{2}}\sinh(2\eta)\cosh(\frac{u}{2}-\kappa\eta) \qquad \text{(A.2)}
\end{aligned}
$$

for $A^{(2)}_{2n}, A^{(2)}_{2n-1}$. Where $\epsilon_i = 1$ for $A^{(2)}_{2n}, B^{(1)}_n, D^{(1)}_n$ models, $\epsilon_i = 1$ if $1 \le i \le n$ and $\epsilon_i = -1$ if $n+1 \le i \le 2n$ for $A^{(2)}_{2n-1}, C^{(1)}_n$ models.

B Derivations of diagonal operators acting on eigenvector

Acting the diagonal operators $x(u) = A(u), \tilde{A}_{aa}(u), \tilde{A}_2(u)$ on the m-particle state and having carried out a very involved analysis as that in Ref.[13], we can obtain the following expression

$$
\begin{aligned}
x(u)|\Upsilon_m(v_1,\cdots,v_m)\rangle &= |\Psi_x(u,\{v_m\})\rangle\\
&+\sum_{i=1}^{m} h^x_1(u,v_i,d)|\Psi^{(1)}_{m-1}(u,v_i;\{v_m\})_{dd}\rangle\\
&+\sum_{i=1}^{m} h^x_2(u,v_i,d)|\Psi^{(2)}_{m-1}(u,v_i;\{v_m\})_{dd}\rangle\\
&+\sum_{i=1}^{m} h^x_3(u,v_i,\bar{\alpha}_x)|\Psi^{(3)}_{m-1}(u,v_i;\{v_m\})_{\alpha_x\alpha_x}\rangle\\
&+\sum_{i=1}^{m} h^x_4(u,v_i,\bar{\alpha}_x)|\Psi^{(4)}_{m-1}(u,v_i;\{v_m\})_{\alpha_x\alpha_x}\rangle\\
&+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \delta_{\bar{d}_1 e_2} H^x_{1,d_1}(u,v_i,v_j)|\Psi^{(5)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1 e_2}\rangle\\
&+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} H^x_{2,d_1}(u,v_i,v_j)^{fe}_{c_2 d_1}|\Psi^{(6)}_{m-2}(u,v_i,v_j;\{v_m\})^{ef}_{d_1 c_2}\rangle
\end{aligned}
$$

$$+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} H^x_{3,d_1}(u,v_i,v_j)|\Psi^{(7)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1d_1}\rangle$$
$$+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} H^x_{4,d_1}(u,v_i,v_j)^{de}_{fd_1}|\Psi^{(8)}_{m-2}(u,v_i,v_j;\{v_m\})^{ed}_{d_1f}\rangle, \tag{B.1}$$

where when $x = A, \tilde{A}_{aa}, \tilde{A}_2$,

$$\begin{aligned}
&h^x_1(u,v_i,d) = a^1_2(u,v_i), R^A_1(u,v_i)^{da}_{ad}, a^3_2(u,v_i),\\
&h^x_2(u,v_i,d) = a^1_3(u,v_i), R^A_2(u,v_i)^{da}_{ad}, a^3_3(u,v_i),\\
&h^x_3(u,v_i,\bar{\alpha}_x) = 0, R^A_3(u,v_i,\bar{a}), a^3_4(u,v_i,\bar{d}),\\
&h^x_4(u,v_i,\bar{\alpha}_x) = 0, R^A_4(u,v_i,\bar{a}), a^3_5(u,v_i,\bar{d}),\\
&\alpha_x = 0, a, d,
\end{aligned}$$

$$|\Psi_x(u,\{v_m\})\rangle = \begin{cases} \omega_1(u)\Lambda^m_1(u;v_1,\cdots,v_m)|\Upsilon_m(v_1,\cdots,v_m)\rangle,\\ \Phi^{d_1\cdots d_m}_m(v_1,\cdots,v_m)[\tilde{T}^m(u;\{v_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}]_{aa}F^{b_1\cdots b_m}|0\rangle,\\ \omega_{2n+1}(u)\Lambda^m_3(u;v_1,\cdots,v_m)|\Upsilon_m(v_1,\cdots,v_m)\rangle, \end{cases} \tag{B.2}$$

respectively, and we denote

$$\begin{aligned}|\Psi^{(1)}_{m-1}(u,v_i;\{v_m\})_{fd_1}\rangle &= B_f(u)|\Phi^{d_2\cdots d_m}_{m-1}(v_1,\cdots,\check{v}_i,\cdots,v_m)\\ &\times S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\Lambda^{m-1}_1(v_i;\{\check{v}_i\})\omega_1(v_i)F^{b_1\cdots b_m}|0\rangle,\end{aligned} \tag{B.3}$$

$$\begin{aligned}|\Psi^{(2)}_{m-1}(u,v_i;\{v_m\})_{fd}\rangle &= B_f(u)\Phi^{e_2\cdots e_m}_{m-1}(v_1,\cdots,\check{v}_i,\cdots,v_m)\\ &\times[\tilde{T}^{m-1}(v_i;\{\check{v}_i\})^{e_2\cdots e_m}_{d'_2\cdots d'_m}]_{dd'_1}S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})F^{b_1\cdots b_m}|0\rangle,\end{aligned} \tag{B.4}$$

$$\begin{aligned}|\Psi^{(3)}_{m-1}(u,v_i;\{v_m\})_{ab}\rangle &= E_a(u)\Phi^{d_2\cdots d_m}_{m-1}(v_1,\cdots,\check{v}_i,\cdots,v_m)\\ &\times S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\Lambda^{m-1}_1(v_i;\{\check{v}_i\})\omega_1(v_i)\delta_{\bar{b}d_1}F^{b_1\cdots b_m}|0\rangle,\end{aligned} \tag{B.5}$$

$$\begin{aligned}|\Psi^{(4)}_{m-1}(u,v_i;\{v_m\})_{ab}\rangle &= E_a(u)\Phi^{e_2\cdots e_m}_{m-1}(v_1,\cdots,\check{v}_i,\cdots,v_m)\\ &\times[\tilde{T}^{m-1}(v_i;\{\check{v}_i\})^{e_2\cdots e_m}_{d'_2\cdots d'_m}]_{\bar{b}d'_1}S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})F^{b_1\cdots b_m}|0\rangle,\end{aligned} \tag{B.6}$$

$$\begin{aligned}|\Psi^{(5)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1e_2}\rangle &= F(u)\Phi^{e_3\cdots e_m}_{m-2}(v_1,\cdots,\check{v}_i,\cdots,\check{v}_j,\cdots,v_m)\\ &\times S^{e_2\cdots e_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\Lambda^{m-2}_1(v_i;\{\check{v}_i,\check{v}_j\})\\ &\times\Lambda^{m-2}_1(v_j;\{\check{v}_i,\check{v}_j\})A(v_i)A(v_j)F^{b_1\cdots b_m}|0\rangle,\end{aligned} \tag{B.7}$$

$$\begin{aligned}|\Psi^{(6)}_{m-2}(u,v_i,v_j;\{v_m\})^{ef}_{d_1c_2}\rangle &= F(u)\Phi^{e_3\cdots e_m}_{m-2}(v_1,\cdots,\check{v}_i,\cdots,\check{v}_j,\cdots,v_m)\\ &\times[\tilde{T}^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c_3\cdots c_m}]_{\bar{f}e}S^{c_2\cdots c_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\\ &\times\Lambda^{m-2}_1(v_j;\{\check{v}_i,\check{v}_j\})A(v_j)F^{b_1\cdots b_m}|0\rangle,\end{aligned} \tag{B.8}$$

$$|\Psi^{(7)}_{m-2}(u,v_i,v_j;\{v_m\})_{fd_1}\rangle = F(u)\Phi^{e_3\cdots e_m}_{m-2}(v_1,\cdots,\check{v}_i,\cdots,\check{v}_j,\cdots,v_m)$$
$$\times[\tilde{T}^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c_3\cdots c_m}]_{\bar{f}c_2}S^{c_2\cdots c_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})$$
$$\times\Lambda^{m-2}_1(v_i;\{\check{v}_i,\check{v}_j\})A(v_i)F^{b_1\cdots b_m}|0\rangle, \tag{B.9}$$

$$|\Psi^{(8)}_{m-2}(u,v_i,v_j;\{v_m\})^{ed}_{d_1f}\rangle = F(u)\Phi^{e_3\cdots e_m}_{m-2}(v_1,\cdots,\check{v}_i,\cdots,\check{v}_j,\cdots,v_m)$$
$$\times[\tilde{T}^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{a_3\cdots a_m}]_{\bar{d}e}[\tilde{T}^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{a_3\cdots a_m}_{c_3\cdots c_m}]_{fc_2}$$
$$\times S^{c_2\cdots c_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})F^{b_1\cdots b_m}|0\rangle. \tag{B.10}$$

The explicit expressions of $H^x_{l,d_1}(u,v_i,v_j), l=1,2,3,4$ are listed as below

$$H^A_{1,d_1}(u,v_i,v_j) = a^1_4(u,v_i,\bar{d}_1)(c^1_5(v_i,v_j)+c^1_7(v_i,v_j))$$
$$+a^1_5(u,v_i)(c^2_4(v_i,v_j,\bar{d}_1)+c^2_6(v_i,v_j,\bar{d}_1))+b^1_2(u,v_i)g_1(v_i,v_j,d_1)$$
$$+a^1_1(u,v_i)a^1_2(u,v_j)g_1(u,v_i,d)\hat{r}(v_i-u)^{\bar{d}d}_{d_1\bar{d}_1}, \tag{B.11}$$

$$H^A_{2,d_1}(u,v_i,v_j)^{fe}_{c_2d_1} = a^1_4(u,v_i,\bar{d}_1)(c^1_6(v_i,v_j)+c^1_9(v_i,v_j))\delta_{d_1f}$$
$$+a^1_5(u,v_i)(R^C_3(v_i,v_j)^{fe}_{c_2d_1}+R^C_4(v_i,v_j)^{fe}_{c_2d_1})$$
$$+b^1_3(u,v_i)g_1(v_i,v_j,d_1)\delta_{d_1\bar{c}_2}+a^1_1(u,v_i)a^1_2(u,v_j)g_2(u,v_i,f)\hat{r}(v_i-u)^{fe}_{c_2d_1}, \tag{B.12}$$

$$H^A_{3,d_1}(u,v_i,v_j) = a^1_4(u,v_i,\bar{d}_1)c^1_8(v_i,v_j)+a^1_5(u,v_i)c^2_7(v_i,v_j,\bar{d}_1)$$
$$+b^1_2(u,v_i)g_2(v_i,v_j,d_1)+a^1_1(u,v_i)a^1_3(u,v_j)g_1(u,v_i,d)\hat{r}(v_i-u)^{d\bar{d}}_{\bar{d}_1d_1}, \tag{B.13}$$

$$H^A_{4,d_1}(u,v_i,v_j)^{de}_{fd_1} = a^1_4(u,v_i,\bar{d}_1)c^1_{10}(v_i,v_j)\delta_{d_1d}+a^1_5(u,v_i)R^C_5(v_i,v_j)^{de}_{fd_1}$$
$$+b^1_3(u,v_i)g_2(v_i,v_j,d_1)\delta_{\bar{d}_1f}+a^1_1(u,v_i)a^1_3(u,v_j)g_2(u,v_i,d)\hat{r}(v_i-u)^{de}_{fd_1}, \tag{B.14}$$

$$H^{\tilde{A}_{aa}}_{1,d_1}(u,v_i,v_j) = R^A_5(u,v_i)^{ad_1}_{d_1a}(c^1_5(v_i,v_j)+c^1_7(v_i,v_j))+b^2_2(u,v_i)g_1(v_i,v_j,d_1)$$
$$+R^A_6(u,v_i)(c^2_4(v_i,v_j,\bar{d}_1)+c^2_6(v_i,v_j,\bar{d}_1))+\tilde{r}(u+v_i)^{ad}_{da}\bar{r}(u-v_i)^{ad_1}_{d_1a}R^A_3(u,v_j,\bar{d}_1)e^1_1(v_i,u,d)$$
$$+\tilde{r}(u+v_i)^{ae}_{dg}\bar{r}(u-v_i)^{gf}_{d_1a}g_1(u,v_i,h)R^A_1(u,v_j)^{d\bar{e}}_{\bar{d}_1f}\hat{r}(v_i-u)^{h\bar{h}}_{\bar{e}e}, \tag{B.15}$$

$$H^{\tilde{A}_{aa}}_{2,d_1}(u,v_i,v_j)^{fe}_{c_2d_1} = R^A_5(u,v_i)^{ad_1}_{d_1a}(c^1_6(v_i,v_j)+c^1_9(v_i,v_j))\delta_{d_1f}$$
$$+R^A_6(u,v_i)(R^C_3(v_i,v_j)^{fe}_{c_2d_1}+R^C_4(v_i,v_j)^{fe}_{c_2d_1})+R^F_1(u,v_i)^{fe}_{a\bar{a}}g_1(v_i,v_j,d_1)\delta_{d_1\bar{c}_2}$$
$$+\tilde{r}(u+v_i)^{ac}_{dg}\bar{r}(u-v_i)^{gh}_{d_1a}g_2(u,v_i,f)R^A_1(u,v_j)^{db}_{c_2h}\hat{r}(v_i-u)^{fe}_{bc}$$
$$+\tilde{r}(u+v_i)^{ah}_{dg}\bar{r}(u-v_i)^{g\bar{c}_2}_{d_1a}R^A_3(u,v_j,\bar{c}_2)R^{be}_3(v_i,u)^{fe}_{\bar{d}h}, \tag{B.16}$$

$$H^{\tilde{A}_{aa}}_{3,d_1}(u,v_i,v_j) = R^A_5(u,v_i)^{ad_1}_{d_1a}c^1_8(v_i,v_j)+R^A_6(u,v_i)c^2_7(v_i,v_j,\bar{d}_1)$$
$$+b^2_2(u,v_i)g_2(v_i,v_j,d_1)+\tilde{r}(u+v_i)^{ad}_{da}\bar{r}(u-v_i)^{ad_1}_{d_1a}R^A_4(u,v_j,\bar{d}_1)e^1_1(v_i,u,d)$$
$$+\tilde{r}(u+v_i)^{ae}_{dg}\bar{r}(u-v_i)^{gf}_{d_1a}g_1(u,v_i,h)R^A_2(u,v_j)^{d\bar{e}}_{\bar{d}_1f}\hat{r}(v_i-u)^{h\bar{h}}_{\bar{e}e}, \tag{B.17}$$

$$H^{\tilde{A}_{aa}}_{4,d_1}(u,v_i,v_j)^{de}_{fd_1}=R^A_5(u,v_i)^{ad_1}_{d_1a}c^1_{10}(v_i,v_j)\delta_{d_1d}+R^A_6(u,v_i)R^C_5(v_i,v_j)^{de}_{fd_1}$$
$$+R^F_1(u,v_i)^{de}_{a\bar{a}}g_2(v_i,v_j,d_1)\delta_{d_1\bar{f}}+\tilde{r}(u+v_i)^{ac}_{hg}\bar{r}(u-v_i)^{g\bar{f}}_{d_1a}R^A_4(u,v_j,f)R^{be}_3(v_i,u)^{de}_{\bar{h}c}$$
$$+\tilde{r}(u+v_i)^{ac}_{bg}\bar{r}(u-v_i)^{gh}_{d_1a}g_2(u,v_i,d)R^A_2(u,v_j)^{bk}_{fh}\hat{r}(v_i-u)^{de}_{kc}, \qquad \text{(B.18)}$$

$$H^{\tilde{A}_2}_{1,d_1}(u,v_i,v_j)=a^3_6(u,v_i,\bar{d}_1)(c^1_5(v_i,v_j)+c^1_7(v_i,v_j))+b^3_2(u,v_i)g_1(v_i,v_j,d_1)$$
$$+a^3_7(u,v_i)(c^2_4(v_i,v_j,\bar{d}_1)+c^2_6(v_i,v_j,\bar{d}_1))+a^3_1(u,v_i)a^3_4(u,v_j,\bar{d}_1)e^1_1(v_i,u,d_1)$$
$$+a^3_1(u,v_i)a^3_2(u,v_j)g_1(u,v_i,d)\hat{r}(v_i-u)^{d\bar{d}}_{\bar{d}_1d_1}, \qquad \text{(B.19)}$$

$$H^{\tilde{A}_2}_{2,d_1}(u,v_i,v_j)^{fe}_{c_2d_1}=a^3_6(u,v_i,\bar{d}_1)(c^1_6(v_i,v_j)+c^1_9(v_i,v_j))\delta_{d_1f}$$
$$+a^3_7(u,v_i)(R^C_3(v_i,v_j)^{fe}_{c_2d_1}+R^C_4(v_i,v_j)^{fe}_{c_2d_1})$$
$$+b^3_3(u,v_i)g_1(v_i,v_j,d_1)\delta_{d_1\bar{c}_2}+a^3_1(u,v_i)a^3_4(u,v_j,c_2)R^{be}_3(v_i,u)^{fe}_{c_2d_1}$$
$$+a^3_1(u,v_i)a^3_2(u,v_j)g_2(u,v_i,f)\hat{r}(v_i-u)^{fe}_{c_2d_1}, \qquad \text{(B.20)}$$

$$H^{\tilde{A}_2}_{3,d_1}(u,v_i,v_j)=a^3_6(u,v_i,\bar{d}_1)c^1_8(v_i,v_j)+a^3_7(u,v_i)c^2_7(v_i,v_j,\bar{d}_1)$$
$$+b^3_2(u,v_i)g_2(v_i,v_j,d_1)+a^3_1(u,v_i)a^3_5(u,v_j,d_1)e^1_1(v_i,u,d_1)$$
$$+a^3_1(u,v_i)a^3_3(u,v_j)g_1(u,v_i,d)\hat{r}(v_i-u)^{d\bar{d}}_{\bar{d}_1d_1}, \qquad \text{(B.21)}$$

$$H^{\tilde{A}_2}_{4,d_1}(u,v_i,v_j)^{de}_{fd_1}=a^3_6(u,v_i,\bar{d}_1)c^1_{10}(v_i,v_j)\delta_{d_1d}+a^3_7(u,v_i)R^C_5(v_i,v_j)^{de}_{fd_1}$$
$$+b^3_3(u,v_i)g_2(v_i,v_j,d_1)\delta_{d_1\bar{f}}+a^3_1(u,v_i)a^3_5(u,v_j,\bar{f})R^{be}_3(v_i,u)^{de}_{fd_1}$$
$$+a^3_1(u,v_i)a^3_3(u,v_j)g_2(u,v_i,d)\hat{r}(v_i-u)^{de}_{fd_1}. \qquad \text{(B.22)}$$

In Eqs.(A.11)-(A.22), the repeated indices a and d_1 do not sum. We can check that

$$\frac{H^x_{2,b}(u,v_i,v_j)^{d_1a}_{cb}}{R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{d_1a}_{cb}}=\frac{H^x_{2,d_1}(u,v_i,v_j)^{d_1d_1}_{d_1d_1}}{R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{d_1d_1}_{d_1d_1}}, \qquad \text{(B.23)}$$

$$\frac{H^x_{4,b}(u,v_i,v_j)^{d_1a}_{cb}}{R^{(n-1)}(\tilde{v}_i+\tilde{v}_j)^{d_1a}_{cb}}=\frac{H^x_{4,d_1}(u,v_i,v_j)^{d_1d_1}_{d_1d_1}}{R^{(n-1)}(\tilde{v}_i+\tilde{v}_j)^{d_1d_1}_{d_1d_1}}. \qquad \text{(B.24)}$$

In Eqs.(B.23,B.24), all the repeated indices do not sum. We conclude that Eq.(B.1) can be verified directly by using mathematical induction, although it is a rather hard work. Similar to assumption of algebraic Bethe ansatz, we might assume that "quasi" m-particle states such as $B\Phi_{m-1}|0\rangle$, $E\Phi_{m-1}|0\rangle$, $BB\Phi_{m-2}|0\rangle$, $BE\Phi_{m-2}|0\rangle$, $EB\Phi_{m-2}|0\rangle$, $F\Phi_{m-2}|0\rangle$, $FB\Phi_{m-3}|0\rangle$ etc are linearly independent. Here all the indices are omitted and all the spectrum parameters in the "quasi" n-particle state keep the order $\{v_{i_1},v_{i_2},\cdots,v_{i_k}\}$ with $i_1<i_2<\cdots<i_k$. For example, $B_1(v_1)B_1(v_2)\Phi^{b_3\cdots b_m}_{m-2}(v_3,\cdots,v_m)F^{11b_3\cdots b_m}|0\rangle$ and $B_1(v_1)B_2(v_2)\Phi^{b_3\cdots b_m}_{m-2}(v_3,\cdots,v_m)F^{12b_3\cdots b_m}|0\rangle$ are thought to be linearly independent. Then, by using this assumption, the property of Eq.(45) and some necessary relations, we can prove the conclusions Eq.(B.1) as done in Ref.[13].

In order to obtain the eigenvalue and the corresponding Bethe equations, we need to change Eq.(B.1) by the following four useful relations (the proofs are omitted here) ,

$$S^{d_1\cdots d_m}_{c_1\cdots c_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{c_1\cdots c_m}_{b_1\cdots b_m} = \\ (\rho^{\frac{1}{2}}_{n-1}(0))^{-1}T^{(d_1)}(v_i)[T^{m-1}(v_i;\{\check{v}_i\})^{d_2\cdots d_m}_{c'_2\cdots c'_m}]_{d_1c'_1}S^{c'_1\cdots c'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\}), \tag{B.25}$$

$$S^{e_2\cdots e_m}_{c_2\cdots c_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{c_1\cdots c_m}_{d_1\cdots d_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m} = \\ (\rho_{n-1}(\tilde{v}_i-\tilde{v}_j)\rho^{\frac{1}{2}}_{n-1}(0))^{-1}T^{(c_1)}(v_i)R^{(n-1)}(\tilde{v}_i+\tilde{v}_j)^{c_1e_2}_{h_1f_1}R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{f_1h'_1}_{c'_2d'_1} \\ \times[T^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c'_3\cdots c'_m}]_{h_1h'_1}S^{c'_2\cdots c'_m}_{d'_2\cdots d'_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\}), \tag{B.26}$$

$$S^{e_2\cdots e_m}_{c_2\cdots c_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{c_1\cdots c_m}_{d_1\cdots d_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_j;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m} = \\ (\rho_{n-1}(\tilde{v}_j-\tilde{v}_i)\rho^{\frac{1}{2}}_{n-1}(0))^{-1}T^{(h_1)}(v_j)R^{(n-1)}(\tilde{v}_j-\tilde{v}_i)^{c_1e_2}_{g_1h_1}R^{(n-1)}(\tilde{v}_j+\tilde{v}_i)^{h_1g_1}_{h_2d'_1} \\ \times[T^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c'_3\cdots c'_m}]_{h_2c'_2}S^{c'_2\cdots c'_m}_{d'_2\cdots d'_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\}), \tag{B.27}$$

$$S^{e_2\cdots e_m}_{a_2\cdots a_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{a_1\cdots a_m}_{c_1\cdots c_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{c_1\cdots c_m}_{d_1\cdots d_m}\tau_1(\tilde{v}_j;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m} \\ = (\rho_{n-1}(\tilde{v}_i-\tilde{v}_j)\rho_{n-1}(0))^{-1}T^{(a_1)}(v_i)T^{(f_1)}(v_j)R^{(n-1)}(\tilde{v}_i+\tilde{v}_j)^{a_1e_2}_{h_1f_1}R^{(n-1)}(\tilde{v}_j+\tilde{v}_i)^{f_1h'_1}_{f_2d'_1} \\ \times[T^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{a'_3\cdots a'_m}]_{h_1h'_1}[T^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{a'_3\cdots a'_m}_{c'_3\cdots c'_m}]_{f_2c'_2} \\ \times S^{c'_2\cdots c'_m}_{d'_2\cdots d'_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\}), \tag{B.28}$$

where

$$[T^m(u;\{v_m\})^{d_1\cdots d_m}_{c_1\cdots c_m}]_{ab} \\ = L(\tilde{u},\tilde{v}_1)^{ad_1}_{h_1g_1}L(\tilde{u},\tilde{v}_2)^{h_1d_2}_{h_2g_2}\cdots L(\tilde{u},\tilde{v}_m)^{h_{m-1}d_m}_{h_mg_m}k^-(u)_{h_m} \\ \times L^{-1}(-\tilde{u},\tilde{v}_m)^{h_mg_m}_{f_{m-1}c_m}L^{-1}(-\tilde{u},\tilde{v}_{m-1})^{f_{m-1}g_{m-1}}_{f_{m-2}c_{m-1}}\cdots L^{-1}(-\tilde{u},\tilde{v}_1)^{f_1g_1}_{bc_1}, \tag{B.29}$$

$$T^{(d_1)}(v_i) = k^+_d(v_i)R^{(n-1)}(2v_i-4\eta)^{dd_1}_{d_1d}. \tag{B.30}$$

Let

$$\Lambda^m_2(u;\{v_m\}) = \prod_{i=1}^m \rho_{n-1}(u-v_i)\tilde{\rho}(u,v_i), \tag{B.31}$$

$$\rho_{n-1}(u) = a_{n-1}(u)a_{n-1}(-u), \quad \tilde{\rho}(u,v) = \frac{1}{a_n(u+v)e_n(u-v)}, \tag{B.32}$$

$$|\tilde{\Psi}^{(2)}_{m-1}(u,v_i;\{v_m\})_{fd}\rangle = \frac{\rho^{\frac{1}{2}}_{n-1}(0)\omega(v_i)}{T^{(d)}(v_i)}\Lambda_2^{m-1}(v_i;\{\check{v}_i\})B_f(u)\Phi^{e_2\cdots e_m}_{m-1}(v_1,\cdots,\check{v}_i,\cdots,v_m)$$
$$\times S^{de_2\cdots e_m}_{d_1\cdots d_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}F^{b_1\cdots b_m}|0\rangle, \quad (B.33)$$

$$|\tilde{\Psi}^{(4)}_{m-1}(u,v_i;\{v_m\})_{ab}\rangle = \frac{\rho^{\frac{1}{2}}_{n-1}(0)\omega(v_i)}{T^{(\bar{b})}(v_i)}\Lambda_2^{m-1}(v_i;\{\check{v}_i\})E_a(u)\Phi^{e_2\cdots e_m}_{n-1}(v_1,\cdots,\check{v}_i,\cdots,v_m)$$
$$\times S^{\bar{b}e_2\cdots e_m}_{d_1\cdots d_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}F^{b_1\cdots b_m}|0\rangle, \quad (B.34)$$

$$|\tilde{\Psi}^{(5)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle = F(u)\Phi^{e_3\cdots e_m}_{m-2}(v_1,\cdots,\check{v}_i,\cdots,\check{v}_j,\cdots,v_m)$$
$$\times S^{\bar{d}_1e_3\cdots e_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})$$
$$\times\Lambda_1^{m-1}(v_i;\{\check{v}_i\})\Lambda_1^{m-1}(v_j;\{\check{v}_j\})\omega_1(v_i)\omega_1(v_j)F^{b_1\cdots b_m}|0\rangle, \quad (B.35)$$

$$|\tilde{\Psi}^{(6)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle = F(u)\Phi^{e_3\cdots e_m}_{m-2}(v_1,\cdots,\check{v}_i,\cdots,\check{v}_j,\cdots,v_m)$$
$$\times S^{\bar{d}_1e_3\cdots e_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{c_1\cdots c_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{c_1\cdots c_m}_{b_1\cdots b_m}$$
$$\times\Lambda_2^{m-1}(v_i;\{\check{v}_i\})\omega(v_i)\rho^{\frac{1}{2}}_{n-1}(0)\Lambda_1^{m-1}(v_j;\{\check{v}_j\})\omega_1(v_j)F^{b_1\cdots b_m}|0\rangle, \quad (B.36)$$

$$|\tilde{\Psi}^{(7)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle = F(u)\Phi^{e_3\cdots e_m}_{m-2}(v_1,\cdots,\check{v}_i,\cdots,\check{v}_j,\cdots,v_m)$$
$$\times S^{\bar{d}_1e_3\cdots e_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{c_1\cdots c_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_j;\{\tilde{v}_m\})^{c_1\cdots c_m}_{b_1\cdots b_m}$$
$$\times\Lambda_2^{m-1}(v_j;\{\check{v}_j\})\omega(v_j)\rho^{\frac{1}{2}}_{n-1}(0)\Lambda_1^{m-1}(v_i;\{\check{v}_i\})\omega_1(v_i)F^{b_1\cdots b_m}|0\rangle, \quad (B.37)$$

$$|\tilde{\Psi}^{(8)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle = F(u)\Phi^{e_3\cdots e_m}_{m-2}(v_1,\cdots,\check{v}_i,\cdots,\check{v}_j,\cdots,v_m)$$
$$\times S^{\bar{d}_1e_3\cdots e_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{c_1\cdots c_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{c_1\cdots c_m}_{a_1\cdots a_m}$$
$$\times\tau_1(\tilde{v}_j;\{\tilde{v}_m\})^{a_1\cdots a_m}_{b_1\cdots b_m}\Lambda_2^{m-1}(v_i;\{\check{v}_i\})\Lambda_2^{m-1}(v_j;\{\check{v}_j\})\omega(v_i)\omega(v_j)\rho_{n-1}(0)F^{b_1\cdots b_m}|0\rangle. \quad (B.38)$$

Using relation Eq.(B.25), we can easily change the $|\Psi^{(2)}_{m-1}(u,v_i;\{v_m\})_{fd}\rangle$ and $|\Psi^{(4)}_{m-1}(u,v_i;\{v_m\})_{ab}\rangle$ in Eq.(B.1) into $|\tilde{\Psi}^{(2)}_{m-1}(u,v_i;\{v_m\})_{fd}\rangle$ and $|\tilde{\Psi}^{(4)}_{m-1}(u,v_i;\{v_m\})_{ab}\rangle$, respectively.

Let $e_2=\bar{c}_1$, from Eq.(B.26), we can get

$$R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{\bar{a}_1h'_1}_{c'_2d'_1}[T^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c'_3\cdots c'_m}]_{a_1h'_1}S^{c'_2\cdots c'_m}_{d'_2\cdots d'_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})$$
$$=\frac{\rho_{n-1}(\tilde{v}_i-\tilde{v}_j)\rho^{\frac{1}{2}}_{n-1}(0)}{\rho_{n-1}(\tilde{v}_i+\tilde{v}_j)}\frac{R^{(n-1)}(-\tilde{v}_i-\tilde{v}_j)^{\bar{a}_1a_1}_{\bar{d}d}}{T^{(d)}(v_i)}$$
$$\times S^{\bar{d}e_3\cdots e_m}_{h_2\cdots h_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{dh_2\cdots h_m}_{d_1\cdots d_m}(v_i;\{\check{v}_i\})]\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}. \quad (B.39)$$

One should note that the repeated indices c_1 and a_1 in Eqs.(B.39, B.42, B.45) do not sum. Using Eq.(B.39) and Eq.(B.23), we have

$$H^x_{2,d_1}(u,v_i,v_j)^{fe}_{c_2d_1}|\Psi^{(6)}_{m-2}(u,v_i,v_j;\{v_m\})^{ef}_{d_1c_2}\rangle =$$
$$\tilde{H}^x_{2,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(6)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle, \quad (B.40)$$

where

$$\tilde{H}^{x}_{2,d_1}(u,v_i,v_j)=\frac{1}{\tilde{\rho}(v_i,v_j)a^1_1(v_j,v_i)}\frac{H^{x}_{2,d}(u,v_i,v_j)^{dd}_{dd}}{R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{dd}_{dd}}\frac{R^{(n-1)}(-\tilde{v}_i-\tilde{v}_j)^{d\bar{d}}_{\bar{d}_1d_1}}{\rho_{n-1}(\tilde{v}_i+\tilde{v}_j)T^{(d_1)}(v_i)}. \quad \text{(B.41)}$$

Let $e_2=\bar{c}_1$, from Eq.(B.27), we obtain

$$\begin{aligned}
&[T^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c'_3\cdots c'_m}]_{\bar{c}_1c'_2}S^{c'_2\cdots c'_m}_{d'_2\cdots d'_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{c_1d'_2\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\\
&=\frac{\rho^{\frac{1}{2}}_{n-1}(0)}{\rho_{n-1}(\tilde{v}_j+\tilde{v}_i)}\frac{R^{(n-1)}(-\tilde{v}_i-\tilde{v}_j)^{c_1\bar{c}_1}_{e_2\bar{e}_2}R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{\bar{e}_2e_2}_{\bar{d}d}}{T^{(\bar{e}_2)}(v_j)}\\
&\times S^{\bar{d}e_3\cdots e_m}_{h_2\cdots h_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{dh_2\cdots h_m}_{d_1\cdots d_m}(v_i;\{\check{v}_i\})\}\tau_1(\tilde{v}_j;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}. \qquad \text{(B.42)}
\end{aligned}$$

Then, by Eq.(B.42), we have

$$\begin{aligned}
&H^{x}_{3,d_1}(u,v_i,v_j)|\Psi^{(7)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1d_1}\rangle=\\
&\qquad\tilde{H}^{x}_{3,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(7)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle, \qquad \text{(B.43)}
\end{aligned}$$

where

$$\tilde{H}^{x}_{3,d_1}(u,v_i,v_j)=\frac{H^{x}_{3,d}(u,v_i,v_j)R^{(n-1)}(-\tilde{v}_i-\tilde{v}_j)^{d\bar{d}}_{e\bar{e}}R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{\bar{e}e}_{\bar{d}_1d_1}}{\tilde{\rho}(v_j,v_i)a^1_1(v_i,v_j)\rho_{n-1}(\tilde{v}_j-\tilde{v}_i)\rho_{n-1}(\tilde{v}_j+\tilde{v}_i)T^{(\bar{e})}(v_j)}. \qquad \text{(B.44)}$$

Let $e_2=\bar{a}_1$, from Eq.(B.28), we achieve

$$\begin{aligned}
&R^{(n-1)}(\tilde{v}_j+\tilde{v}_i)^{\bar{a}_1h'_1}_{f_2d'_1}[T^{n-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{a'_3\cdots a'_m}]_{a_1h'_1}[T^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{a'_3\cdots a'_m}_{c'_3\cdots c'_m}]_{f_2c'_2}\\
&\times S^{c'_2\cdots c'_m}_{d'_2\cdots d'_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})=\frac{\rho_{n-1}(\tilde{v}_i-\tilde{v}_j)\rho_{n-1}(0)}{\rho_{n-1}(\tilde{v}_i+\tilde{v}_j)}\\
&\times\frac{R^{(n-1)}(-\tilde{v}_j-\tilde{v}_i)^{\bar{a}_1a_1}_{\bar{d}d}}{T^{(d)}(v_i)T^{(\bar{a}_1)}(v_j)}S^{\bar{d}e_3\cdots e_m}_{a_2\cdots a_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{da_2\cdots a_m}_{g_1\cdots g_m}(v_i;\{\check{v}_i\})]\\
&\times\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{g_1\cdots g_m}_{d_1\cdots d_m}\tau_1(\tilde{v}_j;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}. \qquad \text{(B.45)}
\end{aligned}$$

Using Eq.(B.45) and considering Eq.(B.24), we have

$$\begin{aligned}
&H^{x}_{4,d_1}(u,v_i,v_j)^{de}_{fd_1}|\Psi^{(8)}_{m-2}(u,v_i,v_j;\{v_m\})^{ed}_{d_1f}\rangle=\\
&\qquad\tilde{H}^{x}_{4,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(8)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle, \qquad \text{(B.46)}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{H}^{x}_{4,d_1}(u,v_i,v_j)&=\frac{1}{\tilde{\rho}(v_i,v_j)\tilde{\rho}(v_j,v_i)\rho_{n-1}(\tilde{v}_j-\tilde{v}_i)\rho_{n-1}(\tilde{v}_i+\tilde{v}_j)}\\
&\quad\times\frac{H^{x}_{4,d}(u,v_i,v_j)^{dd}_{dd}}{R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{dd}_{dd}}\frac{R^{(n-1)}(-\tilde{v}_j-\tilde{v}_i)^{d\bar{d}}_{\bar{d}_1d_1}}{T^{(d_1)}(v_i)T^{(d)}(v_j)}. \qquad \text{(B.47)}
\end{aligned}$$

We can easily get

$$\delta_{\bar{d}_1 e_2} H^x_{1,d_1}(u,v_i,v_j)|\Psi^{(5)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1 e_2}\rangle = \\ \tilde{H}^x_{1,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(5)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle, \tag{B.48}$$

where

$$\tilde{H}^x_{1,d_1}(u,v_i,v_j) = \frac{H^x_{1,d_1}(u,v_i,v_j)}{a^1_1(v_i,v_j)a^1_1(v_j,v_i)}. \tag{B.49}$$

After making the notation

$$|\tilde{\Psi}_x(u,\{v_m\})\rangle = \begin{cases} |\Psi_x(u,\{v_m\})\rangle, x = A, \tilde{A}_2 \\ \omega(u)\Lambda^m_2(u;\{v_m\})\Phi^{d_1\cdots d_m}_m(v_1,\cdots,v_m)[T^m(u;\{v_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}]_{aa}F^{b_1\cdots b_m}|0\rangle, x = \tilde{A}_{aa} \end{cases} \tag{B.50}$$

we arrive at the final result Eq.(46).

References

[1] L.D. Faddeev, 1982 *In recent advances in Field theory and Statistical Mechanics*, Les Houches.

[2] V.E.Korepin, N.M.Bogoliubov and A.G.Izergin 1993 *Quantum Inverse Scattering Method and Correlation Function*, Cambridge University Press. Cambridge.

[3] E. K.Sklyanin, 1988 *J. Phys.* **A 21** 2375.

[4] O. Babelon, H.J. De Vega and C.M. Viallet, 1982 *Nucl. Phys.* **B 200** (1982) 266.

[5] H.J.de vega and A. Gonzalez-Ruiz, 1994 *Nucl. Phys.* **B 417** 553 [hep-th/9404141].

[6] P.B.Ramos and M.J.Martins, 1996 *Nucl. Phys.* **B 474** 678 [hep-th/9604072].

[7] M.J.Martins and P.B.Ramos, 1997 *Nucl. Phys.* **B 500** 579 [hep-th/9703023].

[8] M.J.Martins and P.B.Ramos, 1998 *Nucl. Phys.* **B 522** 413 [solv-int/9712014].

[9] M.J.Martins, *Unified algebraic Bethe ansatz for two-dimensional lattice models,* [solv-int/9901002].

[10] X.W. Guan, 2000 *J. Phys.* **A 33** 5391.

[11] A.Foerster, X.W.Guan, J.Links, I.Roditi and H.Q.Zhou, 2001 *Nucl. Phys.* **B 596** 525.

[12] X.W.Guan, A.Foerster, U.Grimm, R.A.Römer and M.Schreiber, 2001 *Nucl. Phys.* **B 618** 650.

[13] G.L.Li, K.J.Shi and R.H.Yue, 2003 *Nucl.Phys.* **B 670** 401.

[14] G.L.Li, K.J.Shi and R.H.Yue, 2004 *Nucl. Phys.* **B 687** (2004) 220.
G.L.Li, K.J.Shi and R.H.Yue, 2005 *J. High Energy Phys.* **0507** 001 [hep-th/0505001].

[15] V. Kurak, A. Lima-Santos, 2004 *Nucl.Phys.* **B 699** 595 [nlin.SI/0406050].
V. Kurak, A. Lima-Santos, 2005 *J. Phys.* **A 38** 2359 [nlin.SI/0407006].

[16] N.Yu.Reshetikhin, 1987 *Lett. Math.Phys.* **14** 235.

[17] L.Mezincescu and R.I. Nepomechie, 1992 *Nucl. Phys.* **B 372** 597 [hep-th/9110050].

[18] S.Artz, L.Mezincescu and R.I. Nepomechie, 1995 *Int. J. Mod.Phys.* **A 10** 1937 [hep-th/9409130].
S.Artz, L.Mezincescu and R.I. Nepomechie, 1995 *J. Phys.* **A 28** 5131 [hep-th/9504085].

[19] A. Lima-santos, 2006 *J.Stat.Mech* **0607** P003 [nlin.SI/0602003].

[20] V.V.Bazhanov, 1985 *Phys. Lett.* **B 519** 321.
V.V.Bazhanov, 1987 *Commun. Math. Phys.* **113** 471.

[21] M.Jimbo, 1986 *Commun. Math. Phys.* **102** 537.

[22] R.J.Baxter, 1982 *Exactly Solved Models in Statitical Mechanics*, Academic Press, London.

[23] A.Lima-santos, 2003 **B 654** 466 [nlin.SI/0210046].
A.Lima-santos and R. Malara R, 2003 *Nucl.Phys.* **B 675** 661 [nlin.SI/0307046].

[24] J.P.Cao, H.Q.Lin, K.J.Shi and Y.P.Wang, 2003 *Nucl.Phys.* **B 663** 487.

[25] W.Galleas and M.J.Martins, preprint nlin.SI/0407027.

[26] C.S.Melo, G.A.P.Ribeiro and M.J.Martins, 2005 *Nucl. Phys.* **B 711** 565.

[27] W.L. Yang and Y.Z. Zhang, 2005 *J. High Energy Phys.* **0501** 021 [hep-th/0411190].