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1994 Inverse Problems 10 431

(http://iopscience.iop.org/0266-5611/10/2/016)

Inverse Problems 10 (1994) 431–447. Printed in the UK

Fréchet differentiability of boundary integral operators in inverse acoustic scattering

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Received 16 July 1993

Abstract. Using integral equation methods to solve the time-harmonic acoustic scattering problem with Dirichlet boundary conditions, it is possible to reduce the solution of the scattering problem to the solution of a boundary integral equation of the second kind. We show the Fréchet differentiability of the boundary integral operators which occur. We then use this to prove the Fréchet differentiability of the scattered field with respect to the boundary. Finally we characterize the Fréchet derivative of the scattered field by a boundary value problem with Dirichlet conditions, in an analogous way to that used by Firsch.

1. Introduction

In this paper we deal with the time-harmonic acoustic obstacle scattering problem with Dirichlet boundary condition [3]. There exist different methods of solving this standard problem of mathematical physics. Here we refer to the integral equation approach which can be found in [3].

It is especially interesting in the framework of *inverse problems* to study the dependence of the solutions to the scattering problems on the domain of the scatterer. Let Γ denote the boundary of a suitable domain $D \subset \mathbb{R}^3$. The scattering operator R^s maps the boundary Γ onto the solution

$$u^s = R^s(\Gamma) \tag{1}$$

of the direct scattering problem for a fixed entire incident field u^i. The inverse problem consists of looking for a solution of (1) given u^s on an exterior domain or the far field $u^\infty = Fu^s$ of u^s, respectively. In order to invert equation (1) we are interested in properties of R^s. R^s is nonlinear and equation (1) is ill-posed, which makes it difficult to solve. In this paper we prove the Fréchet differentiability of R^s and describe two possibilities of computing the derivative. In principle this allows the application of Newton-type methods to the inversion of equation (1) [4, 6, 7].

Using boundary integral equation methods to solve the scattering problem, following Colton and Kress, one can derive a representation of R^s in terms of acoustic single- and double-layer potentials and weakly singular boundary integral operators. We briefly recall this method in section 2. We use section 3 to state some facts about the Fréchet derivative of integral operators. In section 4 we prove the Fréchet differentiability with respect to the domain and derive the explicit form of the Fréchet derivative of the integral operators used in section 2 which are considered as operators in the space of continuous functions on Γ. This Fréchet differentiability implies 'Γ-differentiability' and the 'domain derivative' defined in [4, 6]. Using well-known properties of the Fréchet derivative it is then possible to obtain the Fréchet differentiability of the scattering operator R^s. In section 5 we characterize the derivative of u^s with respect to the boundary as a solution of a Dirichlet boundary value problem.

Our method of establishing the Fréchet differentiability of the scattered field is new to scattering theory. In principle, the method can be carried over to other boundary value problems, for example to the time-harmonic acoustic scattering problem with Neumann boundary conditions or to time-harmonic electromagnetic boundary value problems. For the case of the Dirichlet scattering problem the differentiability has already been verified by Kress (cf [3]) and by Kirsch [4] using variational methods. Also with the help of the variational approach the characterization of the derivative was obtained by Kirsch [4].

2. The scattering map R^s and the inverse scattering problem

For each normed space we denote by K_L the open ball with radius L and centre 0. Let $D \subset K_L \subset \mathbb{R}^3$ be a bounded domain with boundary ∂D of class C^2, $B \supset \overline{K_L}$ an open set and $k \in \mathbb{C}$ with $\operatorname{Im} k \geqslant 0$. A function $w \in C^1(\mathbb{R}^3 \setminus \overline{K_L})$ satisfies the *Sommerfeld radiation condition* if

$$\hat{x} \cdot (\operatorname{grad} w)(x) - \mathrm{i}kw(x) = o\left(1/|x|\right) \qquad |x| \to \infty \tag{2}$$

holds uniformly on $\Omega = \left\{\hat{x} \in \mathbb{R}^3, |\hat{x}| = 1\right\}$. We denote by

$$\Phi(x, y) = \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \qquad x, y \in \mathbb{R}^3;\ x \neq y$$

the *fundamental solution* of the *Helmholtz equation*

$$\Delta u + k^2 u = 0. \tag{3}$$

$\Phi(, y)$ solves the Helmholtz equation in $\mathbb{R}^3 \setminus \{y\}$ and satisfies the *Sommerfeld radiation condition* uniformly for $y \in K_L$. We denote by ν the exterior unit normal vector on the surface ∂D. For $\varphi \in C(\partial D)$ the *acoustic single-layer potential*

$$u(x) := \int_{\partial D} \Phi(x, y)\varphi(y)\,\mathrm{d}s(y) \qquad x \in \mathbb{R}^3 \setminus \partial D \tag{4}$$

and the *acoustic double-layer potential*

$$v(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)}\varphi(y)\,\mathrm{d}s(y) \qquad x \in \mathbb{R}^3 \setminus \partial D \tag{5}$$

are solutions to the Helmholtz equation in $\mathbb{R}^3 \setminus \partial D$ and satisfy the Sommerfeld radiation condition. We now consider the *Dirichlet obstacle scattering problem*: For a given solution $u^{\mathrm{i}} \in C^1(B)$ to the Helmholtz equation, find a function $u^s \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$, which satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{D}$ and the Sommerfeld radiation condition with boundary values $u^{\mathrm{i}} + u^s = 0$ on ∂D. Following Colton and Kress [3] we look for a solution to the Dirichlet obstacle scattering problem using a *combined single- and double-layer potential*

$$u^s(x) = \int_{\partial D} \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \mathrm{i}\eta \Phi(x, y) \right\} \varphi(y) \, \mathrm{d}s(y) \qquad x \in \mathbb{R}^3 \setminus \partial D \tag{6}$$

$\eta \in \mathbb{R}$, $\eta \neq 0$. Using the classical jump relations for the single- and double-layer potential [2], the potential (6) can be seen to solve the Dirichlet scattering problem if the density $\varphi \in C(\partial D)$ is a solution to the boundary integral equation

$$(I + K - \mathrm{i}\eta S)\varphi = -2u^{\mathrm{i}}. \tag{7}$$

Here the operators

$$(S\varphi)(x) := 2 \int_{\partial D} \Phi(x, y) \varphi(y) \, \mathrm{d}s(y) \qquad x \in \partial D \tag{8}$$

and

$$(K\varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \varphi(y) \, \mathrm{d}s(y) \qquad x \in \partial D \tag{9}$$

are linear with weakly singular kernels, and therefore are compact operators $C(\partial D) \to C(\partial D)$. Existence and boundedness of the inverse of the operator $I + K - \mathrm{i}\eta S$ can be obtained by Riesz–Fredholm theory for equations of the second kind with compact operators [5]. We are interested in the values of the scattered field on a set $M \subset \mathbb{R}^3 \setminus \overline{D}$. Therefore we combine the potential (6) with the restriction to M that $P : C(\partial D) \to C(M), \varphi \mapsto u^s|_M$ is a linear bounded mapping. Using the restriction operator $R : C(B) \to C(\partial D), u^{\mathrm{i}} \mapsto u^{\mathrm{i}}|_{\partial D}$ we can write the solution of the Dirichlet scattering problem in the form

$$u^s = -2P(I + K - \mathrm{i}\eta S)^{-1} R u^{\mathrm{i}}. \tag{10}$$

The inverse Dirichlet scattering problem consists in determining a domain D, which satisfies (10) for a given number of incident fields u^{i} with corresponding scattered fields u^s.

In order to use *Newton-type methods* to solve this inverse scattering problem we have to study the differentiability properties of the mapping $\partial D \mapsto u^s$. For this we first study the differentiability properties of the operators which occur in equation (10), and then use the chain and product rule to derive the differentiability of the mapping $\partial D \mapsto u^s$.

First we have to transform the operators onto a fixed reference boundary. Similarly to [4, 6, 8] we use the mapping $\phi_r : \partial D \to \partial D_r : x \mapsto x + r(x)$ where $r \in C^2(\partial D)$ is a twice continuously differentiable vector field and ∂D_r is defined by $\partial D_r := \{x + r(x), x \in \partial D\}$. For a sufficiently small $l > 0$ depending on ∂D, each ∂D_r with $\|r\|_{C^2(\partial D)} \leqslant l$ is again a class-C^2 boundary of a domain D_r. We use $V_l := \{r \in C^2(\partial D), \|r\|_{C^2(\partial D)} < l\}$. We denote by $\nu_r(x)$ the exterior unit normal vector on the boundary ∂D_r at the point $x_r := x + r(x)$; we abbreviate ν_0 to ν.

We denote the space of all bounded linear operators mapping a normed space X into a normed space Y by $B(X, Y)$. Now for each $r \in V_l$ we transform functions $\varphi \in C(\partial D_r)$ into functions $\tilde{\varphi} \in C(\partial D)$ using $\tilde{\varphi}(x) := \varphi(x_r)$. Analogously we transform operators $I : C(\partial D_r) \to C(\partial D_r)$ to operators $\tilde{I} : C(\partial D) \to C(\partial D)$. Since in this way the space $C(\partial D)$ is isomorphic to $C(\partial D_r)$ and $B(C(\partial D), C(\partial D))$ is isomorphic to

$B(C(\partial D_r), C(\partial D_r))$ we usually just write $\tilde{\varphi} = \varphi$ and $\tilde{I} = I$. We will study the Fréchet differentiability of the mappings

$$
\begin{aligned}
S &: V_l \to B(C(\partial D), C(\partial D)) \qquad && r \mapsto \tilde{S}[r] \\
K &: V_l \to B(C(\partial D), C(\partial D)) \qquad && r \mapsto \tilde{K}[r] \\
R &: V_l \to B(C^1(B), C(\partial D)) \qquad && r \mapsto \tilde{R}[r \\
P &: V_l \to B(C(\partial D), C(M)) \qquad && r \mapsto \tilde{P}[r].
\end{aligned}
$$

3. Some remarks on Fréchet differentiability of integral operators

For the well-known properties of the Fréchet derivative of a nonlinear mapping we refer to [1]; here we just give a summary of our notation.

Let Y be a normed space, let X be a Banach space and let $U \subset Y$ be an open set. A mapping $A : U \to X$ is called *Fréchet differentiable in* $r_0 \in U$, if there is a bounded linear mapping $\partial A/\partial r \in B(Y, X)$, a neighbourhood V of 0 in Y and a mapping $A_1 : V \to X$ such that

$$
\begin{aligned}
&A(r_0 + h) = A(r_0) + \frac{\partial A}{\partial r}(h) + A_1(h) \qquad \forall h \in V \\
&A_1(h) = \mathrm{o}(\|h\|).
\end{aligned}
\tag{11}
$$

If A is Fréchet differentiable in U the derivative can be considered as a mapping $U \to B(Y, X), r \to \partial A(r;\ )/\partial r$. If this mapping is again Fréchet differentiable, we speak of the second derivative of A. We have $\partial^2 A/\partial r^2 \in B(Y, B(Y, X))$ and we use $\partial^2 A(r; h)/\partial r^2 := \partial^2 A(r; h, h)/\partial r^2$. The *chain rule* and the *product rule* are valid analogously to the finite-dimensional case. As a consequence of *Taylor's theorem* for twice continuously Fréchet differentiable functions we obtain:

Theorem 1. Let Y be a normed space, let X be a Banach space and let $U \subset Y$ be an open set. Assume that $f : U \to X$ is a twice continuously differentiable function on U and let the second derivative be bounded, i.e. there exists $c > 0$ such that $\|\partial^2 f(r;\ )/\partial r^2\| \leqslant c$ on U. If $r + th \in U$ for all $t \in [0, 1]$ we have the equality

$$
f(r + h) = f(r) + \frac{\partial f}{\partial r}(r; h) + f_1(r, h) \tag{12}
$$

with some function f_1 satisfying

$$
\|f_1(r, h)\| \leqslant \sup_{r \in U} \left\| \frac{\partial^2 f}{\partial r^2}(r;\ ) \right\| \, \|h\|^2. \tag{13}
$$

Proof. An application of Taylor's theorem [1] yields

$$f(r+h) = f(r) + \frac{\partial f}{\partial r}(r;h) + \int_0^1 (1-t)\frac{\partial^2 f}{\partial r^2}(r+th;h)\,\mathrm{d}t. \tag{14}$$

Since we have $\|\partial^2 f()/\partial r^2\| \leqslant c$ on U the statement of the theorem is a direct consequence of the inequality

$$\left\| \int_0^1 (1-t)\frac{\partial^2 f}{\partial r^2}(r+th;h)\,\mathrm{d}t \right\| \leqslant \sup_{r\in U} \left\| \frac{\partial^2 f}{\partial r^2}(r;\ ) \right\| \|h\|^2. \tag{15}$$

□

In order to show the Fréchet differentiability of $(I+K-\mathrm{i}\eta S)^{-1}$ we need the following theorem.

Theorem 2. Let Y be a normed space, $U \subset Y$ an open set and X a Banach algebra with neutral element e. Let $A : U \to X$ be Fréchet differentiable in $y_0 \in U$. Assume there is a neighbourhood W of y_0 such that for all $y \in W$ the element $A(y)$ is invertible in X and the mapping $y \mapsto (A(y))^{-1}$ is continuous in y_0. Then $A^{-1}(y)$ is Fréchet differentiable in y_0 with Fréchet derivative

$$\frac{\partial}{\partial r}(A^{-1})(y_0;h) = -A^{-1}(y_0)\left(\frac{\partial A}{\partial r}(y_0;h)\right)A^{-1}(y_0). \tag{16}$$

Proof. Here we follow [3]: define

$$z(y_0,h) := A^{-1}(y_0+h) - A^{-1}(y_0) + A^{-1}(y_0)\frac{\partial A}{\partial r}(y_0;h)A^{-1}(y_0).$$

We have to show $z(y_0,h) = \mathrm{o}(\|h\|)$. For this we multiply from the left and from the right by $A(y_0)$, and use the continuous invertibility and Fréchet differentiability of A. We obtain $A(y_0)z(y_0;h)A(y_0) = \mathrm{o}(\|h\|)$ and therefore the statement of the theorem. □

We want to show the Fréchet differentiability of integral operators of the form

$$(A[r]\varphi)(x) := \int_{G_2} f(x,y,r)\varphi(y)\,\mathrm{d}\mu(y) \qquad x \in G_1;\ r \in V. \tag{17}$$

Here G_1 and G_2 are subsets of $\mathbb{R}^3$, μ denotes a measure on G_2 and $V \subset Y$ is a subset of a normed space Y. For fixed $r \in V$ and a suitable kernel the operator A is a bounded linear operator $C(G_2) \to C(G_1)$. We consider A as a mapping $V \to B(C(G_2), C(G_1))$. In the next theorem we will show that, for suitable properties of the kernel f, the differentiation of (17) can be reduced to the differentiation of the kernel f, and that the derivative of A is given by the operator

$$(\tilde{A}[r;h]\varphi)(x) := \int_{G_2} \frac{\partial f}{\partial r}(x,y,r;h)\varphi(y)\,\mathrm{d}\mu(y) \qquad x \in G_1;\ r \in V;\ h \in Y. \tag{18}$$

This includes the classical theorem concerning the differentiation of an integral depending on a parameter.

We use the following notation. Let $Y_i, i = 1, \dots, n$ be normed spaces, $U_i \subset Y_i$. We consider a function ξ of n variables $x_1, \dots, x_n$ of the form $\xi : U_1 \times \dots \times U_n \to \mathbb{C}$, $(x_1, \dots, x_n) \mapsto \xi(x_1, \dots, x_n)$. By $\xi_{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n}$ we denote the function

$$U_j \to \mathbb{C} \qquad x_j \mapsto \xi(x_1, \dots, x_n)$$

for fixed $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$. If $\xi_{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n}$ is Fréchet differentiable, we denote the Fréchet derivative by $\partial\xi/\partial x_j$. The derivative $\partial\xi/\partial x_j$ can be considered as a function

$$\frac{\partial \xi}{\partial x_j} : U_1 \times \dots \times U_n \times Y_j \to \mathbb{C}$$

or as a mapping

$$\frac{\partial \xi}{\partial x_j} : U_1 \times \dots \times U_n \to B(Y_j, \mathbb{C}).$$

Theorem 3. Let G_1, G_2 be subsets of $\mathbb{R}^3$, μ a measure on G_2 and $V \subset Y$ an open convex subset of a Banach space Y. Define $\Delta_G := \{(x, y),\ x = y,\ x \in G_1, y \in G_2\}$. Take $r_0 \in V$ and let $f : ((G_1 \times G_2) \setminus \Delta_G) \times V \to \mathbb{C}$ be a continuous function with the following properties:

- for all fixed $x \in G_1, y \in G_2, x \neq y$ the function $f_{x,y} : V \to \mathbb{C}$ is two times continuously Fréchet differentiable;
- $f_{x,r} : G_2 \setminus \{x\} \to \mathbb{C}$ and $(\partial f/\partial r)_{x,r_0,h} : G_2 \setminus \{x\} \to \mathbb{C}$ are integrable for all $x \in G_1$, $r \in V, h \in Y$;
- $A[r]$ and $\tilde{A}[r_0, h]$ given by (17) and (18) are elements of $B(C(G_2), C(G_1))$ for all $r \in V, h \in Y$;
- there is a Lebesgue-integrable function

  $$g : (G_1 \times G_2) \setminus \Delta_G \to \mathbb{R}$$

  with $\int_{G_2} g(x, y)\,\mathrm{d}\mu(y) \leqslant c$ for all $x \in G_1$. For all $x \in G_1$, $y \in G_2$, $x \neq y$ we have the estimate $|(\partial^2 f/\partial r^2)(x, y, r; h)| \leqslant g(x, y)$ uniformly for all $r \in V, h \in Y$, $\|h\| \leqslant 1$.

Then considered as a mapping $V \to B(C(G_2), C(G_1))$, $r \mapsto A[r]$ the operator A is Fréchet differentiable in r_0 and the derivative of A is given by $(\partial A/\partial r)(r_0; h) = \tilde{A}[r_0; h]$ where $\tilde{A}$ is given by (18).

Remark. The theorem covers the case $G_1 = G_2$ and weakly singular f as well as $G_1 \cap G_2 = \emptyset$ and continuous f. Therefore it can be applied to the operators S, K and P.

Proof. For all sufficiently small h we have $r_0 + h \in V$ and the convexity of V yields $r_0 + th \in V$ for all $t \in [0, 1]$. Then, as in theorem 1, the decomposition

$$f(x, y, r_0 + h) = f(x, y, r_0) + \frac{\partial f}{\partial r}(x, y, r_0; h) + f_1(x, y, r_0, h) \tag{19}$$

holds, and we have

$$|f_1(x,y,r_0,h)| \leqslant \sup_{r\in V}\left\|\frac{\partial^2 f}{\partial r^2}(x,y,r;\ )\right\| \|h\|^2 \qquad h\in Y;\ (x,y)\in(G_1\times G_2)\setminus\Delta_G.$$

Because of

$$\left|\frac{\partial^2 f}{\partial r^2}(x,y,r;h)\right| \leqslant g(x,y) \qquad r\in V;\ \|h\|\leqslant 1$$

we find

$$\left\|\frac{\partial^2 f}{\partial r^2}(x,y,r;\ )\right\| \leqslant g(x,y) \qquad r\in V.$$

Therefore we obtain integrability of f_1 and the inequality

$$\int_{G_2}|f_1(x,y,r_0,h)|\,\mathrm{d}\mu(y) \leqslant \int_{G_2}\sup_{r\in U}\left\|\frac{\partial^2 f}{\partial r^2}(x,y,r;\ )\right\| \|h\|^2\,\mathrm{d}\mu(y)$$
$$\leqslant \left(\int_{G_2} g(x,y)\,\mathrm{d}\mu(y)\right)\|h\|^2.$$

We now know that all terms in equation (19) are integrable on G_2, and can use the linearity of the integral to obtain

$$\begin{aligned}(A[r_0+h]\varphi)(x) &= \int_{G_2} f(x,y,r_0+h)\varphi(y)\,\mathrm{d}\mu(y)\\ &= \int_{G_2} f(x,y,r_0)\varphi(y)\,\mathrm{d}\mu(y) + \int_{G_2}\frac{\partial f}{\partial r}(x,h,r_0;h)\varphi(y)\,\mathrm{d}\mu(y)\\ &\quad + \int_{G_2} f_1(x,y,r_0,h)\varphi(y)\,\mathrm{d}\mu(y)\\ &= (A[r_0]\varphi)(x) + (\tilde{A}[r_0;h]\varphi)(x) + (A_1[r_0,h]\varphi)(x)\end{aligned}$$

where the operator A_1 satisfies

$$|(A_1[r_0,h]\varphi)(x)| \leqslant c\|\varphi\|_\infty\|h\|^2.$$

Therefore A is Fréchet differentiable in r_0 considered as a mapping $V\to B(C(G_2),C(G_1))$ with the derivative given by $\partial A/\partial r = \tilde{A}$. □

4. Fréchet differentiability of boundary integral operators

As an application of theorem 3 we want to show the Fréchet differentiability of the operators occurring in section 2.

First we deal with S and K. Using the transformations described in section 2 the operators can be brought into the form

$$(S[r]\varphi)(x) = \int_{\partial D}\frac{h_1(|x_r-y_r|)}{|x_r-y_r|}J_r(y)\varphi(y)\,\mathrm{d}s(y) \tag{20}$$

$$\begin{aligned}(K[r]\varphi)(x) = \int_{\partial D}\langle\nu_r(y), y_r-x_r\rangle&\left\{\frac{h_2(|x_r-y_r|)}{|x_r-y_r|^3}+\frac{h_3(|x_r-y_r|)}{|x_r-y_r|^2}\right\}\\ &\times J_r(y)\varphi(y)\,\mathrm{d}s(y).\end{aligned} \tag{21}$$

where the functions h_1, h_2 and h_3 are analytic complex valued functions, and where $J_r(y)$ denotes the Jacobian of the transformation ϕ_r in $y\in\partial D$.

Theorem 4. The integral operators S and K are Fréchet differentiable in V_l, considered as mappings

$$V_l \to B(C(\partial D), C(\partial D)).$$

The Fréchet derivative is obtained by differentiation of the kernels according to theorem 3.

We base the proof of the theorem on the following lemma:

Lemma 1. The kernels of the integral operators given by (20) and (21) are two times continuously Fréchet differentiable as mappings $V_l \to \mathbb{C}$ for all fixed $x \neq y$, $x, y \in \partial D$. The kernels and their first two derivatives are bounded on V_l by

$$g(x, y) = C\frac{1}{|x-y|} \qquad \text{for all } r \in V_l;\ x, y \in \partial D \tag{22}$$

with some constant $C > 0$.

Proof of theorem 4. We establish the assumptions made in theorem 3. Lemma 1 states the Fréchet differentiability of the kernels of S and K and also gives estimates for their singularity and those of their derivatives: there is a weakly singular majorante g and therefore they are weakly singular. Now by standard arguments S, K and the operators which are built by integration of the derivatives of the kernels are well defined bounded linear operators $C(\partial D) \to C(\partial D)$. Thus we apply theorem 3 to obtain theorem 4. □

Proof of lemma 1. We verify the Fréchet differentiability of the kernels by four elementary steps. We will use the letter c to denote a generic constant.

Step 1. The mapping $g_{x,y} : V_l \to \mathbb{R}^3$ defined by

$$g_{x,y}(r) := x_r - y_r = (x + r(x)) - (y + r(y))$$

is the sum of a constant and a linear mapping and therefore, for all fixed $x, y \in \partial D$, it is Fréchet differentiable with derivative

$$\frac{\partial g_{x,y}}{\partial r}(r; h) = h(x) - h(y) \qquad h \in C^2(\partial D).$$

The derivative does not depend on $r \in V_l$ and therefore it is continuous. Since for $x \neq y$ we have $x_r - y_r \neq 0$ for all $r \in V_l$, using the chain rule, we obtain the Fréchet differentiability of the mapping

$$g_{1,x,y} : V_l \to \mathbb{R} \qquad r \mapsto |x_r - y_r|$$

for all $r \in V_l$, $x \neq y$ and $x, y \in \partial D$. The Fréchet derivative is given by

$$\frac{\partial g_{1,x,y}}{\partial r}(r; h) = \frac{1}{|x_r - y_r|}\langle (x_r - y_r), \big(h(x) - h(y)\big)\rangle \qquad h \in C^2(\partial D). \tag{23}$$

We use the mean-value theorem for the differentiable vector fields $r \in V_l$ on the manifold ∂D to obtain the estimates

$$\gamma_1 |x - y| \leqslant |x_r - y_r| \tag{24}$$

$$|x_r - y_r| \leqslant \gamma_2 |x - y| \tag{25}$$

uniformly on V_l, where γ_1 and γ_2 are constants depending on l and ∂D. Again with the help of the mean-value theorem—this time applied to h—we derive from (24) and (25) the inequalities

$$\left|\frac{\partial g_{1,x,y}}{\partial r}(r;h)\right| \leqslant c\|h\|_{C^2(\partial D)}|x-y| \qquad \forall r \in V_l;\ h \in C^2(\partial D) \tag{26}$$

with some constant c. Proceeding as for $g_{1,x,y}$ we obtain the Fréchet differentiability of the mapping

$$g_{2,x,y} : V_l \to \mathbb{R} \qquad r \mapsto \frac{1}{|x_r - y_r|^n} \tag{27}$$

the derivative

$$\frac{\partial g_{2,x,y}}{\partial r}(r;h) = (-n)\frac{1}{|x_r - y_r|^{n+2}}\langle (x_r - y_r), h(x) - h(y)\rangle \tag{28}$$

and the estimate

$$\left|\frac{\partial g_{2,x,y}}{\partial r}(r;h)\right| \leqslant c\frac{1}{|x_r - y_r|^n}\|h\|_{C^2(\partial D)} \tag{29}$$

with some constant c. We also want to compute the second derivatives of the terms and to give similar estimates. To do so we have to consider the first derivatives as mappings $V_l \to B(C^2(\partial D), \mathbb{R})$. Using the same arguments as above we obtain

$$\frac{\partial^2 g_{1,x,y}}{\partial r^2}(r;h) = \frac{(-1)}{|x_r - y_r|^3}\langle (x_r - y_r), (h(x) - h(y))\rangle^2 + \frac{1}{|x_r - y_r|}\langle (h(x) - h(y)), (h(x) - h(y))\rangle \tag{30}$$

$$\frac{\partial^2 g_{2,x,y}}{\partial r}(r;h) = n(n+1)\frac{1}{|x_r - y_r|^{n+2}}\left(\frac{\partial g_{1,x,y}}{\partial r}(r;h)\right)^2 - n\frac{1}{|x_r - y_r|^{n+1}}\ \frac{\partial^2 g_{1,x,y}}{\partial r^2}(r;h) \tag{31}$$

and the estimates

$$\left|\frac{\partial^2 g_{1,x,y}}{\partial r^2}(r;h)\right| \leqslant c\|h\|^2_{C^2(\partial D)}|x-y| \qquad r \in V_l;\ h \in C^2(\partial D) \tag{32}$$

and

$$\left|\frac{\partial^2 g_{2,x,y}}{\partial r^2}(r;h)\right| \leqslant c\|h\|^2_{C^2(\partial D)}\frac{1}{|x_r - y_r|^n} \qquad r \in V_l;\ h \in C^2(\partial D). \tag{33}$$

The estimates show that the degree of the singularity in $|x-y|$ of the functions under consideration does not increase when we differentiate. We also want to prove this for the other components of the kernels.

Step 2. Consider the term $\langle \nu_r(x), x_r - y_r \rangle$ and use local coordinates (u, v). With $x = x(u_1, v_1)$ and $y = y(u_2, v_2)$ we have the estimate $\tilde{\gamma}_1 |(u_1, v_1) - (u_2, v_2)| \leqslant |x - y| \leqslant \tilde{\gamma}_2 |(u_1, v_1) - (u_2, v_2)|$ for $x \in U(y)$, where $U(y)$ is a neighbourhood of y and $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are constants [2]. In $U(y)$ we can write

$$\langle \nu_r(x), x_r - y_r \rangle = \frac{1}{g_{3,y}(r)} \left\langle \left\{ \left(\frac{\partial y}{\partial u_2} + \frac{\partial r(y)}{\partial u_2} \right) \times \left(\frac{\partial y}{\partial v_2} + \frac{\partial r(y)}{\partial v_2} \right) \right\}, \{x + r(x) - (y - r(y))\} \right\rangle \tag{34}$$

with

$$g_{3,y}(r) := \left| \left(\frac{\partial y}{\partial u_2} + \frac{\partial r(y)}{\partial u_2} \right) \times \left(\frac{\partial y}{\partial v_2} + \frac{\partial r(y)}{\partial v_2} \right) \right|.$$

The function $g_{3,y}$ is Fréchet differentiable in V_l and there exist constants c_1 and c_2 with $0 < c_1 \leqslant g_{3,y} \leqslant c_2$ and $0 < c_1 \leqslant \partial g_{3,y}/\partial r \leqslant c_2$ $\forall r \in V_l$. Therefore $1/g_{3,y}$ is also Fréchet differentiable in V_l and the derivative is bounded. Using the chain rule, clearly the other terms of (34) are Fréchet differentiable. For the derivative

$$f(u_1, v_1) := \frac{\partial}{\partial r} \left(\left\langle \left(\frac{\partial y}{\partial u_2} + \frac{\partial r(y)}{\partial u_2} \right) \times \left(\frac{\partial y}{\partial v_2} + \frac{\partial r(y)}{\partial v_2} \right), x + r(x) - (y - r(y)) \right\rangle \right)(r; h) \tag{35}$$

we want to show that

$$|f(u_1, v_1)| \leqslant L |(u_1, v_1) - (u_2, v_2)|^2 \tag{36}$$

uniformly for $r \in V_l$ and $h \in K_1 \subset C^2(\partial D)$. The estimate (36) is a direct consequence of Taylor's theorem applied to the twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, if we are able to show that $\mathrm{grad}_{(u_1, v_1)} f|_{u_1 = u_2, v_1 = v_2} = 0$. This can be verified by a straightforward but lengthy calculation. Now collecting all terms and using the product rule for the differentiation of (34) we obtain the estimate

$$\left| \frac{\partial}{\partial r} \{\nu_r(x) \cdot (x_r - y_r)\}(r; h) \right| \leqslant c \|h\|_{C^2(\partial D)} |x - y|^2 \tag{37}$$

for all $r \in V_l$. For the second derivative we obtain the analogous result

$$\left| \frac{\partial^2}{\partial r^2} \{\nu_r(x) \cdot (x_r - y_r)\}(r; h) \right| \leqslant c \|h\|^2_{C^2(\partial D)} |x - y|^2 \tag{38}$$

for all $r \in V_l$.

Step 3. We obtain the differentiability of $J_r(y)$ using the representation

$$J_r(y) = \left| \frac{\partial}{\partial u_1}(y + r(y)) \times \frac{\partial}{\partial u_2}(y + r(y)) \right| \Big/ \left| \frac{\partial}{\partial u_2} y \times \frac{\partial}{\partial v_2} y \right|$$

which is valid locally. The derivatives of J_r are uniformly bounded for $r \in V_l$, $y \in \partial D$.

Step 4. The statement of lemma 1 can now be verified using the estimates of steps 1–3, the chain and product rule. □

Corollary 1. The operator $(I+K-\mathrm{i}\eta S)^{-1}$ is Fréchet differentiable considered as a mapping $V_l \to B(C(\partial D), C(\partial D))$ and the Fréchet derivative is given by

$$\frac{\partial\big((I+K-\mathrm{i}\eta S)^{-1}\big)}{\partial r} = -(I+K-\mathrm{i}\eta S)^{-1}\frac{\partial(K-\mathrm{i}\eta S)}{\partial r}(I+K-\mathrm{i}\eta S)^{-1}.$$

Proof. The statement follows by combining theorems 2 and 4. □

We transform the operator P onto the reference surface ∂D

$$\begin{aligned}(P[r]\varphi)(x) &= u[r,\varphi](x)\\ &= \int_{\partial D}\left\{\frac{h_1(|x-y_r|)}{|x-y_r|} - \mathrm{i}\eta\langle\nu(y_r), x-y_r\rangle\right.\\ &\quad\times\left.\left[\frac{h_2(|x-y_r|)}{|x-y_r|^3}+\frac{h_3(|x-y_r|)}{|x-y_r|^2}\right]\right\}J_r(y)\varphi(y)\,\mathrm{d}s(y) \qquad x\in M.\end{aligned} \tag{39}$$

and establish the following result.

Theorem 5. The integral operator $P : V_l \to B(C(\partial D), C(M))$ is Fréchet differentiable and the derivative can be computed by differentiation of the kernel of P.

Analogously to the proof of theorem 4 we base the proof on the following lemma which can be shown analogously to lemma 1. It is actually more simple since the kernels have no singularities.

Lemma 2. The kernel of the operator P given by (39) is two times continuously Fréchet differentiable as a mapping $V_l \to \mathbb{C}$ for fixed $x \in M, y \in \partial D$. The derivatives are continuous on $M \times \partial D \times V_l$ and bounded by a constant $C \in \mathbb{R}$.

Proof of theorem 5. We verify the assumptions of theorem 3. The differentiability of the kernels and their continuity is stated in lemma 2. Therefore P and the operators which are built by integration of the derivatives of the kernel are well defined bounded linear operators $C(\partial D) \to C(M)$. Since $\mu(\partial D) < \infty$ the constant C is an integrable majorante of the kernels and their derivatives. Now theorem 3 can be applied to obtain the statement of theorem 5. □

Now consider the operator R. We can write

$$(R[r]u^{\mathrm{i}})(x) = u^{\mathrm{i}}(x_r) = u^{\mathrm{i}}(x+r(x)) \qquad x\in\partial D.$$

Theorem 6. The operator $R : V_l \to B(C^1(B), C(\partial D))$ is Fréchet differentiable with derivative

$$\left\{\frac{\partial R}{\partial r}[r;h]u^{\mathrm{i}}\right\}(x) = (\operatorname*{grad}_x u^{\mathrm{i}})(x_r) \cdot h(x) \qquad x \in \partial D.$$

Proof. The proof is a simple application of the chain rule. □

Corollary 2. The nonlinear mapping $R^{\mathrm{s}} : V_l \to C(M)$, $r \mapsto u^{\mathrm{s}}|_M$ is Fréchet differentiable and the derivative is given by

$$\begin{aligned}\frac{\partial(R^{\mathrm{s}})}{\partial r} = -2\frac{\partial P}{\partial r}(I+K-\mathrm{i}\eta S)^{-1}Ru^{\mathrm{i}} \\ + 2P(I+K-\mathrm{i}\eta S)^{-1}\frac{\partial(K-\mathrm{i}\eta S)}{\partial r}(I+K-\mathrm{i}\eta S)^{-1}Ru^{\mathrm{i}} \\ - 2P(I+K-\mathrm{i}\eta S)^{-1}\frac{\partial R}{\partial r}u^{\mathrm{i}}.\end{aligned} \tag{40}$$

5. Characterization of the derivative of R^{s}

The actual numerical evaluation of $\partial(R^{\mathrm{s}})/\partial r$ using corollary 2 is rather lengthy. Therefore we characterize the derivative of R^{s} as the solution of a Dirichlet boundary value problem [4].

Theorem 7. The Fréchet derivative $\partial R^{\mathrm{s}}(r;h)/\partial r$ of R^{s} is given by the solution to the exterior Dirichlet problem for the domain D with boundary values

$$-\langle h(x), \operatorname{grad} u(x_r)\rangle = -\langle h(x), \nu_r(x)\rangle \frac{\partial u}{\partial \nu_r}(x) \qquad x \in \partial D \tag{41}$$

where $u = u^{\mathrm{i}} + u^{\mathrm{s}}$ is the solution of the scattering problem.

Proof. We show that $\partial R^{\mathrm{s}}(r;h)/\partial r$ given by corollary 2 is the solution of the exterior Dirichlet problem with boundary values given by (41). $\partial R^{\mathrm{s}}(r;h)/\partial r$ solves the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{D_r}$ and satisfies the Sommerfeld radiation condition because differentiation with respect to $x \in \mathbb{R}^3 \setminus \overline{D_r}$ and the Fréchet differentiation with respect to r may be interchanged. We have to compute the boundary values of $\partial R^{\mathrm{s}}(r;h)/\partial r$.

The strip

$$D_r^{\tau_0} := \{x \in \mathbb{R}^3,\ \min_{y\in\partial D_r} |x-y| < \tau_0\}$$

is bijectively mapped onto the set $\{(x,\tau), x \in \partial D, -\tau_0 < \tau < \tau_0\}$ by

$$x_r^{\tau} := x + r(x) + \nu_r(x)\cdot\tau$$

for fixed $r \in V_l$ and for $\tau < \tau_0$, τ_0 sufficiently small. For brevity in this proof we will write $S[r] = S$, $K[r] = K$, $P[r] = P$ and $R[r] = R$.

Step 1. We compute the boundary values of $P(I+K-\mathrm{i}\eta S)^{-1}(\partial R/\partial r)u^{\mathrm{i}}$, i.e. the last term of (40). Since $\lim_{\tau\to 0}(2P\varphi)(x_r^\tau) = ((I+K-\mathrm{i}\eta S)\varphi)(x)$, $x \in \partial D$ we obtain

$$\lim_{\tau\to 0}\left(-2P(I+K-\mathrm{i}\eta S)^{-1}\frac{\partial R}{\partial r}(r;h)u^{\mathrm{i}}\right)(x_r^\tau) = -\left(\frac{\partial R}{\partial r}(r;h)u^{\mathrm{i}}\right)(x)$$

$$= -\langle h(x), \operatorname*{grad}_x u^{\mathrm{i}}(x_r)\rangle.$$

Step 2. We want to show that for the limiting value of the first two terms in (40) we have

$$\lim_{\tau\to 0}\left\{-2\frac{\partial P}{\partial r}(r;h)(I+K-\mathrm{i}\eta S)^{-1}Ru^{\mathrm{i}}(x_r^\tau)\right.$$
$$\left.+2(P(I+K-\mathrm{i}\eta S)^{-1}\frac{\partial(K-\mathrm{i}\eta S)}{\partial r}(r;h)(I+K-\mathrm{i}\eta S)^{-1}Ru^{\mathrm{i}})(x_r^\tau)\right\}$$
$$= -\langle h(x), \operatorname{grad}\{u^{\mathrm{s}}\}(x_r)\rangle. \tag{42}$$

Using the chain rule we derive

$$\frac{\partial}{\partial r}\{2P\varphi\}(r;h)(x_r^\tau) = \frac{\partial}{\partial r}\{(2P\varphi)(x_r^\tau)\}(r;h) - \langle h(x), \operatorname*{grad}_x\{2P\varphi\}(x_r^\tau)\rangle$$
$$-\left\langle \tau\cdot\frac{\partial\nu_r(x)}{\partial r}(r;h), \operatorname*{grad}_x\{2P\varphi\}(x_r^\tau)\right\rangle. \tag{43}$$

We now take $\varphi := (I+K-\mathrm{i}\eta S)^{-1}Ru^{\mathrm{i}}$ and use $u^{\mathrm{s}} = -2P(I+K-\mathrm{i}\eta S)^{-1}Ru^{\mathrm{i}}$ to obtain for the first term of (42)

$$\left(\frac{\partial}{\partial r}\{-2P\}(r;h)(I+K-\mathrm{i}\eta S)^{-1}Ru^{\mathrm{i}}\right)(x_r^\tau)$$
$$= \frac{\partial}{\partial r}\{(-2P\varphi)(x_r^\tau)\}(r;h)$$
$$-\langle h(x), \operatorname*{grad}_x\{u^{\mathrm{s}}\}(x_r^\tau)\rangle - \left\langle \tau\cdot\frac{\partial\nu_r(x)}{\partial r}(r;h), \operatorname*{grad}_x\{u^{\mathrm{s}}\}(x_r^\tau)\right\rangle.$$

Since solutions u^{i} to the Helmholtz equation are analytic, and since $(I+K-\mathrm{i}\eta S)^{-1}$ maps $C^{1,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$, we have $\varphi \in C^{1,\alpha}(\partial D)$. Therefore the term

$$-\langle h(x), \operatorname*{grad}_x\{u^{\mathrm{s}}\}(x_r^\tau)\rangle - \left\langle \tau\cdot\frac{\partial\nu_r(x)}{\partial r}(r;h), \operatorname*{grad}_x\{u^{\mathrm{s}}\}(x_r^\tau)\right\rangle$$

has the limiting value $-\langle h(x), \operatorname{grad}_x\{u^{\mathrm{s}}\}(x_r)\rangle$ for $\tau \to 0$ [2]. We know $\lim_{\tau\to 0}(2P(I+K-\mathrm{i}\eta S)^{-1}\varphi)(x_r^\tau) = \varphi(x)$, $x \in \partial D$. To show (42) we still have to verify that

$$\lim_{\tau\to 0}\left\{\frac{\partial}{\partial r}\{(-2)(P\varphi)(x_r^\tau)\}(r;h) + \frac{\partial}{\partial r}(K-\mathrm{i}\eta S)(r;h)\varphi(x)\right\} = 0. \tag{44}$$

For the sake of simplicity we will establish this only for the theoretical potential case $k = 0$. The case $k \neq 0$ can be handled analogously. We split the potential P into two parts: the double-layer potential P_1 and $-i\eta$ times the single-layer potential P_2. First we show

$$\lim_{\tau\to 0}\left\{\frac{\partial}{\partial r}\{(-2)(P_2\varphi)(x_r^\tau)\}(r;h) + \frac{\partial}{\partial r}S(r;h)\varphi(x)\right\} = 0. \tag{45}$$

We compute

$$\begin{aligned}
\frac{\partial}{\partial r}&\{(-2)(P_2\varphi)(x_r^\tau)\}(r;h)\\
&= 2\int_{\partial D}\frac{\langle x_r^\tau - y_r, h(x) - h(y)\rangle}{|x_r^\tau - y_r|^3}J_r(y)\varphi(y)\,\mathrm{d}s(y)\\
&\quad - 2\int_{\partial D}\frac{1}{|x_r^\tau - y_r|}\left\{\frac{\partial}{\partial r}J_r(y)\right\}(r;h)\varphi(y)\,\mathrm{d}s(y)\\
&\quad + \tau\cdot 2\int_{\partial D}\frac{\langle x_r^\tau - y_r, (\partial\nu_r(x)/\partial r)(r;h)\rangle}{|x_r^\tau - y_r|^3}J_r(y)\varphi(y)\,\mathrm{d}s(y).
\end{aligned}\tag{46}$$

The continuity of the first two terms of the right-hand side of (46) for $\tau \to 0$ and their limiting value

$$\begin{aligned}
2\int_{\partial D}&\frac{\langle x_r - y_r, h(x) - h(y)\rangle}{|x_r - y_r|^3}J_r(y)\varphi(y)\,\mathrm{d}s(y)\\
&\quad - 2\int_{\partial D}\frac{1}{|x_r - y_r|}\left\{\frac{\partial}{\partial r}J_r(y)\right\}(r;h)\varphi(y)\,\mathrm{d}s(y)\\
&= \left[\left(-\frac{\partial}{\partial r}S(r;h)\right)\varphi\right](x)
\end{aligned}$$

is a consequence of theorem 2.7 of [2]. The third integral in (46) can be written in the form

$$\left\langle\frac{\partial\nu_r(x)}{\partial r}(r;h), \left(\operatorname*{grad}_x\int_{\partial D}\frac{1}{|x - y_r|}J_r(y)\varphi(y)\,\mathrm{d}s(y)\right)\bigg|_{x_r^\tau}\right\rangle. \tag{47}$$

Since $\varphi \in C^{0,\alpha}(\partial D)$ the term (47) is bounded for $\tau > 0$ as a consequence of theorem 2.17 of [2]. We find

$$\lim_{\tau\to 0}\tau\cdot 2\left\langle\frac{\partial\nu_r(x)}{\partial r}(r;h), \left(\operatorname*{grad}_x\int_{\partial D}\frac{1}{|x - y_r|}J_r(y)\varphi(y)\,\mathrm{d}s(y)\right)\bigg|_{x_r^\tau}\right\rangle = 0$$

and hence we have proved (45).

Now we have to show

$$\lim_{\tau\to 0}\left\{\frac{\partial}{\partial r}\{(-2)(P_1\varphi)(x_r^\tau)\}(r;h) + \left(\frac{\partial K}{\partial r}(r;h)\varphi\right)(x)\right\} = 0. \tag{48}$$

The case of the double-layer potential turns out to be more complicated. We use the decomposition $v_r(x_r^\tau) = \varphi(x)w_r(x_r^\tau) + u_r(x_r^\tau)$, where v_r denotes the double-layer potential on ∂D_r with density φ given by (5). w_r denotes the double-layer potential with constant density 1 and u_r is defined by

$$u_r(x_r^\tau) := \int_{\partial D} \frac{\partial \Phi(x_r^\tau, y_r)}{\partial \nu_r(y)} [\varphi(y) - \varphi(x)] J_r(y)\, ds(y). \tag{49}$$

We obtain

$$\frac{\partial}{\partial r}\big\{(-2)(P_1\varphi)(x_r^\tau)\big\}(r;h) = (-2)\frac{\partial}{\partial r}\big\{\varphi(x)w_r(x_r^\tau) + u_r(x_r^\tau)\big\}$$
$$= (-2)\frac{\partial}{\partial r}\big\{u_r(x_r^\tau)\big\}$$

and

$$\frac{\partial}{\partial r}\big\{K\varphi\big\}(r;h)(x) = 2\frac{\partial}{\partial r}\big\{\varphi(x)w_r(x_r) + u_r(x_r)\big\}$$
$$= 2\frac{\partial}{\partial r}\big\{u_r(x_r)\big\}$$

since $w_r(x_r^\tau) = 1$ if $\tau > 0$ and $w_r(x_r^\tau) = 0.5$ if $\tau = 0$ for all $r \in V_l$. We have to verify the continuity of $\partial\{u_r(x_r^\tau)\}/\partial r$ for $\tau \to 0$. We compute

$$\begin{aligned}\frac{\partial}{\partial r}\{u_r(x_r^\tau)\} = &\int_{\partial D} \frac{\partial}{\partial r}\langle \nu_r(y), x_r^\tau - y_r\rangle(r;h) \frac{1}{|x_r^\tau - y_r|^3} J_r(y)[\varphi(y) - \varphi(x)]\, ds(y)\\ &+ \int_{\partial D} \langle \nu_r(y), x_r^\tau - y_r\rangle \frac{(-3)\langle x_r^\tau - y_r, (\partial/\partial r)(x_r^\tau - y_r)(r;h)\rangle}{|x_r^\tau - y_r|^5}\\ &\times J_r(y)[\varphi(y) - \varphi(x)]\, ds(y)\\ &+ \int_{\partial D} \langle \nu_r(y), x_r^\tau - y_r\rangle \frac{1}{|x_r^\tau - y_r|^3} \frac{\partial}{\partial r}\{J_r(y)\}(r;h)[\varphi(y) - \varphi(x)]\, ds(y)\end{aligned} \tag{50}$$

and

$$\frac{\partial}{\partial r}(x_r^\tau - y_r)(r;h) = h(x) - h(y) + \frac{\partial \nu_r(x)}{\partial r}(r;h)\cdot\tau \tag{51}$$

with

$$\left|\frac{\partial}{\partial r}(x_r^\tau - y_r)(r;h)\right| \leqslant c|x_r^\tau - y_r|. \tag{52}$$

Using the estimate (37) we obtain the continuity of $\partial\{u_r(x_r^\tau)\}/\partial r$ as a consequence of the following lemma 3. □

Lemma 3. For $\varphi \in C(\partial D)$ define

$$\tilde u_1(x_r^\tau) := \int_{\partial D} \langle \nu_r(y), x_r^\tau - y_r\rangle K_{r,h}(x_r^\tau, y_r) \frac{1}{|x_r^\tau - y_r|^3} J_r(y)[\varphi(y) - \varphi(x)]\, ds(y) \tag{53}$$

and

$$\tilde u_2(x_r^\tau) := \int_{\partial D} \frac{\partial}{\partial r}\langle \nu_r(y), x_r^\tau - y_r\rangle(r;h) \frac{1}{|x_r^\tau - y_r|^3} J_r(y)[\varphi(y) - \varphi(x)]\, ds(y) \tag{54}$$

where the kernel K is continuously differentiable with respect to x, $x \neq y$, $K_{r;h}$ is bounded and we have $|\partial K(x,y)/\partial x| \leqslant C/|x-y|$ for all $x \neq y$. Then $\tilde u_1$ and $\tilde u_2$ are continuous in $\partial D_r^{\tau_0}$.

Proof. Using

$$|\langle \nu_r(y), x_r - y_r\rangle| \leqslant L|x_r - y_r|^2 \tag{55}$$

[2] and (37), we observe that the integrals exist as improper integrals for $\tau = 0$ and represent continuous functions on ∂D_r. It suffices to show that

$$\lim_{\tau\to 0} \tilde{u}_i(x_r^\tau) = \tilde{u}_i(x_r) \qquad i = 1, 2$$

uniformly on ∂D_r. We carry out the proof for $\tilde{u}_1$.

Define

$$\Psi_{r;h}(x, y) := \langle \nu_r(y), x - y\rangle\ K_{r;h}(x, y)\frac{1}{|x-y|^3} J_r(y).$$

Using (55) for sufficiently small τ we obtain

$$\begin{aligned}|x_r^\tau - y_r|^2 &= |x_r - y_r|^2 + 2\langle x_r - y_r, x_r^\tau - x_r\rangle + |x_r^\tau - x_r|^2\\ &\geqslant \tfrac{1}{2}\{|x_r - y_r|^2 + |x_r^\tau - x_r|^2\}.\end{aligned}$$

Then with the decomposition

$$\nu_r(y)(x_r^\tau - y_r) = \nu_r(y)(x_r^\tau - x_r) + \nu_r(y)(x_r - y_r)$$

for all $q < Q$, by projecting onto the tangent plane, we obtain

$$\begin{aligned}\int_{S_{x,q}} |\Psi_{r;h}(x_r^\tau, y_r)|\,\mathrm{d}s(y) &\leqslant C\left\{\int_0^q \mathrm{d}\varrho + |x_r^\tau - x_r| \int_0^\infty \frac{\varrho\,\mathrm{d}\varrho}{(\varrho^2 + |x_r^\tau - x_r|^2)^{3/2}}\right\}\\ &= C(q+1) \leqslant C(Q+1)\end{aligned} \tag{56}$$

with $S_{x,q} := \partial D \cap K_q(x)$ and some constant C depending on ∂D and r. From the mean-value theorem we see that

$$|\Psi_{r;h}(x_r^\tau, y_r) - \Psi_{r;h}(x_r, y_r)| \leqslant C_2 \frac{|x_r^\tau - x_r|}{|x_r - y_r|^3}$$

for $2|x_r^\tau - x_r| \leqslant |x_r - y_r|$ and therefore

$$\int_{\partial D\setminus S_{x,q}} |\Psi_{r;h}(x_r^\tau, y_r) - \Psi_{r;h}(x_r, y_r)|\,\mathrm{d}s(y) \leqslant C_3 \frac{|x_r^\tau - x_r|}{q^3} \tag{57}$$

with some constants C_2 and C_3. Now we can combine (56) and (57) to obtain

$$|\tilde{u}_1(x_r^\tau) - \tilde{u}_1(x_r)| \leqslant C\left\{\sup_{|y-x|\leqslant q} |\varphi(y) - \varphi(x_r)| + \frac{|x_r^\tau - x_r|}{q^3}\right\} \tag{58}$$

for some constant C. Given $\epsilon > 0$ we can choose $q > 0$ such that

$$|\varphi(y) - \varphi(x)| < \epsilon/2C$$

for all $y, x \in \partial D$ with $|y - x| < q$ since φ is uniformly continuous on ∂D. Then taking $\delta < (\epsilon/2C)q^3$, we see that

$$|u(x_r^\tau) - u(x_r)| < \epsilon$$

for all $|x_r^\tau - x_r| < \delta$ and the first part of the lemma is proved. The second part can be proved imitating the preceding proof but using $(\partial/\partial r)\{\nu_r(y)(x_r^\tau - y_r)\}(r;h)$ instead of $\nu_r(y)(x_r^\tau - y_r)$. □

Remark. We finally want to look at the statement of theorem 7 from a heuristic point of view. The boundary values of the Fréchet derivative of the scattered field are the sum of the two terms $-\langle h(x), (\operatorname{grad} u^{\mathrm{i}})(x_r)\rangle$ and $-\langle h(x), (\operatorname{grad} u^{\mathrm{s}})(x_r)\rangle$. The first term is the boundary values of the Dirichlet problem with a given function $-\langle h(x), (\operatorname{grad} u^{\mathrm{i}})(x_r)\rangle$ on the boundary. This term comes from the change of u^{s} when the incident field is varied. The second term can be considered locally as the change of u^{s} when the boundary is translated in direction $h(x)$.

Acknowledgments

The author would like to thank Professor Rainer Kress for many helpful discussions throughout the course of this work. In addition, thanks are extended to Dr Peter Hähner for carefully reading this paper.

References

[1] Berger M S 1977 *Nonlinearity and Functional Analysis* (New York: Academic)
[2] Colton D and Kress R 1983 *Integral Equation Methods in Scattering Theory* (New York: Wiley)
[3] Colton D and Kress R 1992 *Inverse Acoustic and Electromagnetic Scattering* (Berlin: Springer)
[4] Kirsch A 1993 The domain derivative and two applications in inverse scattering theory *Inverse Problems* **9** 81–96
[5] Kress R 1989 *Linear Integral Equations* (New York: Springer)
[6] Pironneau O 1984 *Optimal Shape Design for Elliptic Systems* (Berlin: Springer)
[7] Roger A 1981 Newton Kantorovich algorithm applied to an electromagnetic inverse problem *IEEE Trans. Antenn. Propag.* **AP-29** 232–8
[8] Simon J 1980 Differentiation with respect to the domain in boundary value problems *Num. Func. Anal. Opt.* (2)