Robust stability and stabilization of uncertain linear positive systems via Integral Linear Constraints: L_1- and L_∞-gains characterization

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Abstract

Copositive linear Lyapunov functions are used along with dissipativity theory for stability analysis and control of uncertain linear positive systems. Unlike usual results on linear systems, linear supply-rates are employed here for robustness and performance analysis using L_1- and L_∞-gains. Robust stability analysis is performed using Integral Linear Constraints (ILCs) for which several classes of uncertainties are discussed. The approach is then extended to robust stabilization and performance optimization. The obtained results are expressed in terms of robust linear programming problems that are equivalently turned into finite dimensional ones using Handelman's Theorem. Several examples are provided for illustration.

Index Terms

Positive linear systems; robustness; integral linear constraints; robust control; robust linear programming; relaxation.

I. INTRODUCTION

Linear internally positive systems are a particular class of linear systems whose state takes only nonnegative values. Such models can represent many real world processes, from biology [1], passing through ecology and epidemiology [2], to networking [3]. Compartmental models, used e.g. in biological, medical, epidemiological applications, are also generally expressed as (non)linear nonnegative systems [1]. Several works have been devoted to their analysis and control, see e.g. [4], [5], [6], [7], [8], [9], [10], [11], [12], [13].

Quadratic Lyapunov functions of the form $V(x) = x^T P x$, with $P = P^T$ positive definite, are the most commonly used to study the stability of linear systems. Dissipativity theory with *quadratic storage functions* and *quadratic supply-rates* are also widely used for robustness analysis, e.g. through the full-block S-procedure [14], and norms computation, e.g. the $\mathcal{H}_\infty$-norm and generalized $\mathcal{H}_2$-norm. This quadratic framework also allows to apply powerful analysis techniques such as those based on *Integral Quadratic Constraints* (IQCs) [15] (exploiting the KYP-Lemma [16], [17], [18] and the Plancherel Theorem), and *well-posedness theory* [19], [20], [21] which is mainly based on topological separation

[19]. An interesting point in these approaches is that many of the obtained results can be represented as optimization problems involving Linear Matrix Inequalities (LMIs) [22], a wide class of convex optimization problems solvable in polynomial-time using e.g. interior-point algorithms [23].

Unlike general linear systems, the stability of linear positive systems can be losslessly analyzed by considering a positive definite *diagonal matrix P*. Such functions are then called *diagonal Lyapunov functions* and the corresponding stability notion *diagonal stability* [24]. This particular form dramatically simplifies the structure of the stability and stabilization problems, allowing for instance the design of structured and decentralized controllers in a simple way (convex formulation). Convexity does not hold in the general linear systems case where the design of structured controllers is known to be a NP-hard problem [25]. Recently, a KYP-Lemma for positive systems using diagonal Lyapunov functions has been obtained in [10] by using the results reported in [9]. In the same vein, exact losslessness conditions for robust stability analysis have been further obtained in [11] where several classes of uncertainties are covered.

There exists however another class of Lyapunov functions leading to necessary and sufficient conditions. They are referred to as *linear copositive Lyapunov functions* [6], [7], [26], [27], [28], [29] and write $V(x) = \lambda^T x$, where λ is a vector with positive entries. In such a case, the resulting stability condition can be expressed as a linear programming problem (convex again) and solved in an efficient way. Since the Lyapunov function is linear, there is no more relationship with the vector 2-norm and the L_2-norm as in the quadratic case, but rather with the vector 1-norm and the L_1-norm. This framework is then more suitable for the analysis of the L_1-gain of positive systems and its consideration in robustness and performance characterization.

In this paper, the stability analysis and control of uncertain positive systems is considered in the L_1-induced norm and L_∞-induced norm using linear copositive Lyapunov functions and dissipativity theory [30], [6] with linear supply-rates. Stability analysis and control synthesis results for unperturbed systems are first provided to set up the ideas and introduce the important tools. It is shown that computing the L_1- and L_∞-gains for linear positive systems is tantamount to solving a linear programming problem, with a complexity growing linearly with respect to the system size. The L_1-gain is determined via a direct application of dissipativity theory while the L_∞-gain is computed as the L_1-gain of the transposed of the original system. It is also shown that the computed gains are valid regardless of the sign of inputs and states, relaxing then the concept of positive system to certain conditions on the system matrices only. The consideration of nonnegative states and inputs are for computational and theoretical considerations only, i.e. the use of linear Lyapunov functions and linear supply-rates. Convex necessary and sufficient stabilization conditions using full, structured and bounded state-feedback controllers and accounting for performance bounds are then obtained. While the question of determining under which algebraic conditions a system can be made positive in closed-loop is still open, the proposed methodology can be applied to implicitly characterize them, together with a performance constraint.

Robust stability analysis is performed using Linear Fractional Transformations (LFTs), a classical tool of robust analysis [31], very few used in the context of positive systems [32], [11], [12]. The advantage

of using LFT is that parameter-varying systems depending rationally and polynomially in the parameters can be expressed as a simple interconnection of a time-invariant linear system and an uncertain matrix depending linearly on the parameters, simplifying then the analysis. The use of LFT also extends to any type of uncertainties and nonlinearities, e.g. delay operators, uncertain stable transfer functions, more general bounded uncertainties and static nonlinearities. In the context of uncertain positive systems, the overall system is rewritten as a *positive interconnection* of a nominal system and a matrix of uncertain positive operators. These uncertain operators are characterized through Integral Linear Constraints (ILCs), the linear counterpart of IQCs. Although the provided framework does not enjoy the availability of the KYP Lemma nor the Plancherel Theorem, a frequency domain analysis can still be used in order to select the scalings accurately. Several classes of uncertainties are discussed and it is shown that for linear time-invariant uncertainties, ILCs fully characterize their static-gain matrices. Based on this fact, several general robustness properties are discussed.

Robust stability analysis results are finally derived using dissipativity theory and formulated as robust linear programming problems. Exact stability conditions are provided in the particular case of LTI positive uncertainties with fixed static-gain matrix. To improve tractability, Handelman's Theorem [33] is used to produce linear programs involving a finite number of constraints. A procedure to reduce the number of extra variables introduced by Handelman's Theorem is also proposed to reduce the computational complexity of the approach. The results are finally extended to robust stabilization. It is shown that the presence of scalings does not destroy the convexity of the problem as it is often the case in full-block S-procedure [14] or IQC approaches. Several examples demonstrate the efficiency of the approach and its exactness for a certain class of uncertainties including delays.

Outline: Section II introduces the problem, fundamental definitions and results. Section III is devoted to the stability analysis of unperturbed systems and Section IV deals with their stabilization. This is extended to robust stability analysis and robust stabilization in Sections V and VI respectively. In Section VII, Handelman's Theorem is used to relax the robust optimization problems formulated in the previous sections. The results are then finally illustrated through examples in Section VIII.

Notations: $\mathbb{1}_n \in \mathbb{R}^n$ denotes the column vector containing entries equal to 1. For general real matrices or vectors $A, B \in \mathbb{R}^{n \times m}$, the inequality $A < (\leq)B$ is componentwise. Let $x \in \mathbb{R}^n$, the vector σ-norm, σ positive integer, $\sigma < \infty$, is denoted by $||x||_\sigma = \left(\sum_{i=1}^n |x_i|^\sigma\right)^{1/\sigma}$, while the vector ∞-norm is defined by $||x||_\infty = \max_{i \in \{1,...,n\}} |x_i|$. Given $v : [0, \infty) \to \mathbb{R}^n$, the L_σ-norm $||v||_{L_\sigma}$, σ positive integer, $\sigma < \infty$, and the L_∞-norm $||v||_{L_\infty}$ are defined by $||v||_{L_\sigma} = \left(\int_0^\infty ||v(t)||_\sigma^\sigma dt\right)^{1/\sigma}$ and $||v||_{L_\infty} = \text{ess sup}_{t \geq 0} ||v(t)||_\infty$, respectively. The spaces of signals $v : [0, \infty) \to \mathbb{R}^n$ having finite L_σ-norm is denoted by L_σ^n. When the dimension of the signal has no importance, we will use the shorthand L_σ. For a matrix $X \in \mathbb{R}^{n \times m}$, $[X]_{ij}$, $[X]_{r,i}$ and $[X]_{c,i}$ denote the (i, j) scalar entry, the ith row and the ith column. A linear map $\ell(x) = c^T x$ is said to be copositive if $c^T x \geq 0$ for all $x \geq 0$. We also define the sets $\mathbb{R}^n_{++} = \{\alpha \in \mathbb{R}^n : \alpha > 0\}$, $\mathbb{R}^n_+ := \{\alpha \in \mathbb{R}^n : \alpha \geq 0, ||\alpha|| \neq 0\}$ and $\bar{\mathbb{R}}^n_+ := \{\alpha \in \mathbb{R}^n : \alpha \geq 0\}$.

II. PRELIMINARIES

Let us consider general LTI systems of the form:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\
z(t) &= Cx(t) + Du(t) + Fw(t) \\
x(0) &= x_0
\end{aligned}
\tag{1}
$$

where $x, x_0 \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^q$ are respectively the system state, the initial condition, the control input, the exogenous input and the controlled output. When the system (1) with $u \equiv 0$ is asymptotically stable, it defines an operator from $L_\sigma \ni w \to z \in L_\sigma$, σ positive integer. Such an operator framework is suitable for defining, computing and optimizing norms of systems [34], and determining robustness and performance properties.

Definition 1: Let us consider the uncontrolled version of system (1), i.e. $u \equiv 0$. The system is said to be *positive* if the following conditions hold:

  i) The matrix A is Metlzer, i.e. it has nonnegative off-diagonal entries,
  ii) The matrices E, C, F are nonnegative, i.e. they only have nonnegative entries.

A consequence of the above definition is that a) when $w \equiv 0$, we have $x(t) \in \bar{\mathbb{R}}_+^n$ for all $t \geq 0$ and all $x_0 \in \bar{\mathbb{R}}_+^n$; b) when $x_0 = 0$, we have $z(t) \in \bar{\mathbb{R}}_+^q$ for all $t \geq 0$ and all $w(t) \in \bar{\mathbb{R}}_+^p$. These facts justify the denomination of positive system. Note that the terminology also degenerates to the case of autonomous positive systems for which only the matrix A must be Metzler.

Definition 2: A linear map $V(x) = \lambda^T x$ with $V(0) = 0$ is said to be a linear copositive Lyapunov function for the positive system $\dot{x}(t) = Ax(t)$ if both $V(x) > 0$ and $\dot{V}(x) < 0$ hold for all $x \in \mathbb{R}_+^n$.

Definition 3 (L_σ-gains of operators): Given an operator $\Sigma : L_\sigma^p \mapsto L_\sigma^q$, σ positive integer, the L_σ-gain $||\Sigma||_{L_\sigma - L_\sigma}$ is defined as

$$
||\Sigma||_{L_\sigma - L_\sigma} := \sup_{||w||_{L_\sigma} = 1} ||\Sigma w||_{L_\sigma}.
$$

Equivalently, it is the smallest $\theta \geq 0$ such that

$$
||\Sigma w||_{L_\sigma} \leq \theta ||w||_{L_\sigma}
$$

holds for all $w \in L_\sigma$.

Definition 4: The L_1-gain of an asymptotically stable linear time-invariant system H with transfer function $\widehat{h}(s) = C(sI - A)^{-1}E + F$ mapping p inputs to q outputs is given by [34]:

$$
||H||_{L_1 - L_1} = \max_{j \in \{1,\ldots,q\}} \left\{ \sum_{i=1}^p \int_0^{+\infty} |h_{ij}(t)| dt \right\}
\tag{2}
$$

where $h_{ij}(t)$ is the impulse response from input j to output i. In the same way, the L_∞-gain is given by

$$
||H||_{L_\infty - L_\infty} = \max_{i \in \{1,\ldots,p\}} \left\{ \sum_{j=1}^q \int_0^{+\infty} |h_{ij}(t)| dt \right\}.
\tag{3}
$$

The L_1-gain quantifies the gain of the *most influent input* since the max is taken over the columns. In contrast, the L_∞-gain of a system is the max taken over the rows and then characterizes the *most sensitive*

output. Note that in the SISO case, the 2 norms obviously coincide, and so do all the L_σ-norms, as pointed out in [13]. Another important fact, needed later, is the correspondence between the L_1-induced and L_∞-induced norms using the notion of transposed system:

Proposition 1: Let us consider a system H with transfer function $\widehat{h}(s) = C(sI - A)^{-1}E + F$ and its corresponding transposed system H^* having transfer function $\widehat{h}^*(s) = E^T(sI - A^T)C^T + F^T$. The L_∞-gain of H is related to the L_1-gain of H^* through the equality:

$$||H||_{L_\infty - L_\infty} = ||H^*||_{L_1 - L_1}. \tag{4}$$

Proof: The proof follows from the definitions of the transposed system and the norms. ∎

Proposition 2: Given an asymptotically stable positive system H with impulse response $h(t) \in \mathbb{R}^{q \times p}$, $t \geq 0$, we have

$$||H||_{L_1 - L_1} = \max\left\{\mathbb{1}_q^T \widehat{h}(0)\right\} \quad \text{and} \quad ||H||_{L_\infty - L_\infty} = \max\left\{\widehat{h}(0)\mathbb{1}_p\right\} \tag{5}$$

where $\widehat{h}$ is the Laplace transform of h.

Proof: Since A is Metlzer and Hurwitz, then we have $e^{At} \geq 0$ for all $t \geq 0$. It is then immediate to see that the impulse response $h(t) = Ce^{At}E + F$ is nonnegative as well. From the asymptotic stability of the system and (2) we hence have

$$\begin{aligned}
\int_0^{+\infty} h(t)dt &= \left. \int_0^{+\infty} h(t)e^{-st}dt \right|_{s=0} \\
&= \widehat{h}(0)
\end{aligned}$$

and we get the result for the L_1-gain. The L_∞-gain expression follows from Proposition 1. ∎

The above results then show that the L_1- and L_∞-gains of positive systems are intimately related to each others, and to the static-gain matrix of the system. The formulas given in Proposition 2 also suggest that a theoretical analysis of the gains is possible.

III. STABILITY AND PERFORMANCE ANALYSIS OF UNPERTURBED SYSTEMS

In this section, nonconservative stability and performance analysis criteria for unperturbed systems are derived. It is assumed throughout this section that the system (1) is positive and that the control input is identically zero, i.e. $u \equiv 0$. Similar results can be found in [35], [32], [12].

A. L_1-gain characterization and computation

Lemma 1 (L_1-gain characterization): Let us consider system (1) and assume it is positive. Then, the following statements are equivalent:

i) System (1) is asymptotically stable and the L_1-gain of the transfer $w \mapsto z$ is smaller than γ.

ii) System (1) is asymptotically stable and $\mathbb{1}_q^T \widehat{h}(0) < \gamma \mathbb{1}_p^T$ where $\widehat{h}(s)$ is the transfer function associated to system (1).

iii) System (1) is asymptotically stable and $\mathbb{1}_q^T(F - CA^{-1}E) < \gamma \mathbb{1}_p^T$.

iv) There exists $\lambda \in \mathbb{R}^n_{++}$ such that the linear program

$$\lambda^T A + \mathbb{1}_q^T C < 0 \tag{6a}$$

$$\lambda^T E - \gamma \mathbb{1}_p^T + \mathbb{1}_q^T F < 0 \tag{6b}$$

is feasible.

Proof: It is immediate that ii) and iii) are equivalent since $\widehat{h}(0) = F - CA^{-1}E$. We then prove that iv) and i) are equivalent and, finally, consider the equivalence of iii) and iv).

Proof of iv)$\Rightarrow$i)

The proof relies on dissipativity theory for nonnegative systems [30], [6]. Let us consider the copositive linear storage function $V(x) = \lambda^T x$ with $\lambda \in \mathbb{R}^n_{++}$ and the supply rate $s(w, z) = \gamma ||w||_1 - ||z||_1$, $\gamma > 0$. Then, according to dissipativity theory, if the functional

$$\mathcal{H}(x, w, z) = V(x(t)) - \int_0^t s(w(\eta), z(\eta))d\eta. \tag{7}$$

is decreasing along the trajectories solution of the system (1), then the system is dissipative with respect to the supply-rate $s(w, z)$ and the L_1-gain is bounded from above by γ. Moreover, since $\lambda > 0$ holds, then asymptotic stability of the system also follows. The derivative of $\mathcal{H}$ along the trajectories of (1) is given by

$$\begin{aligned}
\dot{\mathcal{H}} &= \lambda^T \dot{x}(t) - \gamma \mathbb{1}_p^T w(t) + \mathbb{1}_q^T z(t) \\
&= \begin{bmatrix} \lambda^T A + \mathbb{1}_q^T C & \lambda^T E - \gamma \mathbb{1}_p^T + \mathbb{1}_q^T F \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}.
\end{aligned}$$

Since the signals $x(t)$ and $w(t)$ are nonnegative, then $\dot{\mathcal{H}}$ is negative on $\mathbb{R}^{n+p}_+$ if and only if the left factor is negative, or equivalently if the conditions (6) hold.

Proof of i)$\Rightarrow$iv)

The necessity comes from Theorems 5.1, 5.3 and 6.2 of [6]. This can also be viewed from the fact that the S-procedure is lossless for any number of constraints in the linear case [36], [37]. An alternative proof given in [12] relies on Farkas' Lemma (which is actually an alternative way of seeing the linear S-procedure [37, Remark 4]).

Proof of iv)$\Rightarrow$iii)

A similar proof is given in [12]. Assume 4) holds, then the matrix A is Metlzer and Hurwitz, implying in turn that A^{-1} is a nonpositive matrix, i.e. $A^{-1} \le 0$. By right-multiplying inequality (6a) by A^{-1}, we obtain $\lambda^T > -\mathbb{1}_q^T CA^{-1}$ and after substitution into (6b), we get the inequality $\mathbb{1}_q^T(F - CA^{-1}E) < \gamma \mathbb{1}_p^T$ which coincides with the one of statement iii).

Proof of iii)$\Rightarrow$iv) The proof relies on an explicit construction of a Lyapunov function such that asymptotic stability of the system implies the feasibility of the linear program of statement iv). Assume iii) holds, then there exist $\nu \in \mathbb{R}^n_{++}$ and $\varepsilon > 0$ such that $\nu^T A < 0$ and

$$\mathbb{1}_q^T F + (\varepsilon \nu^T - \mathbb{1}_q^T CA^{-1})E < \gamma \mathbb{1}_p^T. \tag{8}$$

Noting that the term inside parentheses is positive since $\nu > 0$ and $A^{-1} \le 0$, then we can let $\lambda^T := \varepsilon\,\nu^T - \mathbb{1}_q^T C A^{-1}$. By right-multiplying this inequality by A, we obtain

$$\lambda^T A + \mathbb{1}_q^T C = \varepsilon\,\nu^T A \tag{9}$$

which is negative from assumptions $\nu^T A < 0$ and $\varepsilon > 0$. Since the left-hand term of (9) is identical to the one of (6a), the implication of the feasibility of (6a) is proved. Reorganizing now the terms of the inequality (8), we get $\lambda^T E - \gamma\mathbb{1}_p^T + \mathbb{1}_q^T F < 0$, which is identical to (6b). The proof is complete. ∎

Remark 1: It is interesting to point out that despite of being computed with the assumption of nonnegative input signals $w \in L_1$ and nonnegative state values, *the determined L_1-gain is valid for any input signal in L_1 and any initial state $x_0 \in \mathbb{R}^n$.* This is due to the fact that the L_1-gain has alternative definition (2), which depends on the nonnegativity of impulse response only. Nonnegative input and output signals are considered for theoretical and computational convenience only.

Lemma 1 can be used to compute the exact L_1-gain of any asymptotically stable positive linear system as follows:

Algorithm 1 (Computing the L_1-gain): The gain coincides with the optimal value of the following linear programming problem:

$$\min_{\lambda,\gamma} \ \gamma \quad \text{s.t.} \quad \lambda \in \mathbb{R}_{++}^n, \ \gamma > 0 \text{ and (6) hold.}$$

The complexity of the above linear programming problem is quite low since it is a linear program involving $n + 1$ decision variables and $2n + p + 1$ constraints. The complexity then grows linearly with respect to the size of the system.

B. L_∞-gain characterization and computation

The following result is the L_∞ counterpart of Lemma 1.

Lemma 2 (L_∞-gain characterization): Let us consider system (1) that we assume to be positive. Then, the following statements are equivalent:

i) System (1) is asymptotically stable and the L_∞-gain of the transfer $w \mapsto z$ is smaller than γ.

ii) System (1) is asymptotically stable and $\widehat{h}(0)\mathbb{1}_p < \gamma\mathbb{1}_q$ where $\widehat{h}(s)$ is the transfer function associated to system (1).

iii) System (1) is asymptotically stable and $(F - CA^{-1}E)\mathbb{1}_p < \gamma\mathbb{1}_q$.

iv) There exists $\lambda \in \mathbb{R}_{++}^n$ such that the linear program

$$A\lambda + E\mathbb{1}_p < 0 \tag{10a}$$

$$C\lambda - \gamma\mathbb{1}_q + F\mathbb{1}_p < 0 \tag{10b}$$

is feasible.

Proof: The proof relies on Proposition 1. By substituting the matrices of the transposed system into the conditions (6) of Lemma 1, we get the conditions (10). The equivalence between the statements is immediate from Lemma 1. ∎

Similarly as for the L_1-gain, it is possible to compute the L_∞-gain through an optimization problem.

Algorithm 2 (Computing the L_∞-gain): The gain coincides with the optimal value of the following linear programming problem:

$$\min_{\lambda,\gamma} \ \gamma \quad \text{s.t.} \quad \lambda \in \mathbb{R}^n_{++}, \ \gamma > 0 \text{ and } (10) \text{ hold.}$$

Unlike the L_1-gain case, the number of constraints is $2n + q + 1$ while the number of variables remains the same. Again, the computational complexity grows linearly with respect to the size of the system.

IV. Stabilization of unperturbed systems

This section is devoted to the stabilization of unperturbed linear systems via state-feedback control laws of the form:

$$u(t) = Kx(t) \tag{11}$$

where the controller gain K belongs to one of the following sets:

1) The set of unconstrained controllers $\mathcal{K} := \mathbb{R}^{m \times n}$.
2) The set of structured controllers $\mathcal{K}_c := \{K \in \mathbb{R}^{m \times n} : [K]_{ij} = 0, \ (i,j) \in \mathcal{S}_c\}$, where $\mathcal{S}_c$ is the set of indices corresponding to 0 entries in the controller gain.
3) The set of bounded controllers $\mathcal{K}_b := \{K \in \mathbb{R}^{m \times n} : K^- \le K \le K^+\}$, where K^- and K^+ are the lower and upper bounds on the controller gain K, respectively.

Unlike general LTI systems, the design of structured and bounded controllers is not a NP-hard problem [25], [38], [39]. By indeed using a diagonal Lyapunov function, necessary and sufficient conditions for the design structured controllers can be easily obtained. We show here that it is also the case in the current linear setting and that it readily extends to the stabilization of positive systems with guaranteed L_∞-gain.

Lemma 3 (Stabilization with $K \in \mathcal{K}$): Let us consider the closed-loop system (1)-(11), where (1) is not necessarily a positive system, with transfer function $\widehat{h}_{cl}(s, K) := (C + DK)(sI - A - BK)^{-1}E + F$. Then, the following statements are equivalent:

i) There exists a controller matrix K such that the closed-loop system (1)-(11) is positive, asymptotically stable and the L_∞-gain of the transfer $w \mapsto z$ is less than $\gamma > 0$.

ii) There exists a controller matrix K such that the closed-loop system (1)-(11) is positive, asymptotically stable and verifies $\widehat{h}_{cl}(0, K)\mathbb{1}_p < \gamma \mathbb{1}_p$.

iii) There exist $\lambda \in \mathbb{R}^n_{++}$ and $\mu_i \in \mathbb{R}^m$, $i = 1, \ldots, n$ such that the linear program

$$A\lambda + B \sum_{i=1}^{n} \mu_i + E\mathbb{1}_p < 0 \tag{12a}$$

$$C\lambda + D \sum_{i=1}^{n} \mu_i - \gamma\mathbb{1}_q + F\mathbb{1}_p < 0 \tag{12b}$$

$$[A]_{ij} \lambda_j + [B]_{r,i}\mu_j \ge 0, \ i,j = 1, \ldots, n, \ i \ne j \tag{12c}$$

$$[C]_{ij} \lambda_j + [D]_{r,i}\mu_j \ge 0, \ i = 1, \ldots, q, \ j = 1, \ldots, n \tag{12d}$$

is feasible. In such a case, a suitable K is given by

$$K = \begin{bmatrix} \lambda_1^{-1}\mu_1 & \ldots & \lambda_n^{-1}\mu_n \end{bmatrix}. \tag{13}$$

Proof: To prove the equivalence it is enough to show that statements 1) and 3) are equivalent. The rest follows from the proofs of Lemmas 1 and 2. The closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= (A + BK)x(t) + Ew(t), \\ z(t) &= (C + DK)x(t) + Fw(t). \end{aligned} \tag{14}$$

Substituting the closed-loop system into (10) yields

$$\begin{bmatrix} (A + BK)\lambda + E\mathbb{1}_p \\ (C + DK)\lambda - \gamma\mathbb{1}_q + F\mathbb{1}_p \end{bmatrix} < 0. \tag{15}$$

Noting that $K\lambda = \sum_{i=1}^{n} \lambda_i[K]_{c,i}$, it turns out that the change of variable $\mu_i = [K]_{c,i}\lambda_i$ linearizes the problem and yields (12a)-(12b). To ensure the positivity constraint on the closed-loop system we need to impose $A + BK$ and $C + DK$ to be Metzler and nonnegative, respectively. Using the same procedure as in [7], these constraints are captured by inequalities (12c)-(12d). ∎

The above theorem can hence be viewed as an extension of [7] where no performance criterion is considered. The following proposition discusses the cases $K \in \mathcal{K}_c$ and $K \in \mathcal{K}_b$.

Proposition 3 (Cases $K \in \mathcal{K}_c$ and $K \in \mathcal{K}_b$): These cases can be easily handled by adding supplementary constraints to the linear program of Lemma 3.

- The design of a controller gain $K \in \mathcal{K}_c$ is ensured by considering the additional linear constraints $\begin{bmatrix} \mu_1 & \ldots & \mu_m \end{bmatrix}_{ij} = 0$, for all $(i, j) \in \mathcal{S}_c$.
- Prescribed bounds on the coefficients of the controller gain are imposed, i.e. $K \in \mathcal{K}_b$ by considering the additional linear constraints $[K^-]_{r,i}\lambda_i \le \mu_i \le [K^+]_{r,i}\lambda_i$, $i = 1, \ldots, m$, $K^-, K^+ \in \mathbb{R}^{m \times n}$, $K^- \le K^+$.

Note that in both cases the necessity of the approach is preserved. The above results are thus nonconservative. Other constraints like asymmetric bounds on the control input and the consideration of bounded states can also be easily considered [7].

V. ROBUST STABILITY ANALYSIS AND ROBUST PERFORMANCE

Let us focus now on uncertain linear systems subject to real parametric uncertainties $\delta \in \boldsymbol{\delta} := [0, 1]^N$, $N > 0$, of the form

$$\begin{aligned} \dot{x}(t) &= A_\delta(\delta)x(t) + B_\delta(\delta)u(t) + E_\delta(\delta)w_1(t) \\ z_1(t) &= C_\delta(\delta)x(t) + D_\delta(\delta)u(t) + F_\delta(\delta)w_1(t) \\ x(0) &= x_0 \end{aligned} \tag{16}$$

where $x \in \bar{\mathbb{R}}_+^n$, $x_0 \in \bar{\mathbb{R}}_+^n$, $u \in \mathbb{R}^m$, $w_1 \in \bar{\mathbb{R}}_+^p$ and $z_1 \in \bar{\mathbb{R}}_+^q$ are the system state, the initial condition, the control input, the exogenous input and the performance output respectively. We assume in this section that the above system is positive, that is, for all $\delta \in \boldsymbol{\delta}$, the matrix $A_\delta(\delta)$ is Metzler, and $E_\delta(\delta), C_\delta(\delta), F_\delta(\delta)$ are nonnegative matrices. We also assume that the system matrices are continuous functions of δ.

While this section mainly focuses on positive systems subject to real parametric uncertainties, the proposed methodology basically applies to any type of positive interconnections involving, for instance, delays, positive infinite-dimensional operators, time-invariant and time-varying uncertain positive operators, static sign-preserving nonlinearities, etc. A justification for focusing on parametric uncertainties lies in the particular property of positive systems that (as we shall see later in Section V-B) only the static-gain matrix is critical for evaluating stability of interconnections. This property implies that many problems involving uncertain positive transfer functions equivalently reduce to problems involving constant parametric uncertainties.

A. *Positive Linear Fractional Transformation*

Using LFT, the system (16) is rewritten as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + E_0 w_0(t) + E_1 w_1(t) \\
z_0(t) &= C_0 x(t) + F_{00} w_0(t) + F_{01} w_1(t) \\
z_1(t) &= C_1 x(t) + F_{10} w_0(t) + F_{11} w_1(t) \\
w_0(t) &= \Delta(\delta) z_0(t)
\end{aligned}
\tag{17}
$$

where the loops signals $z_0(t), w_0(t) \in \bar{\mathbb{R}}_+^{n_0}$ have been added. It is important to keep in mind that we are considering robustness in the L_1-norm, the tractability of which relying on the nonnegativity of the loop signals w_0 and z_0. Representation (17) is not unique, even when minimal, so the arising question concerns the constructability of a so-called *positive Linear Fractional Representation*(LFR), i.e. an LFR of the form (17) with nonnegative loop signals and positive operators, corresponding to the positive system (16). Positive interconnections are recurrent in the analysis of positive systems and have already been considered in several works [4], [40], [12], [32]. By picking suitable matrices for the LFR, it is always possible to make the loop signals nonnegative.

B. *Handling positive uncertainties via Integral Linear Constraints*

In robust stability analysis theories addressing two interconnected systems, stability conditions very often consist of two separate conditions: one for each subsystem. The first one, generally very precise, is used to characterize the nominal system which usually enjoys nice properties like linearity, time-invariance, etc. The second condition, often more difficult to derive when high precision is sought, is used to characterize the uncertain part. Several approaches have been developed to study the stability of interconnections like small-gain results [31] and generalizations [14], well-posedness/quadratic separation [41], [21] (based on topological separation arguments [19]) and IQCs [15]. IQCs are very powerful objects that are able to implicitly describe, with high accuracy, uncertain operators through the characterization of their input/output signals. They unfortunately do not fit in the current framework since they are based on quadratic forms, while the proposed approach considers linear ones. Inspired from this idea, Integral Linear Constraints (ILCs) are considered in this paper.

1) Definitions of ILCs and preliminary results:

Let us consider first an uncertain operator $\Sigma : L_1 \mapsto L_1$ belonging to a known set $\boldsymbol{\Sigma}$. The main idea is to implicitly characterize the set $\boldsymbol{\Sigma}$ using ILCs as[1]

$$\boldsymbol{\Sigma} \subseteq \left\{ \Sigma : \int_0^{+\infty} \varphi_{1,i}^T \left(|w(t)| + \varphi_{2,i}^T |z(t)| \right) dt \geq 0, \ z = \Sigma w, \ i = 1, \ldots \right\}. \tag{18}$$

The vectors $\varphi_{1,i}$ and $\varphi_{2,i}$ are called *scalings* and must be chosen according to the set $\boldsymbol{\Sigma}$. Note that the absolute values disappear when considering a set of positive operators and nonnegative input signals.

One important feature of IQCs lies in the fact that, by virtue of the Plancherel Theorem, it is possible to express the inequality in the frequency domain, domain in which the tuning of the scalings may be easier and/or more precise. Finally, by virtue of the Kalman-Yakubovich-Popov Lemma and the S-procedure, the frequency domain conditions are converted back to time-domain conditions, taking the form of tractable LMI-problems. The main concern is that the Plancherel Theorem does not exist in L_1 and we cannot expect to switch from time to frequency domain in the same spirit as in L_2. It is however still possible to consider the frequency domain as stated in the following result:

Lemma 4: The following equivalent statements hold:

i) The ILC

$$\int_0^{+\infty} \left(\varphi_1^T w(t) + \varphi_2^T z(t) \right) dt \geq 0 \tag{19}$$

holds for every pairs of positive signals (w, z) such that $z = \Sigma w$, where Σ is a positive operator.

ii) The algebraic inequality

$$\varphi_1^T \widehat{w}(0) + \varphi_2^T \widehat{z}(0) \geq 0 \tag{20}$$

holds where $\widehat{w}$ and $\widehat{z}$ are the Laplace transform of the signals w and z, respectively.

Proof: The proof follows from noting that

$$\begin{aligned} \int_0^{+\infty} \left(\varphi_1^T w(t) + \varphi_2^T z(t) \right) dt &= \left. \int_0^{+\infty} \left[\left(\varphi_1^T w(t) + \varphi_2^T z(t) \right) e^{-st} \right] dt \right|_{s=0} \\ &= \varphi_1^T \widehat{w}(0) + \varphi_2^T \widehat{z}(0). \end{aligned} \tag{21}$$

■

Assuming that the pair of nonnegative signals (w, z) is related by the equality $z = \Sigma_{LTI}^+ w$ where Σ_{LTI}^+ is a *linear time-invariant* positive operator, we have the following corollary:

Corollary 1: The following equivalent statements hold:

i) The ILC

$$\int_0^{+\infty} \left(\varphi_1^T w(t) + \varphi_2^T z(t) \right) dt \geq 0 \tag{22}$$

holds for every pairs of nonnegative signals (w, z), $z = \Sigma_{LTI}^+ w$, where Σ_{LTI}^+ is a linear time-invariant positive operator.

ii) The algebraic inequality

$$\varphi_1^T + \varphi_2^T \widehat{\Sigma}_{LTI}^+(0) \geq 0 \tag{23}$$

---

[1] the absolute value is componentwise.

holds where $\widehat{\Sigma}^+_{LTI}(s)$ is the transfer function corresponding to the linear time-invariant positive operator Σ^+_{LTI}.

Proof: Since Σ^+_{LTI} is a linear time-invariant positive operator, then the left-hand side of the inequality (20) can be written as $\left[\varphi_1^T + \varphi_2^T \widehat{\Sigma}(0)\right]\widehat{w}(0)$. The nonnegativity of the letter expression is equivalent to the nonnegativity of the term into brackets since $\widehat{w}(0) = \int_0^\infty w(t)dt \geq 0$. This concludes the proof. ∎

2) Generic robustness results and remarks:

Lemma 4 and Corollary 1 show that, similarly as for IQCs, the scalings may be selected both in the time-domain and the frequency domain. In the LTI case, the scalings just have to be selected according to the static-gain matrix of the system, which shows that the problem reduces to a problem with constant parametric uncertainties since the static-gain matrix is a (possibly uncertain) real matrix. Another conclusion is that, since only the zero-frequency is important, there is thus no need for capturing the entire frequency domain by using frequency-dependent scalings as in the μ-analysis [42], IQC-based techniques [15] or approaches using dynamic D-scalings [43]. It is hence expected that a constant scaling matrix acting on the static-gain matrix should be enough for obtaining interesting results.

Along these lines, it is proved in [12] that the use of the scaled L_1-gain yields necessary and sufficient conditions for the characterization of any interconnection of LTI positive systems. Similar results are obtained in the L_2 framework in [11]. These results can be understood through the fact that the stability of interconnections of LTI positive systems is equivalent to the stability of interconnections of their static-gain matrix, i.e. higher order dynamics have no impact on stability. The critical stability information is hence concentrated in the static part of the dynamical systems.

Based on the above remarks, generic robustness properties are formalized below:

Theorem 1: The following statements are equivalent:

 i) The linear time-invariant positive operator Σ satisfies the ILC (22).
 ii) The static-gain matrix $\widehat{\Sigma}(0)$ satisfies (23).
 iii) The linear time-invariant positive operator Σ' satisfies the ILC (22), where $\widehat{\Sigma}'(s) = \widehat{\Sigma}(s) + s\Theta(s)$ and $s\Theta(s)$ is any positive asymptotically stable proper transfer function.

Proof: The equivalence between i) and ii) follows from Corollary 1. The proof of ii) $\Rightarrow$ i) follows from choosing $\Theta(s) = 0$. To show the converse, it is enough to remark that $\widehat{\Sigma}'(0) = \widehat{\Sigma}(0)$ and hence Σ' satisfies the ILC (22) for any positive asymptotically stable strictly proper transfer function $\Theta(s)$. ∎

The above result interestingly shows that the set of uncertainties that satisfies the ILC is typically very large since there is basically no limit on the magnitude of the coefficients of $\Theta(s)$ acting on powers of s.

C. Main results

Based on the results and discussions of the previous sections, the main results on robust stability analysis of linear positive systems subject to uncertain parametric uncertainties can be stated. Both L_1- and L_∞-induced norms are considered. The conditions are expressed as robust linear optimization/feasibility problems that are difficult to solve directly. An exact solving scheme based on Handelman's Theorem is proposed in Section VII.

Theorem 2: Assume there exist a vector $\lambda \in \mathbb{R}^n_{++}$ and polynomials $\varphi_1(\delta), \varphi_2(\delta) \in \mathbb{R}^{n_0}$ such that the robust linear program

$$
\begin{aligned}
\lambda^T A + \varphi_1(\delta)^T C_0 + \mathbb{1}_q^T C_1 &< 0 \\
\lambda^T E_0 + \varphi_2(\delta)^T + \varphi_1(\delta)^T F_{00} + \mathbb{1}_q^T F_{10} &< 0 \\
\lambda^T E_1 - \gamma \mathbb{1}_p^T + \varphi_1(\delta)^T F_{01} + \mathbb{1}_q^T F_{11} &< 0
\end{aligned}
\tag{24}
$$

$$
\varphi_1(\delta)^T + \varphi_2(\delta)^T \Delta(\delta) \geq 0
\tag{25}
$$

is feasible for all $\delta \in \boldsymbol{\delta}$. Then, the uncertain linear positive system (16) is asymptotically stable and the L_1-gain of the transfer $w_1 \to z_1$ is smaller than $\gamma > 0$.

Proof: The proof follows exactly the same lines as for Lemma 1. Note however that the supply-rate

$$
s(w, z) = -\varphi_1(\delta)^T z_0(t) - \varphi_2(\delta)^T w_0(t) + \gamma \mathbb{1}_p^T w_1(t) - \mathbb{1}_q^T z_1(t)
\tag{26}
$$

has to be considered here. ∎

It is important to mention that the above condition is unlikely a necessary condition since the linear Lyapunov function does not depend on the parameters. It is indeed well-known that quadratic stability (parameter independent Lyapunov function) is more conservative than robust stability (parameter dependent Lyapunov function).

The following result addresses the case when the ILC can be saturated using constant scalings:

Theorem 3: The following statements are equivalent:

i) System (17) with $\Delta = \Delta_0$, $\Delta_0 \geq 0$ constant, is asymptotically stable and the L_1-gain of the transfer $w \to z$ is smaller than γ.

ii) System (17) with $\widehat{\Delta}(s)$ LTI, asymptotically stable, positive and $\widehat{\Delta}(0) = \Delta_0 \geq 0$ constant, is asymptotically stable and the L_1-gain of the transfer $w \to z$ is smaller than γ.

iii) There exist vectors $\lambda \in \mathbb{R}^n_{++}$, $\varphi_1, \varphi_2 \in \mathbb{R}^{n_0}$ such that the linear program

$$
\begin{aligned}
\lambda^T A + \varphi_1^T C_0 + \mathbb{1}_q^T C_1 &< 0 \\
\lambda^T E_0 + \varphi_2^T + \varphi_1^T F_{00} + \mathbb{1}_q^T F_{10} &< 0 \\
\lambda^T E_1 - \gamma \mathbb{1}_p^T + \varphi_1^T F_{01} + \mathbb{1}_q^T F_{11} &< 0
\end{aligned}
\tag{27}
$$

$$
\varphi_1^T + \varphi_2^T \Delta_0 = 0
\tag{28}
$$

is feasible.

Proof: The equivalence between i) and ii) follows from Theorem 1. It is easy to show from dissipativity theory that iii) implies i). This can also be proved by substituting φ_1^T by $-\varphi_2^T \Delta_0$ in the inequalities (27) and further substituting φ_2^T by $-\lambda^T E_0 (I - \Delta_0 F_{00})^{-1} - \mathbb{1}_q^T F_{10}(I - \Delta_0 F_{00})^{-1}$. This leads to the inequalities

$$
\begin{aligned}
\lambda^T (A + E_0(I - \Delta_0 F_{00})^{-1}\Delta_0 C_0) + \mathbb{1}_q^T (C_1 + F_{10}(I - \Delta_0 F_{00})^{-1}\Delta_0 C_0) &< 0 \\
\lambda^T (A + E_0(I - \Delta_0 F_{00})^{-1}\Delta_0 F_{01}) - \gamma \mathbb{1}_p^T + \mathbb{1}_q^T (F_{11} + F_{10}(I - \Delta_0 F_{00})^{-1}\Delta_0 F_{01}) &< 0
\end{aligned}
\tag{29}
$$

which are exactly the conditions for asymptotic stability and bounded L_1-gain. By reversing the calculations, we can show that i) implies iii). ∎

This result can indeed be extended to the L_∞-case, this is omitted for brevity.

Theorem 4: Assume there exist a vector $\lambda \in \mathbb{R}^n_{++}$ and polynomials $\varphi_1(\delta), \varphi_2(\delta) \in \mathbb{R}^{n_0}$ such that the robust linear program

$$
\begin{aligned}
A\lambda + \bar{C}_0\varphi_1(\delta) + E_1 \mathbb{1}_p &< 0 \\
\tilde{E}_0^T\lambda + \varphi_2(\delta) + \bar{F}_{00}^T\varphi_1(\delta) + \tilde{F}_{10}^T\mathbb{1}_p &< 0 \\
C_1\lambda - \gamma\mathbb{1}_q + \bar{F}_{01}\varphi_1(\delta) + F_{11}\mathbb{1}_p &< 0
\end{aligned}
\tag{30}
$$

$$
\varphi_1(\delta)^T + \varphi_2(\delta)^T\Delta(\delta) \geq 0, \ \delta \in \boldsymbol{\delta}
\tag{31}
$$

is feasible for all $\delta \in \boldsymbol{\delta}$. Then, the uncertain linear positive system (16) is asymptotically stable and the L_∞-gain of the transfer $w_1 \to z_1$ is smaller than $\gamma > 0$.

Proof: The proof is based on the use the LFT system corresponding of the transposed system of (16) given by:

$$
\begin{aligned}
\dot{x}(t) &= A^Tx(t) + \tilde{E}_0w_0(t) + C_1^Tw_1(t) \\
z_0(t) &= \bar{C}_0x(t) + \bar{F}_{00}w_0(t) + \bar{F}_{01}w_1(t) \\
z_1(t) &= E_1^Tx(t) + \tilde{F}_{10}w_0(t) + F_{11}^Tw_1(t) \\
w_0(t) &= \Delta(\delta)z_0(t)
\end{aligned}
\tag{32}
$$

where the matrices $\tilde{F}_{10}$ and $\tilde{E}_0$ are specific matrices of the transposed system. All the other matrices are those of systems (16) and (17). ■

Remark 2: It must be stressed here that the Linear Fractional Transformation does not commute with the operation of transposition. In other words, the transposed of an LFT system does not coincide, in general, with the LFT of the transposed system. Some matrices may indeed remain unchanged (non transposed) while some others are different. This has motivated the use of the 'bar' and 'tilde' notations in (32). For instance, when the system depends polynomially on the parameters, we may have the equalities $\bar{F}_{00} = F_{00}, \bar{F}_{01} = F_{01}$ and $\bar{C}_0 = C_0$.

VI. ROBUST STABILIZATION

The robust stabilization problem with L_∞-performance is solved in this section. It is interesting to note that the presence of scalings never destroys the convexity of the approach as this may occur in some robust control approaches based on IQC's [15] or the full-block S-procedure [14].

As in Section IV, it is not necessary that the open-loop be positive, the only requirement is the nonnegativity of disturbance input matrices, i.e. $E(\delta), F(\delta)$ nonnegative for all $\delta \in \boldsymbol{\delta}$. The LFR of the closed-loop transposed system is given by:

$$
\begin{aligned}
\dot{\tilde{x}}(t) &= \mathcal{A}(K)\tilde{x}(t) + \mathcal{E}_0(K)\tilde{w}_0(t) + \mathcal{E}_1(K)\tilde{w}_1(t) \\
\tilde{z}_0(t) &= \mathcal{C}_0\tilde{x}(t) + \mathcal{F}_{00}\tilde{w}_0(t) + \mathcal{F}_{01}\tilde{w}_1(t) \\
\tilde{z}_1(t) &= \mathcal{C}_1\tilde{x}(t) + \mathcal{F}_{10}\tilde{w}_0(t) + \mathcal{F}_{11}\tilde{w}_1(t) \\
\tilde{w}_0(t) &= \Delta(\delta)^T\tilde{z}_0(t)
\end{aligned}
\tag{33}
$$

where $\mathcal{A}(K) = (A^0 + B^0K)^T$, $\mathcal{E}_1(K) = (C_1^0 + D^0K)^T$ and

$$
\mathcal{E}_0(K) = \begin{bmatrix} (A^1 + B^1K)^T & \ldots & (A^\eta + B^\eta K)^T & (C_1^1 + D^1K)^T & \ldots & (C_1^\theta + D^\theta K)^T \end{bmatrix}.
\tag{34}
$$

The integers $\eta, \theta > 0$ are related to the system dependency on the parameters.

Theorem 5: The controlled system (16)-(11) with $K \in \mathcal{K}$ is asymptotically stable if there exist vectors $\lambda \in \mathbb{R}_{++}^n$, $\varphi_1(\delta), \varphi_2(\delta) \in \mathbb{R}^{n_0}$, $\mu_i \in \mathbb{R}^m$, $i = 1, \ldots, n$ and a scalar $\gamma > 0$ such that the robust linear program

$$
\begin{aligned}
A^0 \lambda + B^0 \mu + \mathcal{C}_0^T \varphi_1(\delta) + \mathcal{C}_1^T \mathbb{1}_p &< 0 \\
\tilde{\mathcal{E}}_0(\lambda, \mu) + \varphi_2(\delta) + \mathcal{F}_{00}^T \varphi_1(\delta) + \mathcal{F}_{10}^T \mathbb{1}_p &< 0 \\
C_1^0 \lambda + D^0 \mu - \gamma \mathbb{1}_q + \mathcal{F}_{01}^T \varphi_1(\delta) + \mathcal{F}_{11} \mathbb{1}_p &< 0 \\
\varphi_1^T + \varphi_2^T \Delta(\delta) &\geq 0
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
[A_\delta(\delta)]_{ij} \lambda_j + [B_\delta(\delta)]_{r,i} \mu_j &\geq 0, \; i, j = 1, \ldots, n, \; i \neq j \\
[C_\delta(\delta)]_{ij} \lambda_j + [D_\delta(\delta)]_{r,i} \mu_j &\geq 0, \; i = 1, \ldots, q, \; j = 1, \ldots, n
\end{aligned}
\tag{36}
$$

is feasible for all $\delta \in \boldsymbol{\delta}$ with $\mu = \sum_{i=1}^n \mu_i$, $\tilde{\mathcal{E}}_0 = \begin{bmatrix} \tilde{\mathcal{E}}_0^1 & \tilde{\mathcal{E}}_0^2 \end{bmatrix}$ and

$$
\begin{aligned}
\tilde{\mathcal{E}}_0^1(\lambda, \mu) &= \begin{bmatrix} A^1 \lambda + B^1 K \mu & \ldots & A^\eta \lambda + B^\eta K \mu \end{bmatrix}, \\
\tilde{\mathcal{E}}_0^2(\lambda, \mu) &= \begin{bmatrix} C_1^1 \lambda + D^1 K \mu & \ldots & C_1^\theta \lambda + D^\theta K \mu \end{bmatrix}.
\end{aligned}
\tag{37}
$$

Moreover, in such a case, the controller K is given by

$$
K = \begin{bmatrix} \lambda_1^{-1} \mu_1 & \ldots & \lambda_n^{-1} \mu_n \end{bmatrix}
\tag{38}
$$

and the closed-loop system satisfies $||z_1||_{L_\infty} \leq \gamma ||w_1||_{L_\infty}$.

Proof: The proof is similar to the one of the results in Section IV. ∎

The constraints (35) are polynomial in the uncertain parameters δ. This is however not the case of the constraints (36) that are rational when the system is rational. They can however be easily turned into polynomial constraints by finding a common denominator for the left-hand side. Since the sign of the common denominator is fixed (otherwise the system would be ill-posed), the rational constraints reduce to polynomial constraints on the numerators.

VII. SOLVING ROBUST LINEAR PROGRAMS

In this section, we address the problem of solving the robust linear optimization problems arising in Theorems 2, 4 and 5. A solving scheme based on Handelman's Theorem [33] is proposed.

A. Handelman's Theorem

For completeness, let us first recall Handelman's Theorem:

Theorem 6 (Handelman's Theorem): Assume $\mathcal{S}$ is a compact polytope in the Euclidean N-space defined as $\mathcal{S} := \{ x \in \mathbb{R}^N : g_i(x) \geq 0, \; i = 1, \ldots \}$ where the $g_i(x)$'s are linear forms. Assume also that P is a polynomial in N variables which is positive on $\mathcal{S}$, then P can be expressed as a linear combination with nonnegative coefficients (not all zero) of products of members of $\{g_i\}$.

The above theorem states a necessity result regarding the positivity of a polynomial over a compact polytope: if it is positive, then we can write it as a linear combination of products of the linear functions

defining $\mathcal{S}$. In [44], [45] it is shown that Handelman's Theorem implies Pólya's Theorem [46]. Sufficiency is immediate following similar arguments as for the S-procedure [22] or sum-of-squares techniques [47]. A linear combination of positive polynomials on $\mathcal{S}$ (the products of g_i's are positive on $\mathcal{S}$) is indeed a positive polynomial on $\mathcal{S}$. Hence, with a suitable choice of the product terms, it is possible to determine whether a polynomial is positive over a compact polytope of the Euclidian space. To illustrate the above statement, let us consider the following example:

Example 1: Suppose we would like to characterize all the univariate polynomials $p(x)$ of at most degree 2 that are nonnegative on the interval $[-1, 1]$. The basis functions of this interval are given by $g_1(x) = x + 1$ and $g_2(x) = 1 - x$. According to Handelman's Theorem, we know that all such polynomials write as a linear combination of all possible products $g_1(x)^i g_2(x)^j$ with $1 \leq i + j \leq 2$. Hence, we have

$$
\begin{aligned}
p(x) &= \tau_1 g_1(x) + \tau_2 g_2(x) + \tau_3 g_1(x) g_2(x) + \tau_4 g_1(x)^2 + \tau_5 g_2(x)^2 \\
&= \chi_2 x^2 + \chi_1 x + \chi_0
\end{aligned}
\tag{39}
$$

where $\tau_i \geq 0$, $i = 1, \ldots, 5$ and

$$
\begin{aligned}
\chi_2 &= \tau_4 + \tau_5 - \tau_3, \\
\chi_1 &= \tau_1 - \tau_2 + 2\tau_4 - 2\tau_5, \\
\chi_0 &= \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5.
\end{aligned}
\tag{40}
$$

Hence, we can conclude on the very general statement that any univariate polynomial of at most degree 2 that is nonnegative on a bounded interval of the form $[\alpha, \beta]$ can be viewed as a point $\tau = (\tau_1, \ldots, \tau_5) \in \mathbb{R}^5_+$.

B. Equivalent relaxations for robust linear programs

For simplicity, we will focus here on linear optimization problems depending on a single uncertain parameter. The results straightforwardly generalize to the case of multiple uncertain parameters at the expense of complex notations. Let us consider the following semi-infinite feasibility problem

Problem 1: There exists $x \in \mathbb{R}^\eta$ such that the inequality

$$
P(x, \theta) := \sum_{j=0}^{d} P_j(x) \theta^j \leq 0
\tag{41}
$$

holds for all $\theta \in \mathcal{P} := \mathcal{P} = \{\theta \in \mathbb{R} : g_i(\theta) \geq 0, \ i = 1, \ldots, \ g_i's \ \text{linear}\}$ and where the vectors $P_j(x) \in \mathbb{R}^{N_P}$, $j = 0, \ldots, d$, are linear in x.

We then have the following result:

Theorem 7: The following statements are equivalent:

i) Problem 1 is feasible.

ii) There exist an integer $b > 0$ and vectors[2] $Q_k(y) \in \mathbb{R}^{N_P}$, $k = 1, \ldots, b$, such that the finite-dimensional linear program

$$
\begin{aligned}
P_j(x) &= Z_j, \ j = 0, \ldots, d \\
Q_k(y) &\leq 0, \ k = 0, \ldots, b
\end{aligned}
\tag{42a}
\tag{42b}
$$

[2] The notation y is here to emphasize that the vectors Q_k's consist exclusively of independent additional decision variables.

is feasible in $(x, y) \in \mathbb{R}^\eta \times \mathbb{R}_+^{N_p(b+1)}$ where $P(x, \theta) = \sum_{k=0}^{b} Q_k(y) g_i(\theta)^k$, $Z_j := \sum_{k=0}^{b} \upsilon_{jk} Q_k(y)$ and the υ_{jk}'s depend on the coefficients of the basis functions g_i's.

iii) There exist an integer $b > 0$ and vectors $R_k(z) \in \mathbb{R}^{N_P}$, $k = 0, \ldots, b - d$, such that the finite-dimensional linear program

$$R_k(z) \leq 0, \ k = 0, \ldots, d \tag{43a}$$

$$\Upsilon_2^{-1} \left(\begin{bmatrix} P_0(x) \\ \vdots \\ P_d(x) \end{bmatrix} - \Upsilon_1 \begin{bmatrix} R_0(z) \\ \vdots \\ R_{b-d}(z) \end{bmatrix} \right) \leq 0. \tag{43b}$$

is feasible in $(x, z) \in \mathbb{R}^\eta \times \mathbb{R}_+^{N_p(b-d+1)}$ where $\Upsilon := \begin{bmatrix} \Upsilon_1 & \Upsilon_2 \end{bmatrix} = [\upsilon_{ij}]$, $\Upsilon_1 \in \mathbb{R}^{(d+1) \times (b-d)}$, and $\Upsilon_2 \in \mathbb{R}^{(d+1) \times (d+1)}$.

Proof: The equivalence between statements i) and ii) is an immediate consequence of Handelman's Theorem. The equivalence between statements ii) and iii) follows from simple algebraic manipulations allowing to reduce the number of additional decision variables. First remark that the equality constraints can be compactly written as

$$\begin{bmatrix} P_0(x) \\ \vdots \\ P_d(x) \end{bmatrix} = \Upsilon \begin{bmatrix} Q_0(y) \\ \vdots \\ Q_b(y) \end{bmatrix}.$$

Note that the matrix Υ is full-row rank, otherwise it would not be possible to characterize independent polynomial coefficients. Using the decomposition $\Upsilon = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 \end{bmatrix}$ where $\Upsilon_2 \in \mathbb{R}^{(d+1) \times (d+1)}$ is w.l.o.g. a nonsingular matrix, the equality constraints can then be solved to get

$$\begin{bmatrix} Q_1(y) \\ \vdots \\ Q_d(y) \end{bmatrix} = \Upsilon_2^{-1} \left(\begin{bmatrix} P_0(x) \\ \vdots \\ P_d(x) \end{bmatrix} - \Upsilon_1 \begin{bmatrix} Q_{d+1}(y) \\ \vdots \\ Q_b(y) \end{bmatrix} \right). \tag{44}$$

Since the $Q_i(y)$'s are nonnegative vectors, this is then equivalent to say that the right-hand side of the above equality is nonnegative. Finally, posing $R_k = Q_{d+k}$, $k = 1, \ldots b - d$, we obtain the feasibility problem of statement iii). The opposite implication is obtained by reverting the reasoning. The proof is complete. ∎

In the light of the above result, it turns out that it is always possible to represent a robust linear program with polynomial dependence on uncertain parameters within a compact polytope as a more complex finite dimensional linear program. Handelman's Theorem yields a linear program involving $N_P(b + 1) + \eta$ variables, $N_P(d+1)$ equality constraints and $N_P(b+1)$ inequality constraints. After reduction, the problem has $N_P(b-d) + \eta$ decision variables and $N_P(b+2)$ inequality constraints. The complexity of the problem has then be reduced. The same reasoning applies to the case of multiple uncertainties.

The question of choosing which products of basis functions to consider is a difficult problem. A brute force approach would consider all the possible products up to a certain degree. This is an easy task in the case of univariate polynomials. This is however more problematic when considering multivariate

polynomials since the number of basis functions b can be very large. A bound on the necessary order for the products has been provided in [44] and generalizes straightforwardly to vector polynomials.

VIII. EXAMPLES

A. ILCs for several classes of uncertainties

ILCs for some common operators are discussed below.

1) Uncertain SISO tranfer function:

Consider now an uncertain asymptotically stable positive SISO proper transfer function $\widehat{\Sigma}(s, \rho)$ depending on constant uncertain parameters $\rho \in [0, 1]^N$. By virtue of Corollary 1, the ILC can be expressed as $\phi_1^T + \phi_2^T Z \geq 0$ where $Z \in \left\{ \widehat{\Sigma}(0, \rho): \ \rho \in [0, 1]^N \right\}$.

2) Multiplication operator:

Let us consider in this example the multiplication operator Σ which multiplies the input signal by a bounded and time-varying parameter $\delta(t)$ varying arbitrarily within its range of values, i.e. $z(t) = \Sigma(w)(t) = \delta(t)w(t)$. Since the parameter is time-varying, the time-domain version of the ILC must be considered

$$\int_0^{+\infty} [\varphi_1^T + \varphi_2^T \delta(\theta)] w(\theta) d\theta \geq 0. \tag{45}$$

A necessary and sufficient condition is given by $\varphi_1^T + \varphi_2^T \delta(t) \geq 0$ for all $t \geq 0$ where the scalings φ_1 and φ_2 are chosen according to the range of values of $\delta(t)$. Note that it is also possible to select polynomial scalings verifying $\varphi_1(\delta) = -\delta\varphi_2(\delta)$ in order to saturate the ILC and obtain a better characterization of the uncertainty set.

3) Uncertain infinite dimensional system:

The same reasoning also applies to general asymptotically-stable positive LTI infinite-dimensional systems governed by partial differential equations. Let us consider the heat equation given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= \omega^2 \frac{\partial^2 u}{\partial x^2} \\ u(0, t) &= w(t) \\ z(t) &= u(1, t) \end{aligned}$$

where $u(x, t)$ is the state of the system, $x \in [0, 1]$ the space variable, $w(t)$ the input, $z(t)$ the output and $\omega > 0$ an uncertain parameter of the system. The transfer function of the system is given by $G(s) = \alpha e^{-\sqrt{s}/\omega} + \beta e^{\sqrt{s}/\omega}$ where α and β are real constants determined according to initial and boundary conditions. The static-gain $G(0) = \alpha + \beta$ is independent of ω and, according to Corollary 1, it is enough to choose $\varphi_1 = -\varphi_2(\alpha + \beta)$. Stability of interconnections of LTI positive finite-dimensional systems and the heat-equation can hence be proved regardless of $\omega \in [0, +\infty)$.

nb. of systems	(n, p, q)	L_1-gain	L_∞-gain
20	(300,100,150)	$\mu = 12.282,\ \sigma = 1.1406$	$\mu = 14.186,\ \sigma = 1.4151$
100	(50, 20, 30)	$\mu = 0.53973,\ \sigma = 0.27486$	$\mu = 0.50735,\ \sigma = 0.080446$

TABLE I

MEAN COMPUTATION TIME μ [SEC] AND STANDARD DEVIATION σ [SEC] FOR GAIN COMPUTATION

4) Constant delay operator:

Let us consider the constant delay operator $\widehat{\Sigma}(s) = e^{-sh}$, $h \geq 0$, which is indeed a positive operator. The static-gain $\widehat{\Sigma}(0)$ is equal to 1 regardless of the value of h. The ILC can hence be easily saturated by choosing $\varphi_1^T = -\varphi_2^T$. It is shown in Section VIII-D that the consideration of such scalings leads to a necessary and sufficient condition for the stability of positive time-delay systems with constant delays, recovering then the results of [48].

5) Time-varying delay operator:

Let us consider now the time-varying delay operator Σ defined as $z(t) = \Sigma(w)(t) = w(t - h(t))$ where $h(t) \geq 0$ and $\dot{h}(t) \leq \mu < 1$ for all $t \geq 0$. In such a case, the operator can only be characterized through its L_1-gain given by $(1 - \mu)^{-1}$ under the standard assumption of 0 initial conditions [49]. A suitable ILC for this operator is then given by

$$\int_0^\infty \varphi^T \left(w(t) - (1 - \mu)z(t) \right) dt \geq 0$$

with $\varphi > 0$. Hence $\varphi_1 = \varphi$ and $\varphi_2 = -(1 - \mu)\varphi$.

B. Computation of Norms

For this example, many linear positive systems have been randomly generated and their induced-norms computed on a laptop equipped with an Intel U7300 processor of 1.3GHz with 4GB of RAM. The mean computation time and the standard deviation for different systems are gathered in Table I. Note that the number of variables is $n + 1$ and the number of constraints is $2n + p + 1$ and $2n + 1 + q$ for the L_1-gain and the L_∞-gain respectively. Since the number of constraints is larger for the L_∞-gain, it is expected that it takes longer to compute. It seems important to note that the the induced-norms are different since the system is not SISO, so the theoretical analysis in [13] does not hold here.

C. Example 2: Drug distribution

A frequently used model to analyze the distribution or flow of a drug or a tracer through the human body after injection into the bloodstream is given by the following compartmental model [1]:

$$\dot{x}(t) = \begin{bmatrix} -(a_{11} + a_{21}) & a_{12} \\ a_{21} & -a_{12} \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{46}$$

Output matrix C	L_1-gain	L_∞-gain
$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$	$\frac{1}{a_{11}}$	$\frac{1}{a_{11}}$
$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$	$\frac{a_{21}}{a_{11}a_{12}}$	$\frac{a_{21}}{a_{11}a_{12}}$
$C = \mathrm{diag}(k_1, k_2)$	$\frac{|k_1|}{a_{11}} + \frac{|k_2||a_{21}|}{a_{11}a_{12}}$	$\max\left\{ \frac{|k_1|}{a_{11}}, \frac{|k_2||a_{21}|}{a_{11}a_{12}} \right\}$

TABLE II

GAINS OF SYSTEM (46) FOR DIFFERENT OUTPUT MATRICES

where x_1, x_2 and u are the compartment corresponding to the blood plasma, the extravascular space (e.g. tissue) and the drug/tracer injection. Here, the drug or the tracer is injected directly into the bloodstream and the drug is evacuated by the kidneys at rate $a_{11} > 0$. The scalars $a_{12}, a_{21} > 0$ are the transmission coefficients for the drug/tracer between the bloodstream and the extravascular space. Due to the sign pattern, the system is positive and asymptotically stable. When the system is SISO the L_1-gain coincides with the L_∞-gain and they are both equal to the static-gain of the system. Different gains for different output matrices are given in Table II.

D. Theoretical Robustness analysis - Time-delay systems

We illustrate here the nonconservativeness of Theorem 3 made possible by the saturation of the ILC condition. The case of time-delay systems is addressed.

1) Constant time-delay:

Let us consider the linear positive system with constant time-delay:

$$\dot{x}(t) = Ax(t) + A_h x(t - h) \tag{47}$$

for some $h \geq 0$. It is well known that such a system is positive if and only if the matrix A is Metzler and the matrix A_h is nonnegative. Rewriting it in an LFT form we get

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + A_h w_0(t) \\
z_0(t) &= x(t) \\
w_0(t) &= \nabla_c(z_0)(t)
\end{aligned} \tag{48}$$

where ∇_c is the constant delay operator with Laplace transform $\widehat{\nabla}_c(s) = e^{-sh}$. Since the static-gain of the delay operator is equal to 1, we can apply Theorem 3 which yields the stability conditions

$$\begin{aligned}
\lambda^T A + \varphi_1^T &< 0, \\
\lambda^T A_h + \varphi_2^T &< 0, \\
\varphi_1^T + \varphi_2^T &= 0
\end{aligned} \tag{49}$$

that must be feasible for some $\lambda \in \mathbb{R}^n_{++}$ and $\varphi_1, \varphi_2 \in \mathbb{R}^n$. They are equivalent to the conditions

$$\lambda^T A + \varphi_1^T < 0 \text{ and } \lambda^T A_h - \varphi_1^T < 0 \tag{50}$$

which are in turn equivalent[3] to the inequality

$$\lambda^T(A + A_h) < 0 \tag{51}$$

well-known to be a necessary and sufficient condition for asymptotic stability of positive time-delay systems [48], [50]. Note also that this condition is dual to the one derived in [27]. By solving for λ instead of φ_1, we get the condition

$$-\varphi_1^T A^{-1} A_h < \varphi_1^T, \ \ \varphi_1 > 0 \tag{52}$$

which is a linear counterpart of the spectral radius condition for the stability of linear time-delay systems, see e.g. [49].

2) Time-varying delay:

Consider now the time-varying delay operator which has been considered in the case of positive systems in [27]. The L_1-gain of time-varying operator is $(1 - \mu)^{-1}$, $\mu < 1$ as discussed in Section VIII-A5 . In such a case, the stability conditions are given by

$$\begin{aligned} \lambda^T A + \varphi^T &< 0, \\ \lambda^T A_h - (1 - \mu)\varphi^T &< 0 \end{aligned} \tag{53}$$

for some $\lambda, \varphi \in \mathbb{R}_{++}^n$. Identical conditions can be obtained using a linear Lyapunov-Krasovskii functional [50] of the form $V(x_t) = \lambda^T x(t) + \varphi^T \int_{t-h(t)}^t x(s)ds$ with $\lambda, \varphi > 0$. Note that in the robust formulation, (53) is feasible only if $\varphi > 0$. Note also that the conditions can be merged into the novel single inequality

$$\lambda^T((1 - \mu)A + A_h) < 0 \tag{54}$$

where we can see that the time-varying delay penalizes the stability by scaling down the matrix A. The time-invariant case is retrieved for $\mu = 0$.

While the above results are based on the L_1-gain, it seems interesting to analyze the stability using the L_∞-gain. The L_∞-gain of the time-varying delay operator is equal to 1 [51]. We may then use a small-gain argument to prove asymptotic stability of the time-delay system by applying a scaled version of Lemma 2. Hence, the time-delay system is asymptotically stable provided that there exist $\lambda, \varphi \in \mathbb{R}_{++}^n$ such that the conditions

$$A\lambda + A_h\varphi < 0 \ \ \text{and} \ \ \lambda - \varphi^T < 0 \tag{55}$$

hold. They are equivalent to the condition $(A + A_h)\lambda < 0$ (stability for zero delay) or the condition $-A^{-1}A_h\varphi < \varphi$ (small-gain condition) which is the L_∞-gain counterpart of (52). It is important to precise that the stability condition is identical to the one for systems with constant delay. This result is then much stronger than the L_1-based one since the delay-derivative bound does not have any negative impact on the stability. As discussed in [51], while results using the L_1-norm relate to Lyapunov-Krasovskii functionals, those based on the L_∞-norm connect to Lyapunov-Razumikhin functions. Note however that no Lyapunov-Razumikhin results for positive systems have been reported so far.

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[3]To see the equivalence, note that (50) implies (51) by summation. To prove the converse, assume (51) holds and defining $\varphi_1^T = \lambda^T A_h - \lambda^T(A + A_h)/2 > 0$ in (50) makes the conditions negative.

E. Numerical robustness analysis - A systems biology example

Let us consider the example of a simple gene expression process described by the following model [52]:

$$\begin{bmatrix} \dot{x}_r(t) \\ \dot{x}_p(t) \end{bmatrix} = \begin{bmatrix} -\gamma_r & 0 \\ k_p & -\gamma_p \end{bmatrix} \begin{bmatrix} x_r(t) \\ x_p(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{56}$$

where $x_r \geq 0$ is the mean number of mRNA in the cell, $x_p \geq 0$ is the mean number of protein of interest in the cell. Above $u(t) \geq 0$ is the transcription rate of DNA into mRNA, $\gamma_r > 0$ is the degradation rate of mRNA, $k_p > 0$ is the translation rate of mRNA into protein and $\gamma_p > 0$ is the degradation rate of the protein. The parameters are assumed to be uncertain and given by $\gamma_r = \gamma_r^0 + \varepsilon_1 \gamma_r^1$, $k_p = k_p^0 + \varepsilon_2 k_p^1$ and $\gamma_p = \gamma_p^0 + \varepsilon_3 \gamma_p^1$, where $\varepsilon_i \in [-1, 1]$, $i = 1, 2, 3$. We are interested in analyzing the L_∞-gain of the transfer from $u \to x_p$ over the set of all possible systems. In other terms, we would like to analyze the impact of the maximal value of the transcription rate to the maximal value of the mean number of proteins.

The overall system can be rewritten as

$$\begin{bmatrix} \dot{x}_r(t) \\ \dot{x}_p(t) \end{bmatrix} = A_u(\varepsilon) \begin{bmatrix} x_r(t) \\ x_p(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{57}$$

where $\varepsilon = \mathrm{col}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and

$$A_u(\varepsilon) = \begin{bmatrix} -\gamma_r^0 & 0 \\ k_p^0 & -\gamma_p^0 \end{bmatrix} + \varepsilon_1 \begin{bmatrix} -\gamma_r^1 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon_2 \begin{bmatrix} 0 & 0 \\ k_p^1 & 0 \end{bmatrix} + \varepsilon_3 \begin{bmatrix} 0 & 0 \\ 0 & -\gamma_p^1 \end{bmatrix}.$$

Note that the theoretical computation of the L_∞-gain is rather difficult since the static-gain depends rationally and in a nonconvex way on the uncertain parameter vector ε. However, by exploiting the fact that the set $\mathscr{A} := \{A_u(\varepsilon) : \varepsilon \in [-1, 1]^3\}$ is convex, the conditions (6), or equivalently (10), can be checked without using the LFT formulation[4], although in this case both methods are equivalent provided that the scalings are chosen accordingly. Checking indeed the conditions (6) over the whole set $\mathscr{A}$ is equivalent to checking them over the set of its vertices, i.e. $\mathscr{A}_v := \{A_u(\varepsilon) : \varepsilon \in \{-1, 1\}^3\}$. This follows from standard convexity argument.

Let us consider $\gamma_r^0 = 1$, $k_p^0 = 2$ and $\gamma_p^0 = 1$ for numerical application. We also assume that the parameters are known up to a percentage $N \in [0, 1)$ of their nominal value, hence $\gamma_r^1 = N\gamma_r^0$, $\gamma_p^1 = N\gamma_p^0$ and $k_p^1 = Nk_p^0$. The results are summarized in Table III where we can see that the proposed approach is rather accurate.

F. Numerical robustness analysis - A polynomial system example

Let us consider the uncertain system with constant parametric uncertainty $\delta \in [0, 1]$:

$$\begin{aligned} \dot{x}(t) &= (A^0 + \delta A^1 + \delta^2 A^2)x(t) + (E^0 + \delta E^1 + \delta^2 E^2)w_1(t) \\ z_1(t) &= (C^0 + \delta C^1 + \delta^2 C^2)x(t) + (F^0 + \delta F^1 + \delta^2 F^2)w_1(t) \end{aligned} \tag{58}$$

---

[4]Remember that the interest of the LFT is to convexify the problem.

N	L_∞-gain	Theoretical
0	2	2
0.1	2.7162	2.7161
0.3	5.3063	5.3062
0.5	12.0003	12.0000
0.7	37.7783	37.7779

TABLE III

COMPUTED AND THEORETICAL L_∞-GAINS OF THE TRANSFER $u \to x_p$ OF THE UNCERTAIN SYSTEM (57)

$\varphi_1(\delta)$	$\varphi_2(\delta)$	constraints	computed L_1-gain	time [sec]
φ_1^0	φ_2^0	$\varphi_1^0 \geq 0,\ \varphi_1^0 + \varphi_2^0 \geq 0$	133.95	2.7844
$\varphi_1^1 \delta$	φ_2^0	$\varphi_1^1 = -\varphi_2^0$	133.95	3.829
$\varphi_1^1 \delta + \varphi_1^2 \delta^2$	$\varphi_2^0 + \varphi_2^1 \delta$	$\varphi_1^1 = -\varphi_2^0,\ \varphi_1^2 = -\varphi_2^1$	94.167	4.2758

TABLE IV

L_1-GAIN COMPUTATION OF THE TRANSFER $w_1 \to z_1$ USING THEOREM 2 – EXACT L_1-GAIN: 92.8358

with the matrices

$$
A^0 = \begin{bmatrix} -10 & 2 & 4 \\ 3 & -8 & 1 \\ 2 & 1 & -5 \end{bmatrix}, \quad
A^1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ -1 & 2 & -1 \end{bmatrix}, \quad
A^2 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix},
$$

$$
E^0 = \begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}, \quad
E^1 = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad
E^2 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 4 \end{bmatrix},
$$

$$
C^0 = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad
C^1 = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad
C^2 = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 1 \end{bmatrix},
$$

$$
F^0 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad
F^1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad
F^2 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.
$$
(59)

The system can be rewritten in the LFT form (17) with matrices $A_0 = A^0$, $E_0 = \begin{bmatrix} A^1 & A^2 & E^1 & E^2 \end{bmatrix}$, $E_1 = E^0$, $C_1 = C^0$, $F_{10} = \begin{bmatrix} C^1 & C^2 & F^1 & F^2 \end{bmatrix}$, $F_{11} = F^0$ and

$$
C_0 = \begin{bmatrix} I_n \\ 0_n \\ 0_{p \times n} \\ 0_{p \times n} \end{bmatrix}, \quad
F_{00} = \begin{bmatrix} 0_n & 0_n & 0_{n \times p} & 0_{n \times p} \\ I_n & 0_n & 0_{n \times p} & 0_{n \times p} \\ 0_{p \times n} & 0_{p \times n} & 0_p & 0_p \\ 0_{p \times n} & 0_{p \times n} & I_p & 0_{p \times p} \end{bmatrix}, \quad
F_{01} = \begin{bmatrix} 0_{n \times p} \\ 0_{n \times p} \\ I_p \\ 0_p \end{bmatrix}.
$$
(60)

The matrices of the LFT form (32) are given by $\bar{F}_{00} = F_{00}$, $\bar{C}_{00} = C_0$, $\bar{F}_{01} = F_{01}$, $\tilde{E}_0 = \begin{bmatrix} A^{1T} & A^{2T} & C^{1T} & C^{2T} \end{bmatrix}$ and $\tilde{F}_{10} = \begin{bmatrix} E^{1T} & E^{2T} & F^{1T} & F^{2T} \end{bmatrix}$.

Using Theorem 2 and Theorem 4 (together with the complexity reduction technique of Section VII) and different forms for the scalings φ_1 and φ_2, we obtain the results gathered in Tables IV and V. We can see that the L_1-gain is not very well estimated compared to the L_∞-gain for parameter independent

$\varphi_1(\delta)$	$\varphi_2(\delta)$	constraints	computed L_1-gain	time [sec]
φ_1^0	φ_2^0	$\varphi_1^0 \geq 0,\ \varphi_1^0 + \varphi_2^0 \geq 0$	86.195	0.68989
$\varphi_1^1\delta$	φ_2^0	$\varphi_1^1 = -\varphi_2^0$	86.195	1.4629
$\varphi_1^1\delta + \varphi_1^2\delta^2$	$\varphi_2^0 + \varphi_2^1\delta$	$\varphi_1^1 = -\varphi_2^0,\ \varphi_1^2 = -\varphi_2^1$	82.025	1.7509

TABLE V

L_∞-GAIN COMPUTATION OF THE TRANSFER $w_1 \to z_1$ USING THEOREM 4 – EXACT L_∞-GAIN: 82.0249

scalings. Using scalings of degree two considerably reduces the conservatism of the approach. Indeed, we are able to estimate accurately the L_∞-gain while little conservatism persists for the L_1-gain. It is important to note that the above numerical results also hold in the case of time-varying parameters (see Section VIII-A2).

IX. CONCLUSION

By relying on the fact that the L_1-gain and L_∞-gain are related to the static-gain matrix of the positive system, a linear programming approach has been proposed to compute them under the assumption that the input signals are nonnegative. Interestingly, the results turn out to be also valid for inputs and states with no definite sign. Based on this result, stabilization conditions with performance constraints have been obtained for different classes of controllers. Using then the newly introduced Integral Linear Constraints, robust stability analysis results are provided together with generic robustness results, again relying on static-gains of positive systems. It is shown that the results are nonconservative in the case of LTI positive uncertainties with fixed static-gain matrix. Stabilization results are finally obtained and expressed as (robust) linear programming problems. An exact relaxation scheme based on Handelman's Theorem is proposed in order to obtain finite-dimensional feasibility problems. Several examples illustrate the approach.

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REFERENCES

[1] W. M. Haddad, V. Chellaboina, and Q. Hui, *Nonnegative and compartmental dynamical systems*. 41 William Street, Princeton, New Jersey, USA: Princeton University Press, 2010.

[2] J. D. Murray, *Mathematical Biology Part I. An Introduction. 3rd Edition*. Springer, 2002.

[3] R. Shorten, F. Wirth, and D. Leith, "A positive systems model of TCP-like congestion control: asymptotic results," *IEEE Transactions on Networking*, vol. 14(3), pp. 616–629, 2006.

[4] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. John Wiley & Sons, 2000.

[5] P. D. Leenheer and D. Aeyels, "Stabilization of positive linear systems," *Systems & Control Letters*, vol. 44, pp. 259–271, 2001.

[6] W. M. Haddad and V. Chellaboina, "Stability and dissipativity theory for nonnegative dynamic systems: a unified analysis framework for biological and physiological systems," *Nonlinear Analysis: Real World Applications*, vol. 6, pp. 35–65, 2005.

[7] M. Ait Rami and F. Tadeo, "Controller synthesis for positive linear systems with bounded controls," *IEEE Transactions on Circuits and Systems – II. Express Briefs*, vol. 54(2), pp. 151–155, 2007.

[8] R. Bru and S. Romero-Vivó, *Positive systems - Proceedings of the 3rd Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 2009)*. Springer, 2009.

[9] R. Shorten, O. Mason, and C. King, "An alternative proof of the Barker, Berman, Plemmons result on diagonal stability and extensions," *Linear Algebra and Its Applications*, vol. 430, pp. 34–40, 2009.

[10] T. Tanaka and C. Langbort, "KYP Lemma for internally positive systems and a tractable class of distributed h-infinity control problems," in *Americal Control Conference*, Baltimore, Maryland, USA, 2010, pp. 6238–6243.

[11] ——, "Symmetric formulation of the Kalman-Yakubovich-Popov Lemma and exact losslessness condition," in *50th IEEE Conference on Decision and Control*, Orlando, FLorida, USA, 2011, pp. 5645–5652.

[12] Y. Ebihara, D. Peaucelle, and D. Arzelier, "L_1 gain analysis of linear positive systems and its applications," in *50th Conference on Decision and Control, Orlando, Florida, USA*, 2011, pp. 4029–4034.

[13] A. Rantzer, "Distributed control of positive systems," in *50th IEEE Conference on Decision and Control*, 2011, pp. 6608–6611.

[14] C. W. Scherer, "A full-block $\mathcal{S}$-procedure with applications," in *Conference on Decision and Control*, 1997.

[15] A. Rantzer and A. Megretski, "System analysis via Integral Quadratic Constraints," *IEEE Transactions on Automatic Control*, vol. 42(6), pp. 819–830, 1997.

[16] R. Kalman, "Lyapunov functions for the problem of Lur'e in automatic control," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 49(2), pp. 201–205, 1963.

[17] A. Rantzer, "On the Kalman-Yakubovicth-Popov lemma," *System & Control Letters*, vol. 28(1), 1996.

[18] C. W. Scherer and S. Weiland, *Linear Matrix Inequalities in Control*, 2005.

[19] M. G. Safonov, *Stability, Robustness of Multivariable Feedback Systems*. Cambridge, MA, USA: MIT Press, 2000.

[20] T. Iwasaki, "LPV system analysis with quadratic separator," in *Conference on on Decision and Control*, 1998, pp. 3021–3026.

[21] D. Peaucelle, D. Arzelier, D. Henrion, and F. Gouaisbaut, "Quadratic separation for feedback connection of an uncertain matrix and an implicit linear transformation," *Automatica*, vol. 43, pp. 795–804, 2007.

[22] S. Boyd, L. El-Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. PA, SIAM, Philadelphia, 1994.

[23] Y. Nesterov and A. Nemirovskii, *Interior-Point Polynomial Algorithm in Convex Programming*. PA, SIAM, Philadelphia, 1994.

[24] G. P. Barker, A. Berman, and R. J. Plemmons, "Positive diagonal solutions to the lyapunov equations," *Linear and Multilinear Algebra*, vol. 5(3), pp. 249–256, 1978.

[25] V. Blondel and J. Tsitsiklis, "NP-hardness of some linear control problems," *SIAM J. Control and Optimization*, vol. 35(6), pp. 2118–2127, 1997.

[26] O. Mason and R. N. Shorten, "On linear copositive lyapunov functions and the stability of switched positive linear systems," *IEEE Transactions on Automatic Control*, vol. 52(7), pp. 1346–1349, 2007.

[27] M. Ait Rami, "Stability analysis and synthesis for linear positive systems with time-varying delays," in *Positive systems - Proceedings of the 3rd Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 2009)*. Springer, 2009, pp. 205–216.

[28] F. Knorn, O. Mason, and R. N. Shorten, "On linear co-positive Lyapunov functions for sets of linear positive systems," *Automatica*, vol. 45(8), pp. 1943–1947, 2009.

[29] E. Fornasini and M. Valcher, "Linear copositive Lyapunov functions for continuous-time positive switched systems," *IEEE Transactions on Automatic Control*, vol. 55(8), pp. 1933–1937, 2010.

[30] J. Willems, "Dissipative dynamical systems i & ii," *Rational Mechanics and Analysis*, vol. 45(5), pp. 321–393, 1972.

[31] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, New Jersey, USA: Prentice Hall, 1996.

[32] C. Briat, "Robust stability analysis of uncertain linear positive systems via integral linear constraints - L_1- and L_∞-gains characterizations," in *50th IEEE Conference on Decision and Control*, Orlando, Florida, USA, 2011, pp. 3122–3129.

[33] D. Handelman, "Representing polynomials by positive linear functions on compact convex polyhedra," *Pacific Journal of Mathematics*, vol. 132(1), pp. 35–62, 1988.

[34] C. A. Desoer and M. Vidyasagar, *Feedback Systems : Input-Output Properties*. Academic Press, New York, 1975.

[35] A. Zappavigna, P. Colaneri, J. C. Geromel, and R. H. Middleton, "Stabilization of continuous-time switched linear positive systems," in *American Control Conference*, Baltimore, Maryland, USA, 2010, pp. 3275–3280.

[36] V. Yakubovich, "S-procedure in nonlinear control theory (english translation)," *Vestnik Leningrad University*, vol. 4, pp. 73–93, 1977.

[37] U. T. Jönsson, "A lecture on the S-procedure," Division of Optimization and Systems Theory, Royal Institute of Technology, Stockholm, Sweden, Tech. Rep., 2001. [Online]. Available: http://www.math.kth.se/~ulfj/5B5746/Lecture.ps

[38] D. Peaucelle and A. Fradkov, "Robust adaptive $\mathcal{L}_2$-gain control of polytopic MIMO LTI systems - LMI results," *Systems & Control Letters*, vol. 57, pp. 881–887, 2008.

[39] C. Briat and J. Martinez, "Design of $\mathcal{H}_\infty$ bounded non-fragile controllers - a robust control approach," in *European Control Conference 2009, Budapest, Hungary*, 2009.

[40] M. Ait Rami and F. Tadeo, "Positive observation problem for linear discrete positive systems," in *45th IEEE Conference on Decision and Control*, 2006, pp. 4729–4733.

[41] T. Iwasaki and S. Hara, "Well-posedness of feedback systems: insight into exact robustness analysis and approximate computations," *IEEE Transactions on Automatic Control*, vol. 43, pp. 619–630, 1998.

[42] J. Zhou and T. Hagiwara, "H_2 and H_∞ norm computations of linear continuous-time periodic systems via the skew analysis of frequency response operators," *Automatica*, vol. 38, pp. 1381–1387, 2002.

[43] C. W. Scherer and I. E. Köse, "On robust controller synthesis with dynamic D-scalings," in *46th Conference on Decision and Control*, New Orleans, LA, USA, 2007, 2007.

[44] V. Powers and B. Reznick, "A new bound for Pólya Theorem with applications to polynomials positive on polyhedra," *Journal of pure and applied algebra*, vol. 164, pp. 221–229, 2001.

[45] C. Scheiderer, "Positivity and sums of squares: A guide to recent results," in *Emerging Applications of Algebraic Geometry*, ser. IMA Volumes Math. Appl., M. Putinar and S. Sullivant, Eds.   Springer, 2009, vol. 149, pp. 271–324.

[46] G. Hardy, J. Littlewood, and G. Pölya, *Inequalities*.   Cambridge University Press, 1978.

[47] P. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Institute of Technology, Pasadena, California, 2000.

[48] W. M. Haddad and V. Chellaboina, "Stability theory for nonnegative and compartmental dynamical systems with time delay," *Systems & Control Letters*, vol. 51(5), pp. 355–361, 2004.

[49] K. Gu, V. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*.   Birkhäuser, 2003.

[50] T. Kaczorek, "Stability of positive continuous-time linear systems with delays," *Bulletin of the Polish Academy of Sciences - Technical sciences*, vol. 57(4), pp. 395–398, 2009.

[51] C. Briat, "Robust stability analysis in the $*$-norm and Lyapunov-Razumikhin functions for the stability analysis and control of time-delay systems," in *50th IEEE Conference on Decision and Control*, Orlando, Florida, USA, 2011, pp. 6319–6324.

[52] M. Khammash, "Modeling and analysis of stochastic biochemical networks," in *Control Theory and Systems Biology*, P. A. Iglesias and B. P. Ingalls, Eds.   MIT Press, 2010, pp. 29–44.