Geometric structure in the representation theory of p-adic groups

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Abstract

We conjecture the existence of a simple geometric structure underlying questions of reducibility of parabolically induced representations of reductive p-adic groups.

1 Introduction

In the representation theory of reductive p-adic groups, the issue of reducibility of induced representations is an issue of great intricacy: see, for example, the classic article by Bernstein-Zelevinsky [4] on $\mathrm{GL}(n)$ and the more recent article by Muić [7] on G_2. It is our contention, expressed as a conjecture, that there exists a simple geometric structure underlying this intricate theory.

For the moment, our conjecture is *local*, in that it applies only to finite places. To explain our conjecture, we need to refine the usual concept of quotient space.

2 The extended quotient and the reduced quotient

Let Γ be a finite group and let X be a complex affine algebraic variety. Assume that Γ is acting on X as automorphisms of the affine algebraic variety X.

Definition 1. *The quotient variety X/Γ is obtained by collapsing each orbit of X to a point.*

If J is a finite group, $c(J)$ denotes the set of conjugacy classes of J. If $x \in X$, Γ_x denotes the isotropy group of x:

$$\Gamma_x = \{\gamma \in \Gamma \,:\, \gamma x = x\}.$$

The *extended quotient*, denoted $X//\Gamma$ is obtained from X by replacing each orbit $\{\gamma x \,:\, \gamma \in \Gamma\}$ by $c(\Gamma_x)$. To construct $X//\Gamma$, we proceed as follows. Let

$$\widetilde{X} := \{(\gamma, x) \in \Gamma \times X \,:\, \gamma x = x\}.$$

The group Γ acts on $\widetilde{X}$ by:

$$\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha x) \quad \text{with } (\gamma, x) \in \widetilde{X},\ \alpha \in \Gamma.$$

Definition 2. *The* extended quotient*, denoted $X//\Gamma$, is defined as*

$$X//\Gamma := \widetilde{X}/\Gamma$$

i.e. $X//\Gamma$ is the ordinary quotient (as in definition 1) for the action of Γ on $\widetilde{X}$.

The projection

$$\Gamma \times X \longrightarrow X, \quad (\gamma, x) \mapsto x$$

gives a map

$$\pi\colon X//\Gamma \longrightarrow X/\Gamma.$$

If p is an orbit in X, i.e. $p = \{\gamma x \,:\, \gamma \in \Gamma\}$, then the pre-image in $X//\Gamma$ of p is $c(\Gamma_x)$. Thus, in forming $X//\Gamma$, each orbit $\{\gamma x \,:\, \gamma \in \Gamma\}$ has been replaced by $c(\Gamma_x)$.

Definition 3. *The map $\pi\colon X//\Gamma \longrightarrow X/\Gamma$ is* the projection of the extended quotient on the ordinary quotient.

Lemma 1. *The projection $\pi\colon X//\Gamma \to X/\Gamma$ is a finite morphism of algebraic varieties.*

Let $X^\gamma := \{x \in X \,:\, \gamma x = x\}$ and denote by $\mathrm{Z}(\gamma)$ the Γ-centralizer of γ. We have

$$X//\Gamma = \bigsqcup_{\gamma} X^\gamma/\mathrm{Z}(\gamma)$$

where one γ is chosen in each Γ-conjugacy class. Let $e \in \Gamma$ denote the identity element. Since $X^e/\mathrm{Z}(e)$ is the ordinary quotient X/Γ, we have $X/\Gamma \subset X//\Gamma$.

Definition 4. *The* reduced quotient *is defined as*

$$(X/\Gamma)_\rho := \pi(X//\Gamma - X/\Gamma).$$

Lemma 2. *The reduced quotient is an algebraic variety.*

Lemma 3. *The extended quotient is multiplicative: if $\Gamma_1 \times X_1 \to X_1$ and $\Gamma_2 \times X_2 \to X_2$ are as above, then we have*

$$(X_1 \times X_2)//(\Gamma_1 \times \Gamma_2) = (X_1//\Gamma_1) \times (X_2//\Gamma_2).$$

3 Application to the representation theory of p-adic groups

Let F be a local nonarchimedean field, let G be the group of F-rational points in a connected reductive algebraic group defined over F, and let $\mathrm{Irr}(G)$ be the set of irreducible smooth representations of G. We recall the data in the Bernstein programme: $\mathfrak{s} \in \mathfrak{B}(G)$, $\mathfrak{s} = [M, \sigma]_G$ is an inertial class in G (with M a Levi subgroup of G and σ a supercuspidal representation of M), $D^\mathfrak{s}$ is the $\Psi(M)$-orbit of σ in $\mathrm{Irr}(M)$, with $\Psi(M)$ the group of unramified characters of M, $W^\mathfrak{s} = \{w \in \mathrm{N}_G(M)/M \,:\, w \cdot \mathfrak{s} = \mathfrak{s}\}$ and $D^\mathfrak{s}/W^\mathfrak{s}$ is the quotient variety, a component of the Bernstein variety.

We will fix a point $\mathfrak{s} \in \mathfrak{B}(G)$ and write $D = D^\mathfrak{s}, W = W^\mathfrak{s}$. Let $\mathrm{Irr}(G)^\mathfrak{s}$ denote the $\mathfrak{s}$-component of $\mathrm{Irr}(G)$ in the Bernstein decomposition of $\mathrm{Irr}(G)$. We will equip the quotient variety D/W with the Zariski topology, and $\mathrm{Irr}(G)^\mathfrak{s}$ with the Jacobson topology coming from $\mathrm{Prim}\,\mathcal{H}(G)^\mathfrak{s}$. We note that irreducibility is an *open* condition, and so $(D/W)_{\mathrm{red}}$, the set of reducible points in D/W, is a sub-variety of D/W. Let q denote the cardinality of the residue field of F. Let $\mathcal{H}(G)$ denote the Hecke algebra of G, let $\mathcal{H}(G)^\mathfrak{s}$ denote the ideal of $\mathcal{H}(G)$ corresponding to $\mathfrak{s}$ in the Bernstein decomposition of $\mathcal{H}(G)$, and let HP_* denote periodic cyclic homology.

Conjecture 1. *There is an isomorphism*

$$\mathrm{HP}_*(\mathcal{H}(G)^\mathfrak{s}) \cong \mathrm{H}^*(\mathrm{D}/\!/\mathrm{W})$$

and a continuous bijection

$$\mu \colon D/\!/W \to \mathrm{Irr}(G)^\mathfrak{s}$$

such that:

(1) There is an algebraic family $\pi_t \colon D/\!/W \to D/W$ of finite morphisms of algebraic varieties, with $t \in \mathbb{C}^\times$, such that

$$\pi_1 = \pi, \qquad \pi_{\sqrt{q}} = (inf.ch.) \circ \mu.$$

If we let $\mathfrak{X}_t$ be the image of π_t restricted to $D/\!/W - D/W$ then $\mathfrak{X}_t$ is a flat family of algebraic varieties such that

$$\mathfrak{X}_1 = (D/W)_\rho, \qquad \mathfrak{X}_{\sqrt{q}} = (D/W)_{red}.$$

(2) For each irreducible component $\mathbf{c} \subset D/\!/W$ there is a cocharacter $h_\mathbf{c} \colon \mathbb{C}^\times \to D$ such that

$$\pi_t(x) = \pi(h_\mathbf{c}(t) \cdot x)$$

for all $x \in \mathbf{c}$. If $\mathbf{c} = D/W$ then $h_\mathbf{c} = 1$.

Theorem 1. *The conjecture is true for* $G = \mathrm{SL}(2)$*. If* $\mathfrak{s} = [T, 1]_G$ *then* $\mathfrak{X}_t$ *is the* 0*-dimensional variety given by the polynomial* $(x + 1)(x - t^2) = 0$*.*

4 The general linear group

Theorem 2. *The conjecture is true for* $\mathrm{GL}(n)$*.*

Proof. The proof uses Langlands parameters, together with some careful combinatorics. In effect, the L-parameters encode the extended quotient for $\mathrm{GL}(n)$. Let $G = \mathrm{GL}(n) = \mathrm{GL}(n, F)$, $n = mr$, τ be an irreducible supercuspidal representation of $\mathrm{GL}(m, F)$,

$$\mathfrak{s} = [M, \sigma]_G = [\mathrm{GL}(m)^r, \tau^{\otimes r}]_G.$$

We have

$$D = D^{\mathfrak{s}} = (\mathbb{C}^\times)^r, \quad W = W^{\mathfrak{s}} = S_r.$$

Let W_F be the Weil group of F, and let $\mathcal{L}_F = W_F \times \mathrm{SU}(2)$. Let $\mathrm{Hom}_{\mathrm{ss}}(\mathcal{L}_F, \mathrm{GL}(n, \mathbb{C}))$ denote the set of equivalence classes of Frobenius-semisimple smooth homomorphisms from $\mathcal{L}_F$ to $\mathrm{GL}(n, \mathbb{C})$.

For each $n \geq 1$ we have the local Langlands correspondence [6]

$$\mathrm{rec}_F \colon \mathrm{Irr}(\mathrm{GL}(n, F)) \to \mathrm{Hom}_{\mathrm{ss}}(\mathcal{L}_F, \mathrm{GL}(n, \mathbb{C})).$$

We shall write

$$\Phi(G) = \mathrm{Hom}_{\mathrm{ss}}(\mathcal{L}_F, \mathrm{GL}(n, \mathbb{C})).$$

We will denote by p the following partition of r:

$$a_1 + \cdots + a_1 + \cdots + a_l + \cdots + a_l = r_1 a_1 + \cdots r_l a_l = r$$

where a_j is repeated r_j times. Let $\gamma \in S_r$ be the corresponding product of $r_1 + \cdots + r_l$ cycles. The fixed set D^γ is a complex torus of dimension $r_1 + \cdots + r_l$.

We recall that τ is supercuspidal representation of $\mathrm{GL}(m)$. Now let $\mathrm{rec}_F(\tau) = \eta \in \mathrm{Irr}_m(W_F)$. Denote by $R(j)$ the j-dimensional irreducible complex representation of $\mathrm{SU}(2)$. Corresponding to the partition p we have the L-parameter

$$\phi = \eta \otimes R(a_1) \oplus \cdots \oplus \eta \otimes R(a_1) \oplus \cdots \oplus \eta \otimes R(a_l) \oplus \cdots \oplus \eta \otimes R(a_l)$$

where $\eta \otimes R(a_1)$ is repeated r_1 times, $\ldots$, $\eta \otimes R(a_l)$ is repeated r_l times.

Let $\Psi(W_F)$ denote the group of unramified quasicharacters of the Weil group W_F, and consider the complex torus $\Psi(W_F)^{r_1+\cdots+r_l}$. The *orbit* of ϕ in $\Phi(G)$, via the action of this complex torus, is

$$\mathcal{O}(\phi) = \mathrm{Sym}^{r_1}\mathbb{C}^\times \times \cdots \times \mathrm{Sym}^{r_l}\mathbb{C}^\times \subset \Phi(G).$$

Let $\psi_j \in \Psi(W_F)$ with $1 \leq j \leq r_1 + \cdots + r_l$. We will map each L-parameter in the orbit $\mathcal{O}(\phi)$ as follows:

$$\psi_1 \otimes \eta \otimes R(a_1) \oplus \cdots \oplus \psi_{r_1+\cdots+r_l} \otimes \eta \otimes R(a_l) \mapsto (\psi_1(\varpi), \ldots, \psi_{r_1+\cdots+r_l}(\varpi)) \in D^\gamma$$

where ϖ is a uniformizer in F. This induces a *bijection*

$$\mathcal{O}(\phi) \cong D^\gamma / \mathrm{Z}(\gamma).$$

Let $\Phi(G)^\mathfrak{s}$ denote the $\mathfrak{s}$-component of $\Phi(G)$ in the Bernstein decomposition of $\Phi(G)$, so that

$$\Phi(G)^\mathfrak{s} = \mathrm{rec}_F(\mathrm{Irr}(G)^\mathfrak{s}).$$

We now take the disjoint union of the permutations γ, one chosen in each S_r-conjugacy class. This creates a *canonical* bijection

$$\Phi(G)^\mathfrak{s} \cong D/\!/W.$$

The reduced quotient $(D/W)_\rho$ is the hypersurface $\mathfrak{X}_1$ given by the single equation $\prod_{i\neq j}(z_i - z_j) = 0$. The variety $(D/W)_{red}$ is the variety $\mathfrak{X}_q$ given by the single equation $\prod_{i\neq j}(z_i - qz_j) = 0$, according to a classical theorem [4, Theorem 4.2], [8]. The polynomial equation $\prod_{i\neq j}(z_i - tz_j) = 0$ determines a flat family $\mathfrak{X}_t$ of hypersurfaces. The hypersurface $\mathfrak{X}_1$ is the *flat limit* of the family $\mathfrak{X}_t$ as $t \to 1$, as in [5, p.77].

Let $\mathbf{c}$ denote the irreducible variety $D^\gamma/Z(\gamma)$. The cocharacter $h_\mathbf{c}$ is given by

$$h_\mathbf{c} \colon t \mapsto (t^{a_1-1}, \ldots, t^{1-a_1}, \ldots, t^{a_r-1}, \ldots, t^{1-a_r}) \in D.$$

Finally, we have to use the multiplicativity of the extended quotient. □

5 The exceptional group G_2

We have chosen the exceptional group G_2 as an awkward example, requiring many delicate calculations, see [2]. Let $\mathfrak{s} = [T, \chi \otimes \chi]_G$ where $T \simeq F^\times \times F^\times$ is a maximal F-split torus of $G = G_2$ and χ is a ramified quadratic character of $F^\times$. Let $\{\alpha, \beta\}$ be a basis of a set of roots of G with α short and β long. The group $W^\mathfrak{s} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is generated by the fundamental reflections

s_α and $s_{3\alpha+2\beta}$, and we have $D = D^{\mathfrak{s}} = \{\psi_1\chi \otimes \psi_2\chi \,:\, \psi_1, \psi_2 \in \Psi(F^\times)\}$. We obtain

$$D^\gamma/\mathrm{Z}(\gamma) = \begin{cases} \{\{(\lambda,\lambda),(\lambda^{-1},\lambda^{-1})\} \,:\, \lambda \in \mathbb{C}^\times\}, & \text{if } \gamma = s_\alpha, \\ \{\{(\lambda,\lambda^{-1}),(\lambda^{-1},\lambda)\} \,:\, \lambda \in \mathbb{C}^\times\}, & \text{if } \gamma = s_{3\alpha+2\beta}, \\ (1,1) \sqcup (-1,-1) \sqcup (1,-1), & \text{if } \gamma = s_\alpha s_{3\alpha+2\beta}. \end{cases}$$

Denote by $\mathfrak{C}_1$ the line $x - y = 0$ and by $\mathfrak{C}_2$ the hyperbola $xy - 1 = 0$. Setting $\mathrm{pt}_1 := (1,1)$, $\mathrm{pt}_2 := (-1,-1)$, and $\mathrm{pt}_3 := (1,-1)$, we obtain

$$D/\!/W = D/W \sqcup \mathfrak{C}_1 \sqcup \mathfrak{C}_2 \sqcup \mathrm{pt}_1 \sqcup \mathrm{pt}_2 \sqcup \mathrm{pt}_3, \qquad (D/W)_\rho = \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \mathrm{pt}_3.$$

The cocharacter $h_{\mathbf{c}} : \mathbb{C}^\times \to D$ is as follows:

$t \mapsto (t, t^{-1})$ if $\mathbf{c} = \mathfrak{C}_1$, (t^{-1}, t^{-1}) if $\mathbf{c} = \mathfrak{C}_2$, $(1, t^{-2})$ if $\mathbf{c} = \mathrm{pt}_1$ or pt_2, $(1,1)$ if $\mathbf{c} = \mathrm{pt}_3$.

The variety $\mathfrak{X}_t$ is the union of the line $x - t^2 y = 0$, the hyperbola $xy - t^{-2} = 0$ and the point pt_3. This is a flat family. The 2 curves admit 2 intersection points: $(1, t^{-2})$ and $(-1, -t^{-2})$. Now let $t = \sqrt{q}$. At each of the intersection points, the corresponding parabolically induced representation admits 4 irreducible inequivalent constituents. As for the point pt_3: the corresponding parabolically induced representation admits 2 irreducible inequivalent *tempered* constituents, see [7].

Theorem 3. *The conjecture is true for the point* $\mathfrak{s} = [T, \chi \otimes \chi]_G$.

Acknowledgements

The second author was partially supported by an NSF grant.

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