Graphs that admit 3-to-1 or 2-to-1 maps onto the circle

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Abstract

We give a complete and explicit characterization of the connected graphs which admit a continuous 3-to-1 map onto the circle, and of the connected graphs which admit a continuous 2-to-1 map onto the circle. This generalizes earlier work of Heath and Hilton who considered the mappings of trees onto the circle.

1. Introduction

In 1939 Harrold [3] observed that if the image of a $k$-to-1 continuous map is a graph, then the graph must contain a circle. More recently Heath [6] extended this observation, showing that a connected graph with no circles (i.e. a tree) cannot be the $k$-to-1 finitely discontinuous image of any connected metric compactum if $k > 1$. Thus if we wish to study $k$-to-1 continuous maps between graphs, the image graph must contain a circle. In this paper we deal with the simplest such image graph, namely the circle itself.

In [8] Heath and Hilton studied 2-to-1 and 3-to-1 continuous maps from trees onto the circle. In this paper we extend this earlier work to the case when the domain is a connected graph, not just a tree. This problem was raised orally by S.B. Nadler and was also raised in [15]. We do not deal with the case where the domain is not connected; this seems to present considerable further problems.

In [10] Heath and Hilton studied the general existence problem for $k$-to-1 maps between graphs from a different point of view, and in [13] Hilton characterized those pairs $(G, H)$ of graphs for which there is a continuous $k$-to-1 map from $G$ onto $H$ for all large $k$. For further recent papers in this area, see the further references.

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It is easy to jump to the conclusion that a $k$-to-1 continuous map is equivalent in some way to a covering space, but we should like to emphasize that this is not the case.

A few words about our terminology. An edge (or an arc) is a compact topological space homeomorphic to the real unit interval $[0, 1]$. A graph is a compact topological space that is the union of a finite number of edges, each two of which either do not intersect or else intersect in a common endpoint called a vertex, node, junction point or point. The order or degree of a vertex is the number of edges the vertex is in. The order of any point that is not a vertex is two. If $p$ is a point of $G$, the order of $p$ is denoted by $d_G(p)$, or, if $G$ is known, by $d(p)$. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$ respectively. If vertices $v$ and $w$ are joined by an edge, then $w$ is called a neighbour of $v$. An open interval is a connected subset of a graph which is homeomorphic to $(0, 1)$; a closed interval or a path is a connected subset of a graph which is homeomorphic to $[0, 1]$. An edge joining two vertices $v$ and $w$ may be denoted by $(v, w)$ or $[v, w]$ depending on whether or not we wish to include the endpoints. If $p$ is a vertex of $G$ with neighbouring vertices $a$, $b$, $c$, then the subgraph of $G$ consisting of the edges $[a, p]$, $[b, p]$ and $[c, p]$ is termed a triad, and is denoted by $T(p; a, b, c)$. If $P$ is a path and $a$ and $b$ are points of $P$, then $P(a, b)$ and $P[a, b]$ are open and closed intervals of $P$ between $a$ and $b$, respectively. $S$ is a given circle. The word map will be used for a continuous function. In the expression "continuous map" the word continuous is only inserted for emphasis. For standard graph theory terminology (which we have tried to stick to as much as possible) see [1].

2. Graphs which can be mapped 2-to-1 onto the circle

We describe here the connected graphs which can be mapped 2-to-1 onto the circle.

Let a type 0 tree consist of a single vertex, and let a type 1 tree consist of a single edge or a single vertex; let a vertex of a type 1 tree be designated as the root vertex. Let a type 2 tree consist of a main path $P$ onto which are rooted at distinct non-endpoints some (perhaps no) type 1 trees. We allow the possibility here and elsewhere that the main path $P$ might just consist of a single point. Again, let one of the endpoints of $P$ be designated the root vertex. Let $T_0$, $T_1$ and $T_2$ denote the sets of all type 0, type 1 and type 2, respectively, rooted trees.

Let a class 1 unicyclic graph consist of a circle onto which are rooted at distinct points some (perhaps no) type 1 trees. Let $C_1$ denote the set of all class 1 unicyclic graphs. Let a class 2 unicyclic graph consist of a circle onto which are rooted at distinct points some (perhaps no) type 2 trees. Let $C_2$ denote the set of all class 2 unicyclic graphs. Denote the set of graphs of $C_2$ with at most two rooted trees by $C_2(2)$.

We now state the main theorem about connected graphs which can be mapped 2-to-1 onto the circle.
Theorem 1. Let $G$ be a connected graph. Then $G$ can be mapped continuously 2-to-1 onto the circle if and only if $G \in C_1 \cup C_2(2)$.

We illustrate in Fig. 1 members of $T_2$, $C_1$ and $C_2(2)$.

3. Graphs which can be mapped 3-to-1 onto the circle

We describe here the connected graphs which can be mapped 3-to-1 onto the circle. We shall show that such graphs have at most one cycle.

Initially, let us describe the trees that can be mapped 3-to-1 onto the circle. First some further notation. Given two disjoint rooted trees $\tau_1$ and $\tau_2$, let $\tau_1 \oplus \tau_2$ denote the rooted tree obtained by identifying the root vertices of $\tau_1$ and $\tau_2$, but leaving the rest of $\tau_1$ and $\tau_2$ disjoint. Let the root vertex of $\tau_1 \oplus \tau_2$ be the identified root vertex. If $T_i$ and $T_j$ are two sets of rooted trees, let

$$T_i \oplus T_j = \{ \tau_i \oplus \tau_j : \tau_i \in T_i \text{ and } \tau_j \in T_j \}.$$ 

Let $T_3$ be the set of all rooted trees of $T_2$ with up to three endpoints, together with all trees which can be obtained from trees of $T_2$ with at least four endpoints by redesignating as the root vertex an endpoint which is not on the main path.
Let $T_4$ be the set of all rooted trees which consist of a main path, with one endpoint designated as the root vertex, which has rooted on it at distinct non-endpoints a finite number of trees in $T_1 \oplus T_2$.

Let $T_5$ be the set of all rooted trees which consist of a main path, with one endpoint designated as the root vertex, which has rooted on it at distinct non-endpoints a finite number of trees in $(T_2 \oplus T_2) \cup T_3$.

Let $T_6$ be the set of all rooted trees $\tau_6$ obtained from a tree of the form $\tau_4 \oplus \tau_5$ with $\tau_4 \in T_4$ and $\tau_5 \in T_5$, by taking a tree $\tau_1 \oplus \tau_2$ with $\tau_1 \in T_1$ and $\tau_2 \in T_2$, with $\tau_2$ not simply a single vertex unless $\tau_1$ is also a single vertex, rooted on the root vertex of $\tau_4 \oplus \tau_5$ and proceeding as follows. Let the main path of $\tau_2$ have endpoints $a$ and $b$, where $a$ is the vertex rooted on the root vertex of $\tau_4 \oplus \tau_5$. Then designate $b$ as the root vertex of $\tau_6$. Note that if $\tau_1 \oplus \tau_2$ is simply a single vertex then $\tau_6$ is in $T_4 \oplus T_5$. Note also that if $\tau_4$ is simply a single vertex then $\tau_6$ is in $T_5$. Thus $T_6 \subset T_4 \oplus T_5 \subset T_6$.

Note that, more generally, $T_0 \subset T_1 \subset T_2 \subset T_3 \subset T_4 \subset T_5 \subset T_4 \oplus T_5 \subset T_6$. These various trees are illustrated in Fig. 2.

The set of all trees which can be mapped 3-to-1 onto the circle was determined by Heath and Hilton in [8]. They proved the following result.

**Theorem 2.** Let $G$ be a tree. Then $G$ can be mapped continuously 3-to-1 onto the circle if and only if $G \in T_5 \setminus T_0$.

Now let us turn to the mapping of connected unicyclic graphs $G$ onto the circle. There are two kinds of unicyclic graphs which can be mapped 3-to-1 onto the circle. One kind can be mapped 3-to-1 onto the circle in such a way that the image of the circle in $G$ is the circle; the other kind cannot be mapped in this way.

We first consider the first kind of graph. Let a class 3 unicyclic graph consist of a circle onto which are rooted at distinct points some (perhaps no) trees of type $T_3$ or type $T_2 \oplus T_2$. Let $C_3$ denote the set of all class 3 graphs.

**Theorem 3.** Let $G$ be a connected unicyclic graph. Then $G$ can be mapped continuously 3-to-1 onto the circle, with the image of the circle in $G$ being the circle, if and only if $G \in C_3$.

We illustrate a member of $C_3$ in Fig. 3. The description of the remaining graphs which can be mapped 3-to-1 onto the circle is rather complicated.

Let a class 4 unicyclic graph consist of a circle onto which is rooted a tree $\tau_1 \in T_5$ at a point $b_1$, and, possibly, a tree $\tau_2 \in T_1 \oplus T_4$ at a disjoint point $b_2$, and, possibly, a tree $\tau_3 \in T_2$ at a further point $b_3$, and some (perhaps no) trees in $T_1$ rooted at distinct points which are not in $\{b_1, b_2, b_3\}$, with no tree in $T_1$ rooted on the segment $(b_1, b_3)$ of the circle which does not include $b_2$. Let $C_4$ be the set of all class 4 graphs.

Let a class 5 unicyclic graph consist of a circle onto which is rooted a tree $\tau_1 \in T_2 \oplus T_5$ at a point $b_1$, and possibly, a tree $\tau_2 \in T_1 \oplus T_4$ at a distinct point $b_2$, and
some (possibly no) trees in $T_1$ rooted at distinct points which are not in $\{b_1, b_2\}$. Let $C_5$ denote the set of all class 5 graphs.

Let a *class 6* unicyclic graph consist of a circle onto which is rooted a tree $\tau_1 \in T_5$ at a point $b_1$, and, possibly, a tree $\tau_2 \in T_2 \oplus T_4$ at a distinct point $b_2$. Let $C_6$ denote the set of all class 6 graphs.

![Diagrams of graphs $T_1 \oplus T_2$, $T_2 \oplus T_2$, $T_3$, $T_4$, and $T_5$.](image)
Let a class 7 unicyclic graph consist of a circle onto which is rooted a tree $\tau_1 \in T_6$ at a point $b_1$, and, possibly, at a distinct point $b_2$ a tree $\tau_2 \in T_2$. Let $C_7$ denote the set of all class 7 graphs.

The various classes, $C_4$, $C_5$, $C_6$, and $C_7$, of graphs are illustrated in Figs. 4(a)–(d).

We can now state the main result about the connected unicyclic graphs which can be mapped 3-to-1 onto the circle.
Theorem 4. Let \( G \) be a connected unicyclic graph. Then \( G \) can be mapped continuously 3-to-1 onto the circle if and only if \( G \in C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \).

Combining Theorems 2–4, and Lemma 7 together we have:

Theorem 5. Let \( G \) be a connected graph. Then \( G \) can be mapped continuously 3-to-1 onto the circle if and only if \( G \in (T_5 \setminus T_0) \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \).

Fig. 4(a). \( C_4 \).

Fig. 4(b). \( C_5 \).
4. Finitely discontinuous functions

In [6] Heath proved the following interesting and useful result.

**Theorem 6.** Let $G$ and $H$ be graphs and let $k \geq 2$. There is a finitely discontinuous $k$-to-1 function from $G$ onto $H$ if and only if

$$|E(G)| - |V(G)| \leq k(|E(H)| - |V(H)|) \quad if \quad k \geq 3,$$

$$|E(G)| - |V(G)| = k(|E(H)| - |V(H)|) \quad if \quad k = 2.$$
We remark that in [6] it was assumed that $G$ and $H$ are connected, but this assumption is not necessary, and no real change in the proof is entailed.

Of course a circle can be thought of as a circuit with 3 (or more) vertices, so the value of $|E(H)| - |V(H)|$ for a circle is zero. So from Theorem 6 we deduce the following lemma.

**Lemma 7.** Let $G$ be a connected graph and let $k \geq 2$. There is a finitely discontinuous $k$-to-1 function from $G$ onto the circle if and only if $G$ is unicyclic if $k = 2$, and $G$ is a tree or a unicyclic graph if $k \geq 3$.

**Proof.** From Theorem 6 it follows that

$$|E(G)| - |V(G)| \leq 0 \quad \text{if } k \geq 3,$$

$$|E(G)| - |V(G)| = 0 \quad \text{if } k = 2.$$

If $k = 2$, it follows that $G$ is unicyclic. For a connected graph, $|E(G)| - |V(G)| \geq -1$, so for $k \geq 3$ we have $-1 \leq |E(G)| - |V(G)| \leq 0$, from which it follows that $G$ is a tree or $G$ is unicyclic. □

5. Preliminary results

For any graph $G$, let $t_i(G)$ be the number of vertices of $G$ having degree $i$; if the graph $G$ is known, $t_i(G)$ may be shortened to $t_i$.

Given a $k$-to-1 map from a graph $G$ onto $S$, and given a point $y \in S$, let $t_i(y)$ denote the number of points of $f^{-1}(y)$ of order $i$ in $G$.

We first give a lemma that was given incorrectly in [8]. We give the correct proof in full.

**Lemma 8.** If $f$ is a $k$-to-1 map from $G$ onto $S$, then for each $y \in S$,

$$t_1(y) + 2t_2(y) + 3t_3(y) + \cdots \leq 2k.$$

**Proof.** Call a point $x$ such that $x$ is an interior point of an interval that is mapped by $f$ 1-to-1 onto an interval of $S$ a regular point. Arbitrarily close to $y$ on either side of $y$ in $S$ are two points $z$ and $z'$ such that the points mapping to $z$ and $z'$ are all regular. For each point in $f^{-1}(y)$ of order $i \geq 1$, there are at least $i$ regular points in $G$ that map to either $z$ or $z'$. Therefore the $t_i(y)$ points in $f^{-1}(y)$ of order $i$ there are at least $it_i(y)$ regular points in $G$ which map to $z$ or $z'$. The sum of the orders of the regular points mapping to $z$ or $z'$ is $2(2k)$. Therefore

$$2(t_1(y) + 2t_2(y) + 3t_3(y) + \cdots) \leq 2(2k).$$

Lemma 8 now follows. □
We remark that there need not be equality in Lemma 8. For example, in the well-known 3-to-1 map of \([0, 1]\) onto \(S\), there is one point \(y\) with \(t_1(y) = 2\), \(t_2(y) = 1\) and \(t_i(y) = 0\) for \(i \geq 3\), so that in this case \(t_1(y) + 2t_2(y) + 3t_3(y) + \cdots = 4 < 6 = 2k\).

Let \(\Delta(G)\) denote the maximum order of any point of \(G\) [if \(G\) is connected then, except in the case where \(G\) consists of a single edge, \(\Delta(G)\) is the maximum degree in \(G\) in the usual graph-theoretic sense].

**Lemma 9.** If \(G\) has no isolated points and if \(f\) is a \(k\)-to-1 map from \(G\) onto \(S\), then

\[
\Delta(G) \leq k + 1;
\]

moreover, no point of \(G\) of order \(k + 1\) maps to the same point of \(S\) as any point of order \(2\) or more.

**Proof.** Let \(d_G(x) = \Delta(G)\). Since \(f\) is \(k\)-to-1, \(k - 1\) further points of order at least one map to \(f(x)\). Consequently \(\Delta(G) + k - 1 \leq 2k\), so \(\Delta(G) \leq k + 1\); moreover equality is only possible if the order of all the other points in \(f^{-1}(f(x))\) is one. \(\square\)

**Lemma 10.** Let \(G\) have no isolated vertices. If \(f\) is a \(k\)-to-1 map from \(G\) onto \(S\), then

1. \(t_1(y) \geq t_3(y) + 2t_4(y) + \cdots + (k - 1)t_{k+1}(y) (\forall y \in S)\),
2. \(t_1 \geq t_3 + 2t_4 + \cdots + (k - 1)t_{k+1}\).

**Proof.** (i) By Lemmas 8 and 9, and since \(f\) is exactly \(k\)-to-1,

\[
2(t_1(y) + t_2(y) + t_3(y) + t_4(y) + \cdots + t_{k+1}(y)) = 2k \geq t_1(y) + 2t_2(y) + \cdots + (k + 1)t_{k+1}(y).
\]

(i) now follows.

(ii) Let \(S_i\) be the set of points \(y\) of \(S\) such that a point of \(G\) of degree other than 2 is mapped onto \(y\). Then, for \(i \neq 2\), \(t_i = \sum_{y \in S_i} t_i(y)\). Therefore (ii) follows from (i) by summation. \(\square\)

We now remark that the following weaker version of Lemma 7 now follows easily.

**Lemma 11.** Let \(G\) be a connected graph, and let \(f\) be a \(k\)-to-1 map from \(G\) onto \(S\). Then \(G\) has at most one cycle; moreover, if \(G\) has one cycle, then

\[
t_1 = t_3 + 2t_4 + \cdots + (k - 1)t_{k+1}.
\]
Proof. For \( v \in V(G) \), let \( d(v) \) be the degree of \( v \). Since \( \Delta(G) \leq k + 1 \), we have

\[
2|V(G)| - 2|E(G)| = 2 \sum_{i=1}^{k+1} t_i - 2 \sum_{v \in V(G)} d(v) = 2 \sum_{i=1}^{k+1} t_i - 2 \sum_{i=1}^{k+1} it_i = t_1 - (t_3 + 2t_4 + \cdots + (k-1)t_{k+1}) \\
\geq 0,
\]

by Lemma 10(ii). Since \( G \) is connected, \( G \) has at most one cycle.

If \( G \) has one cycle, then \( |V(G)| = |E(G)| \), and so

\[
t_1 - t_3 + 2t_4 + \cdots + (k-1)t_{k+1} = 0. \quad \square
\]

Lemma 12. Let \( G \) be a unicyclic connected graph and let \( f \) be a \( k \)-to-1 map from \( G \) onto \( S \). Let \( y \in S \). Then

\[
t_1(y) = t_3(y) + 2t_4(y) + \cdots + (k-1)t_{k+1}(y).
\]

Proof. Again, let \( S_y \) be the set of points \( y \) of \( S \) such that a point of \( G \) of degree not equal to 2 is mapped onto \( y \). Then, by Lemmas 11 and 10(i),

\[
0 = t_3 + 2t_4 + \cdots + (k-1)t_{k-1} - t_1 = \sum_{v \in S_y} (t_3(y) + 2t_4(y) + \cdots + (k-1)t_{k-1}(y) - t_1(y)) \leq 0.
\]

Therefore, for each \( y \in S_y \),

\[
t_3(y) + 2t_4(y) + \cdots + (k-1)t_{k-1}(y) = t_1(y).
\]

But this holds trivially for \( y \in S \setminus S_y \). Therefore it holds for all \( y \in S \). \( \square \)

Lemma 13. Let \( G \) be a unicyclic connected graph and let \( f \) be a \( k \)-to-1 map from \( G \) onto \( S \). Let \( y \in S \). Then the orders of the points which map to \( y \) add up to \( 2k \).

Proof. By inserting vertices into edges, if necessary, we may assume that all points in \( f^{-1}(y) \) are vertices. Then by Lemma 12,

\[
t_1(y) = t_3(y) + 2t_4(y) + \cdots + (k-1)t_{k+1}(y).
\]

Therefore

\[
2k = 2(t_1(y) + \cdots + t_{k+1}(y)) = t_1(y) + 2t_2(y) + \cdots + (k+1)t_{k-1}(y).
\]

Lemma 13 now follows. \( \square \)
Given two points \( y_1, y_2 \in S \), let \( S_1(y_1, y_2) \) denote one of the two open intervals of \( S \) between \( y_1 \) and \( y_2 \). Given a graph \( G \) with no isolated vertices and a \( k \)-to-1 map \( f \) from \( G \) onto \( S \), for \( 1 \leq i \leq k + 1 \), let \( t_i(y_1, y_2) \) be the number of vertices of degree \( i \) which are mapped into \( S_1(y_1, y_2) \).

**Lemma 14.** Let \( G \) be a unicyclic connected graph, and let \( f \) be a \( k \)-to-1 map from \( G \) onto \( S \). Let \( y_1 \) and \( y_2 \) be two points of \( S \) such that all the points of \( f^{-1}(y_1) \cup f^{-1}(y_2) \) have order 2. Then

\[
t_1(y_1, y_2) = t_3(y_1, y_2) + 2t_4(y_1, y_2) + \cdots + (k - 1)t_{k+1}(y_1, y_2).
\]

**Proof.** The formula is obtained by summing over all points \( y \in S_1(y_1, y_2) \), the result of Lemma 12. \( \square \)

The next lemma shows, roughly speaking, that the contribution from some tree \( \tau \) in \( G \) to the number

\[
t_1 - t_3 + 2t_4 + \cdots + (k - 1)t_{k+1}
\]

is zero.

If \( G \) is a graph which contains a tree \( \tau \) which is joined to the rest of \( G \) solely at a vertex \( p \), let \( G \setminus \tau \) be the graph obtained from \( G \) by removing all of \( \tau \) except for the vertex \( p \).

**Lemma 15.** Let \( G \) be a graph with \( \Delta(G) \leq k + 1 \) which contains a tree \( \tau \) joined to the rest of \( G \) solely at a vertex \( p \), where \( d_G(p) \geq 3 \). Then

\[
t_1(G) - (t_3(G) + 2t_4(G) + \cdots + (k - 1)t_{k+1}(G)) = t_1(G \setminus \tau) - (t_3(G \setminus \tau) + 2t_4(G \setminus \tau) + \cdots + (k - 1)t_{k+1}(G \setminus \tau)).
\]

**Proof.** In the tree \( \tau \),

\[
\sum_{v \in V(\tau)} d_\tau(v) = 2|E(\tau)| = 2|V(\tau)| - 2,
\]

so

\[
t_1(\tau) + 2t_2(\tau) + \cdots + (k + 1)t_{k+1}(\tau) = 2t_1(\tau) + 2t_2(\tau) + \cdots + 2t_{k+1}(\tau) - 2,
\]

so

\[
t_1(\tau) - (t_3(\tau) + 2t_4(\tau) + \cdots + (k - 1)t_{k+1}(\tau)) = 2.
\]

But

\[
t_i(G) = \begin{cases} 
    t_i(G \setminus \tau) + t_1(\tau) & \text{if } i\neq d_G(p), d_G(p) - 1, 1, \\
    t_i(G \setminus \tau) + t_1(\tau) + 1 & \text{if } i = d_G(p), \\
    t_i(G \setminus \tau) + t_1(\tau) - 1 & \text{if } i = d_G(p) - 1, \\
    t_i(G \setminus \tau) + t_1(\tau) - 1 & \text{if } i = 1.
\end{cases}
\]
Writing $d_G(p) = q$, we then have

$$t_1(G) - (t_3(G) + 2t_4(G) + \cdots + (k-1)t_{k+1}(G))$$

$$- \left\{ t_1(G \setminus \tau) - (t_3(G \setminus \tau) + 2t_4(G \setminus \tau) + \cdots + (k-1)t_{k-1}(G \setminus \tau)) \right\}$$

$$= t_1(G) - t_1(G \setminus \tau) - \sum_{j=3}^{k+1} (j-2)(t_j(G) - t_j(G \setminus \tau))$$

$$= t_1(\tau) - 1 - \left( \sum_{j=3}^{k+1} (j-2)t_j(\tau) \right) - (q-2) + (q-3)$$

$$= t_1(\tau) - \left( \sum_{j=3}^{k+1} (j-2)t_j(\tau) \right) - 2$$

$$= 0. \quad \Box$$

The next three lemmas were proved in [8]. Lemma 16 was stated in [8] for trees, but the proof there works for general graphs.

**Lemma 16.** If $f$ is a continuous map from a graph $G$ into $S$, and if $T(p; a, b, c)$ is a triad in $G$, then one of the points, $a$, $b$, $c$ maps the same as some other point of the triad.

**Lemma 17.** Let $k > 1$ and let $f$ be a continuous map from a tree $\tau$ onto the circle $S$. If no point of order two or more in $\tau$ maps to the same point of $S$ as any point of degree at least $k + 1$, then there is a path $P$ in $\tau$ such that

1. $P$ contains all points of $\tau$ of degree at least $k + 1$.
2. The endpoints of $P$ are endpoints of $\tau$.
3. $f(\tau) = f(P)$, and
4. if $y \in f(\tau)$ and some point of $\tau$ of order at least 2 maps to $y$, then some point of $P$ of order at least 2 maps to $y$.

**Lemma 18.** Suppose $f$ is a 3-to-1 map from a tree $\tau$ onto $S$. Then

1. if $\tau$ is not type 2, then there is a set of $m$ points of $\tau$ (m is either 1 or 2) that map the same and whose orders add up to at least $3 + m$, and
2. if $\tau$ is not type 5, then there is a set of $m$ points of $\tau$ (m is either 1, 2, or 3) that map the same and whose orders add up to at least $4 + m$.

6. Basic maps

Let $u$, $v$ be two points, and let $[u, v]$ be the closed arc from $u$ to $v$ going clockwise.

**Lemma 19.** If $R$ is a type 2 rooted tree with root $r$, then there is a map $g: R \to [u, v]$ such that $g$ is 2-to-1 on $[u, v)$, 1-to-1 on $v$ and $g(r) = u$. 
Proof. See Fig. 5(a). □

Lemma 20. Let $R$ be a type 5 rooted tree with root $r$. Then:
1. There is a map $f: R \rightarrow [u, v]$ such that $f(r) = u$, $f$ is 3-to-1 on $(u, v)$, and 2-to-1 on $(u, v)
2. There is a map $f: R \rightarrow [u, v]$ and a point $w \in (u, v)$ such that $f(r) = u$, $f$ is 3-to-1 on $(u, w)$, 2-to-1 on $(u, w)$, and 1-to-1 on $(w, v)$.
3. There is a map $f: R \rightarrow [u, v]$ such that $f(r) = u$, $f$ is 3-to-1 on $(u, v)$, 2-to-1 on $v$ and 1-to-1 on $u$.

Proof. See Figs. 5(b)–(d). □

Lemma 21. If $R$ is a type 4 rooted tree with root vertex $r$, there is a map $f: R \rightarrow [u, v]$ and a vertex $w \in (u, v)$ such that $f(r) = u$, $f$ is 3-to-1 on $(u, w)$, 2-to-1 on $(w, v)$ and 1-to-1 on $v$.

Proof. See Fig. 5(e). □

7. 2-to-1 maps: proofs

By Lemma 7 there can only be a 2-to-1 map $f$ of a graph $G$ onto the circle $S$ if $G$ contains exactly one cycle $C$. By Lemma 9, $d_G(p) \leq 3$ for all points $p$ of $G$. We consider first the case where $f(C) = S$.

Lemma 22. Let $G$ be a connected graph containing a unique cycle $C$. There is a 2-to-1 continuous map $f$ from $G$ onto the circle $S$ with $f(C) = S$ if and only if $G \subseteq C_1$.

Proof. (Necessity) If $v$ is a vertex of $G$ of degree 3 which is not on $C$, then $f^{-1}(f(v)) = \{v, v'\}$ for some point $v'$ of $C$. But then $d(v') \geq 2$ so that $d(v) + d(v') \geq 5 > 4$, contradicting Lemma 13. Therefore $G \subseteq C_1$.

(Sufficiency) Suppose $G \subseteq C_1$. If $G$ is a circle then we obtain a 2-to-1 map from $G$ onto $S$ by wrapping $G$ twice round $S$. If $G$ is not a circle the map is evident from Fig. 6. □

Now we consider the case when $f(C) \neq S$.

Lemma 23. Let $G$ be a connected graph containing a unique cycle $C$. There is a 2-to-1 continuous map $f$ from $G$ onto the circle $S$ with $f(C) \neq S$ if and only if $G \subseteq C_2(2)$, $G \neq C$.

Proof. (Necessity) Clearly $G \neq C$. If there are three or more points $p_1, p_2, p_3$ of degree three on $C$ we can find one vertex $p \in \{p_1, p_2, p_3\}$ having the property that $f(p) = f(v)$
for some \( v \) on \( C, v \neq p \). Then

\[
d(v) + d(p) \geq 5 > 4,
\]

contradicting Lemma 13. So we may assume that there are either one or two vertices of degree 3 on \( C \). Let \( \tau \) be a tree rooted at a point \( a \) on \( C \). If there is a triod \( T(x; a, b, c) \) within \( \tau \) such that \( d(a) = d(b) = d(c) \) then, by Lemma 16, there is a vertex \( p \in \{ a, b, c \} \) which maps to the same point of \( S \) as some point \( y \in T(x; a, b, c), y \neq p \). But then \( d(p) + d(y) \geq 5 > 4 \), contradicting Lemma 13. It follows that \( \tau \in T_2 \). Consequently \( G \in C_2(2) \).

(Sufficiency) The map is evident from Fig. 7. \( \square \)

Theorem 1 follows immediately from Lemmas 22 and 23.

8. 3-to-1 maps: proofs

By Lemma 7, there can only be a 3-to-1 continuous map \( f \) of a connected graph \( G \) onto a circle \( S \) if \( G \) contains at most one circle \( C \). The case when \( G \) is a tree is covered by Theorem 2, which was proved in [8]. So we only need to prove the case when \( G \) contains one circle. We first prove Theorem 3, which is the case when \( f(C) = S \).
Proof of Theorem 3. (Necessity) Let $\tau$ be a tree rooted on $C$ and suppose that $\tau \not\in T_3 \cup (T_2 \oplus T_2)$. By Lemma 18 there is a set of $m \in \{1, 2\}$ points of $\tau$ that map to the same point $p$ of $S$ and whose orders add up to at least $3 + m$. Since each point of $S$ is the image of 3 points of $G$, and since $f(C) = S$ we have that the sum of the orders of the points that map to $p$ is at least

$$\sum_{i=0}^{m} (3 + m) = 7.$$

Since $7 > 6$, this contradicts Lemma 13. Therefore $\tau \in T_3 \cup (T_2 \oplus T_2)$.

(Sufficiency) This is evident from Fig. 8. \qed

We now turn to the proof of Theorem 4. We shall suppose that $f: G \rightarrow S$ is a continuous 3-to-1 map with $f(C) \neq S$. Since $f(C) \neq S$, we may suppose that $f(C)$ is an arc $[r, s]$ of the circle $S$. There is exactly one point of $C$ which maps to $r$; we shall denote this point by $r^{-1}$. We define $s^{-1}$ similarly. By Lemma 9, $\Delta(G) \leq 4$.

We first prove the following lemma.

Lemma 24. Let $f: G \rightarrow S$ be a 3-to-1 map with $f(C) \neq S$. There are two paths $P_1$ and $P_2$ in $G$ which are disjoint, except possibly for one common vertex, $P_1$ containing at least one edge (we allow the possibility that $P_2$ consists of a single vertex) such that, denoting $P = P_1 \cup C \cup P_2$ by $P^*$,

(i) $P^*$ contains all vertices of $G$ of degree 4,
(ii) the endpoints of $P^*$ are endpoints of $G$,
(iii) $f(P^*) = S$,
(iv) if $y \in S$ and some point of $G$ of order at least 2 maps to $y$, then some point of $P^*$ of order at least two maps to $y$,
(v) no vertex of $C$ other than $r^{-1}$ and $s^{-1}$ can have degree 4,
(vi) $r$ is the image of a point different to $r^{-1}$ of $P^*$ of order at least 2, unless $r$ is the root vertex of a path $P' \in \{P_1, P_2\}$ consisting of more than just one vertex, similarly for $s$. 

Fig. 8.
Remark. Let $P^+ = P_1 \cup P_2$. Note that $P^+$ may be disjoint from $C$. In that case, since $G$ is connected there is one point $q$ on $P^+$ at which a unicyclic subgraph containing $C$ is rooted. If $a$ and $b$ are the endpoints of $P^+$ we shall let $P_1$ and $P_2$ denote $P^+ [a, q]$ and $P^+ [q, b]$, respectively. We shall denote the path joining $q$ to $C$ by $P_3$. An example of a graph $G$ and a 3-to-1 map from $G$ to $S$ with $P^+$ disjoint from $C$ is given in Fig. 9. If $P^+$ is not disjoint from $C$ then we shall assume that $P_1$ and $P_2$ are rooted on $C$; $P_2$ may be a single point.

Proof. First let us deal with (v). Suppose there is a point $p$ of $C \setminus \{r^{-1}, s^{-1}\}$ of order 4. There is a point $q_1 \in C \setminus \{r^{-1}, s^{-1}\}$ with $f(p) = f(q_1)$, and a further point $q_2 \in G \setminus \{p, q_1\}$ with $f(p) = f(q_2)$. But then $d(p) + d(q_1) + d(q_2) \geq 4 + 2 + 1 = 7 > 6$ contradicting Lemma 13. Therefore (v) is true.

Next we show that $P^*$ exists satisfying (i)--(iv). We may clearly choose an open segment $e$ of $C$ not containing the roots of any tree and such that $f(G \setminus e) = S$. But $G \setminus e$ is a tree with the property that no point of order 2 or more in $G \setminus e$ maps to the same point of $S$ as any point of order at least 4 (for otherwise we can find three points of $G$ mapping to the same point in $S$ whose orders total at least 7, contradicting Lemma 13). By Lemma 17 we can find a path $P$ satisfying (1)–(4) of Lemma 17. We can choose paths $P_1$ and $P_2$ of $P \cup C$ such that $P \cup C = P_1 \cup C \cup P_2$ (allowing the possibility that $P_2$ is a single vertex), where $P_1$ and $P_2$ have at most their root vertices in common with $C$. [If $P$ is disjoint from $C$, then $P_1$ and $P_2$ are chosen as described in the Remark above.] Denoting $P_1 \cup C \cup P_2$ by $P^*$, it is easy to see that $P^*$ satisfies (i)–(iv) above.

To prove (vi) we first remark that if $P$ and $C$ are disjoint or have one point in common then $P = P^+ = P_1 \cup P_2$ and (vi) is clearly true. Therefore we may assume that $P$ and $C$ have more than just one point in common; so $P_1$ and $P_2$ are both rooted on $C$ at distinct points.

Since $f(C) \neq S$, $f(P_i) \setminus f(C) \neq \emptyset$ for at least one $i \in \{1, 2\}$. Without loss of generality, we may assume that $f(P_1) \setminus f(C) \neq \emptyset$ and that $P_1$ does not just consist of a single
vertex. We may further assume that \( r \) is either the image of a point different to \( r^{-1} \) of \( P_1 \) of order at least 2, or that \( r^{-1} \) is a root vertex of \( P_1 \); in other words, we may assume that (vi) is true for \( r \).

If \( s^{-1} \) is the root vertex of \( P_2 \), where \( P_2 \) consists of more than just one vertex, then (vi) is true. So suppose that \( s^{-1} \) is not the root vertex of \( P_2 \).

Suppose now that \( s^{-1} \) has order 2 or 3. Then since there are three points of \( G \) of combined order 6 (by Lemma 13) which map to \( s \), some point \( \sigma \) of \( G \setminus s^{-1} \) of order at least 2 maps to \( s \). Clearly \( \sigma \notin C \). If the point \( \sigma \) is in \( P_1 \) or \( P_2 \), then (vi) is satisfied by \( s \). So suppose that \( \sigma \) is not in \( P_1 \) or \( P_2 \). Let \( P_4 \) denote a maximal path which includes \( \sigma \) and which has exactly one vertex in common with \( C \).

Since \( s^{-1} \) is not the root vertex of \( P_2 \), and no point of order 2 of \( P_1 \) or \( P_2 \) maps to \( s \), and since \( f(P_1) \setminus C \neq \emptyset \) and \( r \) is the image of a point of \( P_1 \) of order 2 or is the image of the root vertex of \( P_1 \), it follows that either \( f(P_1) \) includes all of \( S \setminus f(C) \), or \( f(P_2) \) includes all of \( S \setminus f(C) \). Without loss of generality, suppose that \( f(P_1) \) includes all of \( f(C) \). Then \( f(P_2) \subset f(P_1) \cup f(C) \). Moreover, except possibly for the root vertex, \( P_2 \) contains no vertex of degree 4 (for otherwise we obtain the same contradiction as in the proof of (vi)). It follows that \( P^{**} = P_1 \cup C \cup P_4 \) satisfies (i)–(iv). But then \( P^{**} \) satisfies (vi) also.

Now suppose that \( s^{-1} \) has order 4. Then the other two points which map to \( s \) both have order 1. It follows that either \( f(P_1) \) includes all of \( S \setminus f(C) \) or \( f(P_2) \) includes all of \( S \setminus f(C) \), or both. We may suppose that \( f(P_1) \) includes all of \( f(C) \). Then \( f(P_2) \subseteq f(P_1) \cup f(C) \). Since \( s^{-1} \) has rooted on it a tree of the form \( \tau_1 \oplus \tau_2 \), one of \( \tau_1 \) and \( \tau_2 \) (say \( \tau_1 \)) must map onto part of \( S \setminus f(C) \). Let \( P_4 \) be a maximal path in \( \tau_1 \). Now let \( P^{**} = P_1 \cup C \cup P_4 \). Then again \( P^{**} \) satisfies (i)–(iv), and also (vi).

Next we go on to consider what the trees rooted on \( C \) containing \( P_1 \) and \( P_2 \) are like. First we need to consider the tree rooted on \( P_1 \cup P_2 \).

**Lemma 25.** Let \( G \) be a unicyclic connected graph, let \( f : G \to S \) be a 3-to-1 map, and let \( f(C) \neq S \). Let \( \tau \) be a tree within \( G \) rooted on \( P_1 \cup P_2 \) and having no other point in common with \( P_1 \cup P_2 \). Then \( \tau \in T_3 \cup (T_2 \oplus T_2) \).

**Proof.** If \( \tau \notin T_3 \cup (T_2 \oplus T_2) \) there is a vertex \( x \in \tau \) and three vertices \( a, b, c \), all of degree at least three, such that
- either the triod \( T(x; a, b, c) \) is disjoint from \( P_1 \cup P_2 \),
- or the triod \( T(x; a, b, c) \) has exactly one point in common with \( P_1 \cup P_2 \), and that point is a vertex of degree 4 (we may assume that if there is such a point then that point is \( a \)).

In the first case it follows from Lemma 16 that there are two points, \( y_1 \) and \( y_2 \), of the triod with \( d(y_1) + d(y_2) \geq 5 \) which map to the same point of \( S \). By Lemma 24(iv), there is a point \( z \in P^* \) with \( d(z) \geq 2 \) and \( f(z) = f(y_1) = f(y_2) \). Then \( d(z) + d(y_1) + d(y_2) \geq 7 > 6 \), contradicting Lemma 13. In the second case, we may find \( y_1 \) and \( y_2 \) such that \( d(y_1) + d(y_2) \geq 5 \) and \( y_1, y_2 \notin P^* \), in which case the same
argument applies, or it may be that we can find \( y_1 \) and \( y_2 \) in the triod with \( y_1 = a \), so that \( d(y_1) = 4 \) and \( d(y_2) \geq 2 \). But then there is a further point \( z \) with \( f(z) = f(y_1) = f(y_2) \). Then \( d(z) \geq 1 \) so that \( d(y_1) + d(y_2) + d(z) \geq 7 \), and we again obtain a contradiction.

Lemma 26. If \( P^+ = P_1 \cup P_2 \) is disjoint from \( C \), or has exactly one point in common with \( C \) (so that \( P^+ \) is a path), the unicyclic graph rooted on \( P^+ \) at the point \( q \) has the following structure. Either

- (Si) it is of the form \( \tau_2 \oplus H \), where \( \tau_2 \in T_2 \) and \( H \) consists of \( C \) with a tree in \( T_2 \) rooted on it, together with a path \( P_3 \) joining \( q \) to a distinct point of \( C \), and on \( P_3 \) may be rooted at distinct non-end points a finite number of trees in \( T_1 \); or
- (Sii) \( q \) is on \( C \), and \( C \) has a tree in \( T_2 \) rooted on it at a point disjoint from \( q \); or
- (Siii) it consists of \( C \) with a tree in \( T_2 \) rooted on it, together with a path \( P_3 \) joining \( q \) to a distinct point of \( C \), and on \( P_3 \) is rooted a tree in \( T_2 \) at a point \( q_1 \), and at distinct non-endpoints of the path joining \( q_1 \) to \( C \) are rooted a finite number of trees in \( T_1 \).

Proof. As in the proof of Lemma 24, the path \( P = P^+ = P_1 \cup P_2 \) has all the vertices of order 4 on it. Lemma 26 therefore follows easily using the triod arguments of Lemma 25.

Lemma 27. Let \( G \) be a unicyclic graph, let \( f: G \to S \) be a 3-to-1 map and let \( f(C) \neq S \). Then one of (I)–(III) applies:

- (I) \( P_1 \) and \( P_2 \) are rooted at distinct points of \( C \). In that case it follows that \( P_1 \) is contained in a tree belonging to \( T_2 \oplus T_5 \) which is rooted on \( C \); similarly with \( P_2 \). Moreover any further tree rooted on \( C \) must be in \( T_3 \cup(T_2 \oplus T_2) \). Here \( P_2 \) could be a single vertex, but not \( P_1 \).
- (II) \( P_1 \) and \( P_2 \) are rooted on the same point of \( C \). In that case it follows that \( P_1 \) and \( P_2 \) are both contained in a tree in \( T_5 \oplus T_5 \) which is rooted on \( C \). Moreover there is at most one other tree rooted on \( C \) that tree is rooted at a distinct point of \( C \) and is in \( T_2 \). Neither \( P_1 \) nor \( P_2 \) are single vertices.
- (III) \( P^+ = P_1 \cup P_2 \) is a path disjoint from \( C \). In that case there is a point \( q \) on \( P^+ \), and \( P^+ \) has the form \( P_1 \oplus P_2 \), where \( q \) is the root vertex of \( P_1 \) and \( P_2 \), and \( P_1 \oplus P_2 \) is contained in a tree in \( T_5 \oplus T_5 \). Finally, also rooted on \( q \) is a unicyclic graph of the types described in Lemma 26, (Si) and (Siii). Neither \( P_1 \) nor \( P_2 \) are single vertices.

Proof. If \( P_1 \) and \( P_2 \) are rooted at distinct points of \( C \), then the triod argument of Lemma 25 shows that any further tree rooted on \( C \) is in \( T_3 \cup(T_2 \oplus T_2) \), and Lemma 25 and a further application of the triod argument show that any further tree rooted on the root vertex of a tree containing \( P_1 \) in \( T_5 \) is in \( T_2 \). Case (I) then follows.

If \( P_1 \) and \( P_2 \) are rooted at the same point of \( C \), then Case (II) follows from Lemmas 25 and 26(Sii).

If \( P^+ = P_1 \cup P_2 \) is a path disjoint from \( C \), then Case (III) follows from Lemmas 25 and 26(Si) and (Sii).
From now on, for \( i = 1, 2 \), if \( P_i \) is rooted on \( C \) we shall denote the root vertex by \( h_i \). If \( P_i \) is rooted on \( C \), we shall let \( \tau_i \) be the maximal tree in \( G \) rooted on \( h_i \) and containing no point of \( C \setminus h_i \). If \( P_1 \) is not rooted on \( C \), we shall let \( \tau_1 \) be the maximal (with respect to inclusion) tree in \( G \) rooted on \( q \), containing no point of \( (P_2 \cup P_3) \setminus \{ q \} \); similarly for \( \tau_2 \).

**Lemma 28.** Let \( G \) be a unicyclic graph, let \( f : G \to S \) be a 3-to-1 map, and let \( f(C) \neq S \). If one of \( \tau_1 \) and \( \tau_2 \) is in \( T_5 \setminus T_4 \), then the other is in \( T_4 \).

**Proof.** We consider the three cases of Lemma 27 separately.

**Case (I):** In this case \( \tau_1 \) and \( \tau_2 \) are rooted at distinct points \( h_1 \) and \( h_2 \) of \( C \).

By Lemma 27, it is enough to show that if \( \tau_1 \in T_5 \setminus T_4 \) then \( \tau_2 \in T_4 \). Suppose this is not the case. Then there exist vertices \( p_1 \) and \( p_2 \) such that, for \( i = 1, 2 \), \( p_i \) is the first vertex reached going along \( P_i \) starting at \( h_i \) on which is rooted a tree \( \tau_i \) in \( (T_1 \setminus T_2) \cup (T_2 \setminus T_1) \).

We note immediately that it is impossible that \( f(p_1) \in f(P_2) \) or \( f(p_2) \in f(P_1) \). For if, for example, we suppose that \( p_1 \) is in an open interval \( I \) of \( P_1 \) which is mapped 1-to-1 into \( S \), and if \( f^{-1}(f(I)) \) contains an open interval of \( P_2 \) containing the endpoint, say \( a_2 \), of \( P_2 \), with \( f(a_2) = f(p_1) \), then we easily obtain a contradiction to the fact that \( f \) is continuous and 3-to-1; this contradiction is obtained whether \( p_1 \) is the root of a tree in \( T_3 \setminus T_2 \) or in \( (T_2 \setminus T_1) \setminus (T_1 \setminus T_2) \). The same is true if \( p_1 \) is in an open interval \( I \) of \( P_1 \) which is mapped into a half open, half closed interval \( J \) of \( S \) with the closed end at \( f(p_1) \), and if \( f(P_2) \cap J \subseteq f(p_1) \) (the case \( f(P_2) \cap J = f(p_2) \) cannot occur in view of Lemma 24(iv)). Finally note that by Lemma 24(ii) above, \( p_1 \) is not an endpoint of \( P_1 \), so that all possibilities with \( f(p_1) \in f(P_2) \) have been covered.

Suppose that, for some \( i \in \{ 1, 2 \} \), \( \tau_i^* \) is in \( T_3 \setminus T_2 \). Consider the triod \( T(x_0; p_i, x_1, x_2) \) within \( \tau_i^* \), where \( p_i \) has \( x_0 \) as a neighbour, and \( x_0 \) has the further neighbours \( x_1 \) and \( x_2 \) \((p_i, x_1 \) and \( x_2 \) all having degree three). By Lemmas 16 and 24(iii),

\[
f([x_0, x_1]) \cap f([x_0, x_2]) = f(x_0).
\]

Since \( f(p_1) \notin f(P_2) \) and \( f(p_2) \notin f(P_1) \), we may choose points \( p_1^*, p_2^* \in P_i \) close to \( p_i \), with \( p_1^* \) closer to \( h_i \) than \( p_i \), and with the property that the points of \( f^{-1}(f(p_1^*)) \) and \( f^{-1}(f(p_2^*)) \) are situated so that one of each set occurs in \((p_i, x_0)\), one of each set occurs in \( P_i \) and either the third point of each set occurs in \((x_0, x_i)\) or the third point of each set occurs in \((x_0, x_2)\). Thus the points of \( f^{-1}(f(p_1^*)) \) and \( f^{-1}(f(p_2^*)) \) have order 2.

The points of \( \tau_i^* \setminus f^{-1}(f(p_i^*)) \) are split naturally into two sets. One set \( \tau_i^* \) consists of those points which are joined by a path, not including any point of \( f^{-1}(f(p_i^*)) \), to a point of \( f^{-1}(f(p_i^*)) \). Clearly \( \tau_i^* \) has no vertices of degree 4, and it is easy to see now that the following observation is true:

1. The number of vertices of degree 1 in \( \tau_i^* \) equals the number of points of degree 3 in \( \tau_i^* \) plus twice the number of vertices of degree 4.

\[
\frac{1}{2} \left( f([x_0, x_1]) \cap f([x_0, x_2]) \right) = f(x_0).
\]
Now suppose that, for some $i \in \{1, 2\}$, $\tau_i$ is in $(T_2 \oplus T_2) \setminus (T_1 \oplus T_2)$. Let $\tau_i = \tau_{i1} \oplus \tau_{i2}$ where $\tau_{i1}, \tau_{i2} \subset T_i$. By Lemmas 16 and 24(iii), $f(\tau_{i1} \setminus \tau_{i2}) = f(p_i)$. There are two points, say $x_{i1}$ and $x_{i2}$, such that $f(x_{i1}) = f(x_{i2}) = f(p_i)$, and since $p_i$ has degree 4, by Lemma 13, $x_{i1}$ and $x_{i2}$ are both endpoints. Clearly one is in $\tau_{i1}$ and the other in $\tau_{i2}$, so we may suppose that $x_{ij}$ is an endpoint of $\tau_{ij}$ for $j = 1, 2$. Let $z_{ij}$ be the vertex of $\tau_{ij}$ adjacent to $x_{ij}$. We may choose points $p_i^*$ and $p_i^*$ with $p_i^*$ close to $p_i$, and $p_i^*$ lying between $p_i$ and $p_i^*$, and with $p_i^* \in P_i(p_i, b_i)$; we can assume that $p_i^* \in f(\tau_{i1})$ and that the points of $f^{-1}(f(p_i^*))$ and $f^{-1}(f(p_i^*))$ are situated so that one of each set is in $(p_i, z_{i1})$, one of each set is in $P_i$, and the third point of each set is in $(x_{i1}, z_{i1})$, and so each point of $f^{-1}(f(p_i^*))$ and $f^{-1}(f(p_i^*))$ has order 2.

As in the case when $Z \in T_3 \setminus T_2$, the points of $\tau_i \setminus f^{-1}(f(p_i^*))$ split naturally into two sets $\tau_{i1}^*$ and $\tau_{i2}^*$. It is easy to see that (1) is true in this case also.

Let $G(p_i^*, p_j^*)$ consist of $\tau_{i1}^*, \tau_{i2}^*, C$ and those points of $G$ which are joined to $C$ by a path which does not include any point of $f^{-1}(f(p_i^*)) \cup f^{-1}(f(p_j^*))$. Clearly $f$ maps $G(p_i^*, p_j^*)$ 3-to-1 onto one of the open arcs of $S$ joining $f(p_i^*)$ to $f(p_j^*)$. Denote this arc by $S_1(p_i^*, p_j^*)$. By Lemma 14, $t_1(p_i^*, p_j^*) = t_3(p_i^*, p_j^*) + 2t_4(p_i^*, p_j^*)$. By (1) and Lemma 15, the contribution to $t_1(p_i^*, p_j^*) - t_3(p_i^*, p_j^*) - 2t_4(p_i^*, p_j^*)$ from $\tau_{i1} \cup \tau_{i2}$ or from any tree rooted on $P_i^*$ between $P_1$ and $P_2$ is zero. However there are two further vertices of degree 3 in $G(p_i^*, p_j^*)$, namely $b_1$ and $b_2$. Thus we have a contradiction. Therefore Lemma 28 is true in Case (I).

Case (II): In this case $\tau_1$ and $\tau_2$ are rooted at the same vertex $b_1 = b_2$ of $C$. We may let $\tau_i \in T_3 \setminus T_4$ and let $\tau_2 \in T_3$. We shall show that $\tau_2 \in T_4$. Assume instead that $\tau_2 \in T_3 \setminus T_4$. Then it follows that $f(\tau_1 \oplus \tau_2) = f(C \cup (\tau_1 \oplus \tau_2))$, so that all of $C$ is mapped onto the same arc of $S$ as part of an edge of one of $\tau_1$ and $\tau_2$. Then the argument proceeds as in Case (I), when $P_1$ and $P_2$ are rooted at distinct vertices of $C$; the sole difference is in the final part of the argument and is that $G(p_i^*, p_j^*)$ has one further vertex of degree 4, namely $b_1$, instead of having two further vertices of degree 3.

Case (III): In this case $\tau_1$ and $\tau_2$ are rooted at $q$, and are disjoint from $C$. Suppose Lemma 28 is untrue in Case (III). Then, by Lemma 27, $\tau_1 \in T_3 \setminus T_4 (i = 1, 2)$. For $i = 1, 2$, let $p_i$ be the first vertex going along $P_i$ starting at $q$ on which is rooted a tree $\tau_i^*$ in $(T_3 \setminus T_2) \cup ((T_2 \oplus T_2) \setminus (T_1 \oplus T_2))$. The argument proceeds as in Case (I), the only difference being that the contribution to $t_1(p_i^*, p_j^*) - t_3(p_i^*, p_j^*) - 2t_4(p_i^*, p_j^*)$ from the unicyclic graph rooted on $q$ is $\leq 2$.

In the next lemma we characterize the graphs $G$ satisfying Case (II) or Case (III) of Lemma 27.

**Lemma 29.** Let $G$ be a unicyclic graph, let $f: G \to S$ be a 3-to-1 map, and let $f(C) \neq S$. If $P_1$ and $P_2$ are rooted on the same vertex of $C$, or if $P_1 = P_1 \cup P_2$ is a path disjoint from $C$, then $G \in C_7$.

**Remark.** If $P_1$ and $P_2$ are rooted on the same vertex of $C$, then $G$ is in the subclass of $C_7$ in which the path $P_2$ reduces to a single vertex (in that case, the optional edge is not present).
Proof. If $P_1$ and $P_2$ are rooted on the same vertex in $C$, then it follows from Lemmas 27(II) and 28 that $G \subset C_I$.

Now suppose that $P^+ = P_1 \cup P_2$ is a path disjoint from $C$. Then Lemma 27(III) applies.

In the case when (Si) applies and on $q$ is rooted $\tau_2 \in T_2$ and also the unicyclic graph $H$, we need to show that in fact $\tau_2$ is either a single vertex or a single edge. We shall suppose that $\tau_2 \subset T_2 \setminus T_1$, and then deduce a contradiction. Since $G$ has no vertices of degree 5, $P_3$ contains at least one edge. Consider the triod $T(x_0, q, x_1, x_2)$ within $\tau_2$, where $q$ has $x_0$ as a neighbour, and $x_0$ has the further neighbours $x_1$ and $x_2$. As in Lemma 28, $f([x_0, x_1]) \cap f([x_0, x_2]) = f(x_0)$; also one of $x_1$ and $x_2$, we may suppose it is $x_2$, satisfies $f(x_1) = f(q)$. We may choose points $q^*$ and $q^{**}$ on $P$ close to $q$ with $q^{**}$ lying between $q^*$ and $q$, with the property that the points of $f^{-1}(f(q^*))$ and $f^{-1}(f(q^{**}))$ all have order 2 and are situated so that one of each set occurs on $P$, one of each set occurs in $(q, x_0)$, and the third point of each set occurs in $(x_0, x_2)$. Similarly there are points $p^*$ and $p^{**}$ on $P$ with $p^{**}$ between $q$ and $p^*$, with the property that the points of $f^{-1}(f(p^*))$ and $f^{-1}(f(p^{**}))$ all have order 2 and are situated so that one of each point of each set is in an edge of $P$, and the other two points of each set are in the same two edges of $C$.

Let $G(q^*, p^*)$ consist of those points of $G$ which are joined to $q^{**}$ or $p^{**}$ by a path not including any point of $f^{-1}(f(q^*))$ or $f^{-1}(f(p^*))$. It is easy to see that $t_1(p^*, q^*) + 2 = t_2(p^*, q^*) + 2t_4(p^*, q^*)$, contradicting Lemma 14. Therefore, if (Si) applies, $\tau_2$ consists of a single edge or a single vertex.

If (Siii) applies, the same kind of argument leads to a contradiction unless the tree in $T_2$ rooted in $P_3$ is in fact in $T_1$. There are two slightly different cases to consider, depending on whether $f^{-1}(f(q))$ includes a point other than $q$ of the unicyclic graph rooted on $q$. We omit the details as they are by now routine.

From this it now follows that in Case (III), $G$ is in $C_7$. □

From now on we need only consider Case (I) of Lemma 27, so that from now $P_1$ and $P_2$ will be paths rooted at distinct points $h_1$ and $h_2$ of $C$.

We show in Lemmas 30–32 and 35 that when $P_1$ and $P_2$ are rooted at distinct points of $C$ then Lemma 28 can be extended somewhat.

**Lemma 30.** Let $G$ be a unicyclic graph, let $f : G \rightarrow S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$. If $\tau_1 = \tau_1^1 \oplus \tau_1^2$ with $\tau_1^1 \in T_5 \setminus T_4$ containing $P_1$ and $\tau_1^2 \in T_2$, $\tau_2 = \tau_2^1 \oplus \tau_2^2$ with $\tau_2^1 \in T_5$ containing $P_2$ and $\tau_2^2 \in T_2$, then in fact $\tau_2^2 \in T_4$.

**Proof.** This is the same as the proof of Lemma 28. □

**Lemma 31.** Let $G$ be a unicyclic graph, let $f : G \rightarrow S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$. If $\tau_1 = \tau_1^1 \oplus \tau_1^2$ with $\tau_1^1 \in T_5 \setminus T_4$ containing $P_1$ and $\tau_1^2 \in T_2 \setminus T_1$, and if $\tau_2 = \tau_2^1 \oplus \tau_2^2$ with $\tau_2^2$ containing $P_2$, then $\tau_2^2 \in T_4$ and $\tau_2^2 \in T_1$. □
Proof. By Lemma 30 it follows that $\tau' \in T_4$. Suppose that $\tau'_2 \not\in T_1$. By Lemma 24(v), we may assume that $b_1 = r^{-1}$ and $b_2 = s^{-1}$. By Lemma 13, the two points other than $r^{-1}$ which map to $r$ have degree 1, and similarly the two points other than $s^{-1}$ which map to $s$ have degree 1. Consequently no point of $\tau'_1 \oplus \tau''_1$ of order 2 maps to $s$, and no point of $\tau'_2 \oplus \tau''_2$ of order 2 maps to $r$. It follows that no point of $\tau'_1 \oplus \tau''_1$ other than $b_1$ maps to $f(C)$, and no point of $\tau'_2 \oplus \tau''_2$ maps to $f(C)$. Let $p^*$ be a point on $P_1$ close to $b_1$, and let $p^{**}$ be between $p^*$ and $b_1$, such that $f^{-1}(f(p^*))$ and $f^{-1}(f(p^{**}))$ each contain three points of order 2, one on the edge of $\tau'_1$ incident with $b_1$, and one on the edge of $\tau''_1$ incident with $b_1$, and one on an edge of $\tau_1$ whose endpoint maps to $r$. Define $q^*$ and $q^{**}$ similarly on $P_2$ close to $b_2$. Let $G(p^*, q^*)$ be the set of points of $G$ joined to at least one point of $f^{-1}(f(p^*)) \cup f^{-1}(f(q^*))$ by a path not containing any point of $G(p^*, q^*)$. It is easy to see that $t_1(p^*, q^*) + 2 = t_3(p^*, q^*) + 2t_4(p^*, q^*)$, contradicting Lemma 14. Therefore Lemma 31 is true. \[\Box\]

Lemma 32. Let $G$ be a unicyclic graph, let $f: G \to S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$. Then either $\tau_2 \in T_5$ or $\tau_2 = \tau'_2 \oplus \tau''_2$ with $\tau'_2 \in T_4 \setminus T_0$ containing $P_2$ and $\tau''_2 \in T_3 \setminus T_0$.

Remark. It is more convenient to prove Lemma 32 now, and to prove the stronger, but very similar statement, Lemma 35, later.

Proof. In view of Lemmas 27 and 28, we need to show that $\tau_2$ cannot be of the form $\tau_2 = \tau'_2 \oplus \tau''_2$ with $\tau'_2 \in T_3 \setminus T_0$ containing $P_2$, $\tau''_2$ being any tree with at least one edge. To see this, suppose the contrary. We may suppose that $b_1 = r^{-1}$ and $b_2 = s^{-1}$. Since $s^{-1}$ has degree 4, the other two vertices mapping to $s$ have degree 1. If $f(C) \cap f(\tau'_2) = s$, then the argument used to prove Lemma 31 works (with just very minor alterations). So suppose that $f(C) \cap f(\tau'_2) \neq s$. Then $f(\tau'_2) \subset f(C)$ and $\tau'_2$ is a single edge. Let $p$ be the first point reached going along $P_2$ starting at $b_2$ at which is rooted a tree $\tau^* \in T_3 \cup (T_2 \oplus T_2)$. Let $p^*$ and $p^{**}$ be two points near $p$, with $p^{**}$ nearer to $b_2$ than $p^*$ is, such that $f^{-1}(f(p^*))$ and $f^{-1}(f(p^{**}))$ each contain three points of order 2, one on $P_1$, one on an edge $(x_0, p)$ of $\tau^*$, and one on an edge $(x_1, x_0)$ of $\tau^*$. Let $q^*$ and $q^{**}$ be two points on $\tau''_2$ near to $b_2$, with $q^*$ between $b_2$ and $q^{**}$, such that $f^{-1}(f(q^*))$ and $f^{-1}(f(q^{**}))$ each contain three points of order 2, one on $\tau'_2$, the other two in $C$. Let $G(p^*, q^*)$ be the set of all points of $G$ which are joined by a path containing no point of $f^{-1}(f(\{p^*, q^*\}))$ to at least one point of $f^{-1}(f(\{p^{**}, q^{**}\}))$. Then it is easy to see that $G(p^*, q^*)$ satisfies the hypothesis, but not the conclusion of Lemma 14. This contradiction proves Lemma 32. \[\Box\]

Lemma 33. Let $G$ be a unicyclic graph, let $f: G \to S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$. Then every tree $\tau$ rooted on $C$ with only the root vertex in common with $P^*$ is in $T_2$. 

Proof. By Lemma 24(v) no point of $C$ other than $r^{-1}$ and $s^{-1}$ can have degree 4. So if there is a tree $\tau$ rooted on $C$ with only the root vertex in common with $P^*$ which is not in $T_2$, then
either (a) $p$ has degree 3 and there is a triod $T(x_0; p, x_1, x_2)$ in $\tau$ with $x_0$ a neighbour of $p$, and $x_1$ and $x_2$ neighbours of $x_0$, with $x_1, x_2$ also having degree 3;
or (b) $p$ has degree 4 and $p \in \{r^{-1}, s^{-1}\}$.
By Lemma 16, there are two points of the triod, say $y_1$ and $y_2$, including one of $p, x_1, x_2$, which map to the same point of $S$. In case (a), $d(y_1) + d(y_2) \geq 5$. In that case, if $p \notin \{y_1, y_2\}$ then it follows from Lemma 24(iv) that there is some point $z$ of $P^*$ of order at least 2 with $f(z) = f(y_1) = f(y_2)$. But then $d(z) + d(y_1) + d(y_2) \geq 7$, contradicting Lemma 13. If $p \in \{y_1, y_2\} \setminus \{r^{-1}, s^{-1}\}$ then it is clear that there is a point $z \in C \setminus \{p\}$ with $d(z) \geq 3$ and $f(z) = f(y_1) = f(y_2)$, and then the same argument works. If $p \in \{y_1, y_2\} \cap \{r^{-1}, s^{-1}\}$, say $p = r^{-1}$, then $p \notin \{b_1, b_2\}$ since $d(p) = 3$ and $\tau \cap P^* = \{p\}$, so by Lemma 24(vi) there is a point $z$ of $P^*$ of order 2 which maps to $r$, and so the same argument works again. In case (b), if $p \notin \{y_1, y_2\}$ the argument is the same as in case (a). If $p \in \{y_1, y_2\}$, then $d(y_1) + d(y_2) \geq 6$. Since the map $f$ is 3-to-1 there is a third point $z$ with $d(z) \geq 1$ and $f(z) = f(y_1) = f(y_2)$, so $d(z) + d(y_1) + d(y_2) \geq 7$, contradicting Lemma 13. Therefore any tree rooted on $C$ with only the root vertex in common with $P^*$ is in $T_2$.

Lemma 34. Let $G$ be a unicyclic graph, let $f : G \rightarrow S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$. Then there is at most one tree $\tau \in T_2 \setminus T_1$ which is rooted on $C$ and which has just the root vertex in common with $P^*$.

Proof. Suppose to the contrary that there are two trees $\tau_1$ and $\tau_2$ in $T_2 \setminus T_1$ rooted on $C$ at $p_1$ and $p_2$, respectively, with only their root vertices in common with $P^*$. Since two points of $C \setminus \{r^{-1}, s^{-1}\}$ are mapped onto each point of $f(C) \setminus \{r, s\}$, it is clear that

$$f(\tau_1) \cap f(\tau_2) \cap [f(C) \setminus \{r, s\}] = \emptyset.$$ 

For $i = 1, 2$, if $p_i \notin \{r^{-1}, s^{-1}\}$ there is a point $q_i$ on $\tau_i$ such that $f^{-1}(f(q_i))$ has three points, each of order 2, two lying on $C$. Let $q_i^*$ be a point on $\tau_i$ between $q_i$ and the root vertex $p_i$. Let $G(q_1, q_2)$ be the set of all points of $G$ which are joined to at least one point of $f^{-1}(f(q_1^*))$ by a path not including any point of $f^{-1}(f(q_1)) \cup f^{-1}(f(q_2^*))$. Then it is easy to see that $t_1(q_1, q_2) + 2 = t_3(q_1, q_2) + 2t_4(q_1, q_2)$, contradicting Lemma 14. If $p_i \in \{r^{-1}, s^{-1}\}$ for one or both $i = 1, 2$ there is only a slight difference in the argument. The points $q_i$ on $\tau_i$ may have the property that $f^{-1}(f(q_i))$ has three points of order 2, two lying on edges of $\tau_i$, the other lying on $P_i$. It may be also that one or both of $p_1, p_2$ have degree 4 and lie in $G(q_1, q_2)$. But in each of these cases, the same contradiction to Lemma 14 is obtained.

We are now in a position to strengthen Lemma 32 as follows.
Lemma 35. Let $G$ be a unicyclic graph, let $f : G \to S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$. If $\tau_1 = \tau_1' \oplus \tau_1''$ with $\tau_1' \in T_2 \setminus T_1$, then either $\tau_2 \in T_3$ or $\tau_2 = \tau_2' \oplus \tau_2''$ with $\tau_2' \in T_4$ containing $P_2$ and $\tau_2'' \in T_1$.

Proof. This follows from Lemmas 32–34. 0

Lemma 36. Let $G$ be a unicyclic graph, let $f : G \to S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$. If $C$ has two vertices of degree 4, then $G \in C_5$.

Proof. By Lemma 24(v), the two vertices of degree 4 are $r^{-1}$ and $s^{-1}$, and by Lemma 13 the other vertices mapping to $r$ and $s$ all have degree 1. It follows easily that $\{b_1, b_2\} \cap \{r^{-1}, s^{-1}\} \neq \emptyset$, so we may suppose that $b_1 = r^{-1}$. If $P_2$ were just a single vertex, then $s^{-1}$ would be the image of an end vertex of $P_1$, contradicting Lemma 24(vi). Therefore $P_2$ is not a single vertex. The same contradiction is obtained if $P_2$ is rooted in $C \setminus \{r^{-1}, s^{-1}\}$. Therefore $P_2$ is a proper path rooted at $s^{-1}$. Note that, since $r^{-1}$ and $s^{-1}$ both have degree 4, no tree in $T_2 \setminus T_1$ can be rooted on $C \setminus \{r^{-1}, s^{-1}\}$. It now follows from Lemmas 31, 33 and 35 that $G \in C_5$. 0

In view of Lemma 36, from now on we need only consider the cases when $C$ has at most one vertex of degree 4.

Lemma 37. Let $G$ be a unicyclic graph, let $f : G \to S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$ and let $C$ have one vertex of degree 4. Let $\tau_1 = \tau_1' \oplus \tau_1''$ where $\tau_1'$ contains $P_1$ and $\tau_1'' \in T_2 \setminus T_1$. Then $G \in C_5 \cup C_6$.

Proof. If $\tau_1' \in T_3 \setminus T_4$, then, by Lemmas 30 and 34, it follows that $G \in C_5$. So now suppose that $\tau_1' \in T_4$. Then by Lemmas 34 and 35, either $\tau_2 \in T_1 \oplus T_2$ and $G \in C_5$, or $\tau_2 \in T_5 \setminus T_4$.

So suppose that $\tau_1 \in T_4$ and $\tau_2 \in T_5 \setminus T_4$. We shall show that $G$ is in $C_6$. To do this, all we need to show is that there are no other trees rooted on $C$. Since $b_1$ has degree 4, $b_1 \in \{r^{-1}, s^{-1}\}$. Let us suppose that $b_1 = r^{-1}$. Let $p$ be the first point reached going along $P_2$ starting at $b_2$ at which there is rooted on $P_2$ a tree $t^* \in T_3 \cup (T_2 \oplus T_2)$. Let $p^*$ and $p^{**}$ be two points of $P_2$ near $p$, with $p^{**}$ nearer to $b_1$ than $p^*$ is, such that each of $f^{-1}(f(p^*))$ and $f^{-1}(f(p^{**}))$ have three points of order 2, one on $P_2$, one on an edge $(x_0, p)$ of $t^*$, and one on an edge $(x_1, x_0)$ of $t^*$. Let the edges of $\tau_1'$ and $\tau_1''$ incident with $r^{-1}$ be $(z', r^{-1})$ and $(z'', r^{-1})$ respectively. Since $\tau_1' \in T_2 \setminus T_1$, both $\tau_1'$ and $\tau_1''$ are in $T_4 \setminus T_1$, and so it follows that either $f([z', r^{-1}]) \ni f(C)$ or $f([z', r^{-1}]) \ni f(C)$; similarly for $f([z'', r^{-1}])$. Suppose that $f([z', r^{-1}]) \ni f(C) = r$ and $f([z'', r^{-1}]) \ni f(C) = r$. Let $q^*$ and $q^{**}$ be points of $(z', r^{-1})$, with $q^{**}$ nearer to $r^{-1}$ than $q^*$ is, such that each of $f^{-1}(f(q^*))$ and $f^{-1}(f(q^{**}))$ has three points of order two, one in $(z', r^{-1})$, one in $(z'', r^{-1})$, and the third in an edge of $\tau' \cup \tau''$ incident with $z'$ or $z''$. 0
Let $G(p^*, q^*)$ be the set of points of $G$ joined to a point of $f^{-1}(f(p^{**})) \cup f^{-1}(f(q^{**}))$ by a path not including any point of $f^{-1}(f(p^{**})) \cup f^{-1}(f(q^{**}))$. Then it is easy to see that $t_3(p^*, q^*) + 2 = t_3(p^*, q^*) + 2t_4(p^*, q^*)$, contradicting Lemma 14. Consequently either $f([z^*, r^{-1}]) = f(C)$ of $f([z^*, r^{-1}]) = f(C)$. Therefore there are no other trees rooted on $C$. It follows that $G$ is in $C_6$. 

Recall that, by Lemma 24(vi), any vertex of $C$ of degree 4 is the root vertex of $P_1$ or $P_2$. It remains to deal with the cases when $C$ has at most one vertex of degree 4, and if $C$ does have a vertex of degree 4, then that vertex is the root of $\tau_1$, where $\tau_1$ is of the form $\tau_1 = \tau'_1 \oplus \tau''_1$, $\tau'_1 \in T_5 \setminus T_0$ and contains $P_1$, and $\tau''_1 \in T_1 \setminus T_0$.

We first deal with the case when $\tau''_1 \in T_5 \setminus T_4$.

**Lemma 38.** Let $G$ be a unicyclic graph, let $f: G \rightarrow S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$ and let $C$ have one vertex, $r^{-1}$, of degree 4. Let $\tau_1$, the tree rooted on $r^{-1}$, be of the form $\tau_1 = \tau'_1 \oplus \tau''_1$, where $\tau'_1 \in T_5 \setminus T_4$ and contains $P_1$, and $\tau''_1 \in T_1 \setminus T_0$. Then $G \in C_5$.

**Remark.** Of course, in the case of Lemma 38, $G$ is only in the subclass of $C_5$ in which the tree in $T_1 \oplus T_4$ rooted on $C$ is actually in $T_4$.

**Proof.** By Lemmas 30, 33 and 34, $G \in C_5$ unless there are trees $\tau_2 \in T_4 \setminus T_1$ containing $P_2$ and $\tau_3 \in T_2 \setminus T_1$ with $\tau_3$ rooted at $b_3$, where $b_3 \in \{b_1, b_2\}$. So suppose instead that these trees $\tau_2$ and $\tau_3$ exist.

Going along $P_1$ starting at $b_1$ there is a first vertex $p \neq b_1$ at which a tree $\tau^* \in T_3 \cup (T_2 \cup T_2)$ is rooted on $P^*$. Let $p^*$ and $p^{**}$ be points of $P^*$ near $p$ with $p^{**}$ nearer to $b_1$ that $p^*$ is, such that $f^{-1}(f(p^*))$ and $f^{-1}(f(p^{**}))$ each have three points of order 2, one on $P^*$, one on an edge $(x_0, p^*)$ of $\tau^*$, the third on an edge $(x_0, x_1)$ of $\tau^*$. Since $b_1$ has degree 4 there is no point of $\tau_2 \cup \tau_3$ of order 2 which maps to $\tau$. It follows that $s^{-1} \in \{b_2, b_3\}$. Suppose that $i \in \{2, 3\}$ and $b_i \neq s^{-1}$. Let $q^*$ and $q^{**}$ be points near $b_i$ on the arc of $C$ joining $b_i$ to $r^{-1}$ which includes $s^{-1}$, with $q^{**}$ nearer to $r^{-1}$ than $q^*$, and such that $f^{-1}(f(q^*))$ and $f^{-1}(f(q^{**}))$ each includes three points of order 2, two on $C$, the third on $\tau_i$. Let $G(p^*, q^*)$ be the set of points of $G$ which are joined to at least one point of $f^{-1}(f(p^*)) \cup f^{-1}(f(q^{**}))$ by a path not including any point of $f^{-1}(f(p^*)) \cup f^{-1}(f(q^*))$. It is easy to see that $t_3(p^*, q^*) + 2 = t_4(p^*, q^*) + 2t_4(p^*, q^*)$, contradicting Lemma 14. 

Next we deal with the case when $C$ has one vertex of degree 4 on which is rooted $\tau_1$, where $\tau_1 = \tau'_1 \oplus \tau''_1$ and $\tau'_1 \in T_5 \setminus T_0$ and contains $P_1$, and $\tau''_1 \in T_1 \setminus T_0$.

**Lemma 39.** Let $G$ be a unicyclic graph, let $f: G \rightarrow S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$, and let $C$ have one vertex, $r^{-1}$, of degree 4. Let $\tau_1$, the tree rooted on $r^{-1}$, be of the form $\tau_1 = \tau'_1 \oplus \tau''_1$, where $\tau'_1 \in T_5 \setminus T_0$ and contains $P_1$, and $\tau''_1 \in T_1 \setminus T_0$. Then $G \in C_4 \cup C_5$.
Proof. If $C$ has at most one tree, other than $\tau_1$, which is not in $T_1$, then by Lemma 32, that tree must be in $T_5 \setminus T_1$, and $G$ is in a subclass of $C_5$. So we may suppose that $C$ has two further trees rooted on it at distinct vertices, one being $\tau_2 \in T_5 \setminus T_1$ containing $P_2$, the other being $\tau_3 \in T_2 \setminus T_1$. Let the root vertex of $\tau_3$ be $b_3$. No point of $\tau_2 \cup \tau_3$ of order two can be mapped by $f$ onto $r^{-1}$. It follows that one of $\tau_2$ and $\tau_3$ is rooted on $s^{-1}$ and the other contains a point of order two which is mapped to $s$. Therefore there are no trees rooted on $C$ between $b_2$ and $b_3$, so $G \in C_4$. [Note that the labels $b_1$ and $b_2$ are placed differently in Fig. 4(a).] \Box

Finally we deal with the case when $C$ has no vertices of degree 4.

**Lemma 40.** Let $G$ be a unicyclic graph, let $f: G \to S$ be a 3-to-1 map, and let $f(C) \neq S$. Let $P_1$ and $P_2$ be rooted at distinct points of $C$. Suppose that $C$ has no vertices of degree 4. Then $G \in C_4$.

**Proof.** By Lemma 34, $C$ has at most three trees not in $T_1$ rooted on it. If $C$ has in fact two or fewer trees rooted on it which are not in $T_1$ rooted on it, then unless one of them is in $T_5 \setminus T_4$, it is again easy to see that $G \in C_4$. So suppose that $G$ has three trees $\tau_1, \tau_2, \tau_3$ rooted on it at $b_1, b_2$ and $b_3$ respectively, with $P_1 \in \tau_1 \in T_5 \setminus T_4$, $P_2 \in \tau_2 \in T_4 \setminus T_1$, and $P_3 \in \tau_3 \in T_2 \setminus T_1$.

It is clear that there are no trees in $T_1$ rooted on $C$ either (a) between $b_1$ and $b_2$, or (b) between $b_1$ and $b_3$, or (c) between $b_2$ and $b_3$. If $\tau_2 \in T_4 \setminus T_2$ then (a) is not possible. If $\tau_2 \in T_2 \setminus T_1$ then (a) is possible; however, in that case, there is no loss of generality in interchanging the labels of $\tau_2$ and $\tau_3$ (and of $b_2$ and $b_3$), and designating a main path of the new $\tau_2$ as $P_2$; then we have case (b).

Therefore we may suppose that either (b) or (c) is true. The lemma follows by showing that in fact (b) is true. Suppose to the contrary that there are trees in $T_2 \setminus T_1$ rooted between $b_1$ and $b_3$; if $\tau_2 \in T_2 \setminus T_1$ suppose also that there are trees in $T_1$ rooted between $b_1$ and $b_2$. Then $\{b_2, b_3\} \cap \{r^{-1}, s^{-1}\} \neq \emptyset$, so we may suppose that $s^{-1} \in \{b_2, b_3\}$, and two points, one of order 2 and one of order 1 in $\tau_2 \cup \tau_3$ map to $S$. Let $q^*$ be a point in $\tau_2 \cup \tau_3$ on the edge incident with $s^{-1}$, near $s^{-1}$, and let $q^{**}$ lie between $s^{-1}$ and $q^*$, and suppose that the three points of each of $f^{-1}(f(q^*))$ and $f^{-1}(f(q^{***}))$ all have order 2.

Let $\tau^* \in (T_2 \oplus T_2) \setminus T_3$ be rooted in $P_1$ at a point $p$, and let $p^*$ and $p^{**}$ be points on $P_1$, defined as usual. Define $G(p^*, q^*)$ as usual. The usual contradiction to Lemma 14 is obtained. Therefore there are no trees in $T_1$ between $b_1$ and $b_3$ if $\tau_2 \in T_4 \setminus T_2$, and if $\tau_2 \in T_2 \setminus T_1$, there are either no trees of $T_1$ between $b_1$ and $b_3$ or no trees of $T_1$ between $b_1$ and $b_2$. Therefore $G \in C_4$. \Box

**Proof of Theorem 4.** (Necessity) The necessity follows from Theorem 3 and Lemmas 29, 36–40.

(Sufficiency) Recall Lemmas 19–21. The maps are evident from Figs. 10(a)–(d), where we show possible maps for the actual graphs depicted in Fig. 4 (note that if
Fig. 10(a). $C_4$.

Fig. 10(b). $C_5$.

Fig. 10(c). $C_6$. 
a graph can be mapped 3-to-1 onto the circle, it can usually be mapped 3-to-1 onto the
circle in many different ways).

The proof of Theorem 4 is completed. □

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