A SIMPLE PROOF OF THE SION MINIMAX THEOREM

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ABSTRACT. For convex subsets $X$ of a topological vector space $E$, we show that a KKM principle implies a Fan-Browder type fixed point theorem and that this theorem implies generalized forms of the Sion minimax theorem.

The von Neumann-Sion minimax theorem is fundamental in convex analysis and in game theory. von Neumann [8] proved his theorem for simplexes by reducing the problem to the 1-dimensional cases. Sion’s generalization [7] was proved by the aid of Helly’s theorem and the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz [5]. In a recent paper, Kindler [4] proved Sion’s theorem by applying the 1-dimensional KKM theorem (i.e., every interval in $\mathbb{R}$ is connected), the 1-dimensional Helly theorem (i.e., any family of pairwise intersecting compact intervals in $\mathbb{R}$ has nonempty intersection), and Zorn’s lemma (or other method).

In this short note, for convex subsets $X$ of a topological vector space $E$, we show that a KKM principle implies a Fan-Browder type fixed point theorem and that this theorem implies a generalized form of the Sion minimax theorem.

**Definition.** If a multimap $G : X \rightharpoonup X$ satisfies

$$
\text{co} A \subseteq G(A) \coloneqq \bigcup_{y \in A} G(y) \quad \text{for all finite subset } A \text{ of } X,
$$

then $G$ is called a KKM map.

**Definition.** A multimap $T : X \rightharpoonup X$ is called a Fan-Browder map provided that

(a) for each $x \in X$, $T(x)$ is convex; and
(b) $X = \bigcup_{y \in N} \text{Int} T^-(y)$ for some finite subset $N$ of $X$.

Here, $\text{Int}$ denotes the interior with respect to $X$ and, for each $y \in X$, $T^-(y) \coloneqq \{x \in X \mid y \in T(x)\}$.

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The statement
Suppose that there exists a real
Define a map
Then we have
Suppose that
spaces, and
by (b). Therefore, the family
Theorem 1. The statement (A) implies (B).
Proof. Define a map \(G : X \rightarrow X\) by \(G(x) := X \setminus \text{Int} T^-(x)\) for each \(x \in X\). Then each \(G(x)\) is (relatively) closed, and
\[
\bigcap_{y \in N} G(y) = X \setminus \bigcup_{y \in N} \text{Int} T^-(y) = X \setminus X = \emptyset
\]
by (b). Therefore, the family \(\{G(x)\}_{x \in X}\) does not have the finite intersection property, and hence, \(G\) is not a KKM map by (A). Thus, there exists a finite subset \(A\) of \(X\) such that \(\co A \not\subset G(A) = \bigcup \{X \setminus \text{Int} T^-(y) \mid y \in A\}\). Hence, there exists an \(x_0 \in \co A\) such that \(x_0 \in \text{Int} T^-(y) \subset T^-(y)\) for all \(y \in A\); that is, \(A \subset T(x_0)\). Therefore, \(x_0 \in \co A \subset T(x_0)\) by (a).

Theorem 2. Let \(X\) and \(Y\) be nonempty convex subsets of two topological vector spaces, and \(f, s, t, g : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}\) be four functions,
\[
\mu := \inf_{y \in Y} \sup_{x \in X} f(x, y) \quad \text{and} \quad \nu := \sup_{x \in X} \inf_{y \in Y} g(x, y).
\]
Suppose that
\begin{align*}
(2.1) & \quad f(x, y) \leq s(x, y) \leq t(x, y) \leq g(x, y) \quad \text{for each} \quad (x, y) \in X \times Y; \\
(2.2) & \quad \text{for each} \quad r < \mu \quad \text{and} \quad y \in Y, \quad \{x \in X \mid s(x, y) > r\} \quad \text{is convex; for each} \quad r > \nu \quad \text{and} \quad x \in X, \quad \{y \in Y \mid t(x, y) < r\} \quad \text{is convex;} \\
(2.3) & \quad \text{for each} \quad r > \nu, \quad \text{there exists a finite subset} \quad \{x_i\}_{i=1}^n \quad \text{of} \quad X \quad \text{such that} \quad Y = \bigcup_{i=1}^n \text{Int} \{y \in Y \mid f(x_i, y) > r\}; \quad \text{and} \\
(2.4) & \quad \text{for each} \quad r < \mu, \quad \text{there exists a finite subset} \quad \{y_j\}_{j=1}^m \quad \text{of} \quad Y \quad \text{such that} \quad X = \bigcup_{j=1}^m \text{Int} \{x \in X \mid g(x, y_j) < r\}.
\end{align*}
Then we have \(\mu \leq \nu\), that is,
\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).
\]

Proof. Suppose that there exists a real \(c\) such that
\[
\nu := \sup_{x, y} \inf_{y} g(x, y) < c < \inf_{y} \sup_{x} f(x, y) =: \mu.
\]
Define a map \(T : X \times Y \rightarrow X \times Y\) by
\[
T(x, y) := \{\bar{x} \in X \mid s(\bar{x}, y) > c\} \times \{\bar{y} \in Y \mid t(x, \bar{y}) < c\}
\]
for each \((x, y) \in X \times Y\). Then each \(T(x, y)\) is convex by (2.2). Moreover, for each \((\bar{x}, \bar{y}) \in X \times Y\), we have
\[
T^{-}(\bar{x}, \bar{y}) = \{ x \in X \mid s(x, \bar{y}) > c \} \times \{ y \in Y \mid t(\bar{x}, y) < c \}
\]
\[
\sup \{ x \in X \mid f(x, \bar{y}) > c \} \times \{ y \in Y \mid g(\bar{x}, y) < c \}
\]
\[
\int \{ x \in X \mid f(x, \bar{y}) > c \} \times \int \{ y \in Y \mid g(\bar{x}, y) < c \}.
\]
Therefore, by (2.3) and (2.4), \(X \times Y\) is covered by
\[
\{ \int T^{-}(x_{i}, y_{j}) \mid 1 \leq i \leq m, 1 \leq j \leq n \}.
\]
Hence, \(T\) is a Fan-Browder map. Since \(X \times Y\) is a convex subset of a topological vector space, (A) and (B) hold. Therefore, by (B), we have an \((x_{0}, y_{0}) \in X \times Y\) such that \((x_{0}, y_{0}) \in T(x_{0}, y_{0})\). Therefore, \(t(x_{0}, y_{0}) < c < s(x_{0}, y_{0})\), a contradiction.

Recall that an extended real-valued function \(f : X \to \mathbb{R}\) on a topological space \(X\) is lower [resp., upper] semicontinuous (l.s.c.) [resp., u.s.c.] if \(\{ x \in X \mid f(x) > r \}\) [resp., \(\{ x \in X \mid f(x) < r \}\)] is open for each \(r \in \mathbb{R}\).

For a convex set \(X\), a extended real-valued function \(f : X \to \mathbb{R}\) is said to be quasiconcave [resp., quasiconvex] if \(\{ x \in E \mid f(x) > r \}\) [resp., \(\{ x \in E \mid f(x) < r \}\)] is convex for each \(r \in \mathbb{R}\).

**Theorem 3.** Let \(X\) and \(Y\) be compact convex subsets of topological vector spaces, and \(f, s, t, g : X \times Y \to \mathbb{R} \cup \{ +\infty \}\) be functions satisfying
\[
(3.1) f(x, y) \leq s(x, y) \leq t(x, y) \leq g(x, y) \quad \text{for each} \quad (x, y) \in X \times Y;
\]
\[
(3.2) \text{for each} \quad x \in X, \quad f(x, \cdot) \text{ is l.s.c. and} \quad t(x, \cdot) \text{ is quasiconvex on} \ Y; \quad \text{and}
\]
\[
(3.3) \text{for each} \quad y \in Y, \quad s(\cdot, y) \text{ is quasiconcave and} \quad g(\cdot, y) \text{ is u.s.c. on} \ X.
\]
Then we have
\[
\min_{y \in Y} \sup_{x \in X} f(x, y) \leq \max_{x \in X} \inf_{y \in Y} g(x, y).
\]

**Proof.** Note that \(y \mapsto \sup_{x \in X} f(x, y)\) is l.s.c. on \(Y\) and \(x \mapsto \inf_{y \in Y} g(x, y)\) is u.s.c. on \(X\). Therefore, the both sides of the inequality exist. Then all the requirements of Theorem 2 are satisfied.

For \(f = s = t = g\) in Theorem 3, we have the following Sion minimax theorem [7]:

**Theorem 4.** Let \(X\) and \(Y\) be compact convex subsets of topological vector spaces and \(f : X \times Y \to \mathbb{R}\) a real function such that
\[
(4.1) \text{for each} \quad x \in X, \quad f(x, \cdot) \text{ is l.s.c. and quasiconvex on} \ Y; \quad \text{and}
\]
\[
(4.2) \text{for each} \quad y \in Y, \quad f(\cdot, y) \text{ is u.s.c. and quasiconcave on} \ X.
\]
Then
\[
(i) \quad f \text{ has a saddle point} \quad (x_{0}, y_{0}) \in X \times Y; \quad \text{and}
\]
\[
(ii) \quad \min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).
\]
Proof. It is well known and easy to see that the minima and maxima in Theorem 4 exist under our topological assumptions. Hence, there exists an \((x_0, y_0) \in X \times Y\) such that
\[
\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} f(x_0, y) \geq \min_{x \in X} f(x_0, y) = \max_{y \in Y} \min_{x \in X} f(x, y).
\]
Moreover, all the requirements of Theorem 3 with \(f = g\) are satisfied. Therefore, the \(\geq\)'s in the above should be = and we have the conclusion. \(\square\)

Remark 1. von Neumann [8] obtained Theorem 4 when \(X\) and \(Y\) are subsets of Euclidean spaces and \(f\) is continuous.

2. (A) also holds for open-valued KKM maps, and (B) also holds when \(T^-\) has closed values. In this case, (A) implies (B) also.

3. For other simple proof of the Sion minimax theorem, see [4].

4. Theorem 2 is motivated from [2, Theorem 8], which is for \(f = s = t = g\).

5. For the history of the KKM theory, see [6].

6. All the results in this paper can be extended to abstract convex spaces without assuming any linear structure.

References