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BURIED SIERPINSKI CURVE JULIA SETS

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Abstract. In this paper we prove the existence of a new type of Sierpinski curve Julia set for certain families of rational maps of the complex plane. In these families, the complementary domains consist of open sets that are preimages of the basin at ∞ as well as preimages of other basins of attracting cycles.

In recent years the families of rational maps of the complex plane given by $z^n + \lambda/z^d$ have been shown to exhibit a rich array of both dynamical and topological phenomena. The principal focus of these studies has most often been the Julia sets for such maps. As is well known, the Julia set is the set on which all of the "interesting" dynamics occurs. For many λ-values, the Julia sets of these maps are also quite interesting from a topological point of view.

For example, for each $n \geq 2$, it is known that there are infinitely many λ-values for which the Julia set is a *generalized Sierpinski gasket* (see [5]), and none of these Julia sets are homeomorphic to each other. As another example, there are infinitely many λ-values in each of these families for which the Julia set is a *Sierpinski curve* ([1]). A Sierpinski curve is any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by simple closed curves that are pairwise disjoint. A result of Whyburn [13] shows that any such set is homeomorphic to the well-known Sierpinski carpet fractal. The interesting topology arises from the fact that a Sierpinski curve is universal in the sense that it contains a homeomorphic copy of any planar, one-dimensional continuum. In the case of the Sierpinski curve Julia sets, all of these sets are homeomorphic, but as shown in [4], infinitely many of them are dynamically distinct in the sense that the corresponding maps are not topologically conjugate on their Julia sets. Moreover, as shown in [1], when $n = 2, d = 2$ or $n = 2, d = 1$, in every neighborhood of the parameter value $\lambda = 0$, there are infinitely many parameter values for which the Julia set is a Sierpinski curve on which the dynamics are distinct. Hence these families undergo a dramatic explosion when λ becomes nonzero.

In each of the above cases where the Julia set is a Sierpinski curve, the complementary domains (or the Fatou components) are always preimages of the immediate basin of attraction of ∞, which is a superattracting fixed point for these maps (provided $n \geq 2$). In this paper, we exhibit a similar infinite collection of dynamically distinct Julia sets, but now the Fatou components are quite different. Instead of being preimages of a single superattracting basin at ∞, we give examples where the

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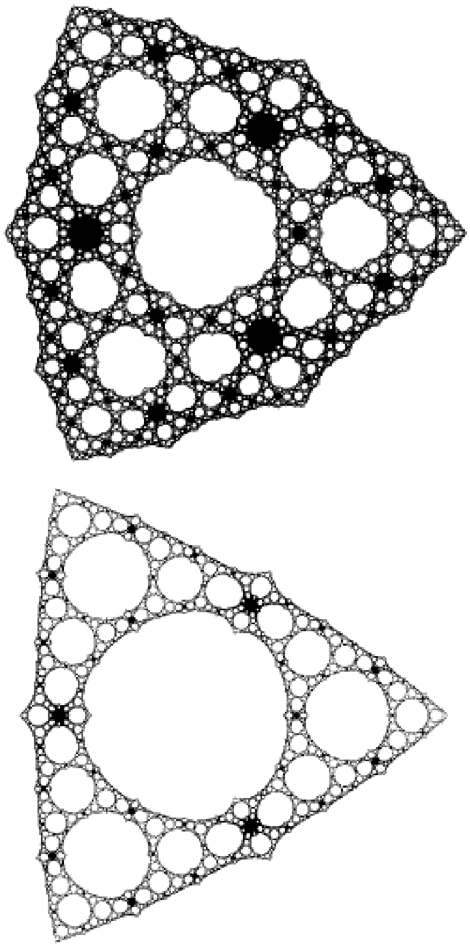

FIGURE 1. The Julia sets for $\lambda = -0.327$ and $\lambda = -0.5066$.

complementary domains consist of a collection of a number of different attracting basins together with the basin at ∞ and all of the preimages of these basins. As before, we prove that the dynamics on these Julia sets are all distinct from one another as well as from those mentioned above, but again, all of these Julia sets are homeomorphic.

For simplicity, we restrict attention in this paper to the special family $F_\lambda(z) = z^2 + \lambda/z$. At the end of the paper, we describe generalizations to other higher degree families of the form $z^n + \lambda/z^d$. In Figure 1, we display the Julia set of F_λ when $\lambda = -0.327$. For this map, there are attracting basins of period 3 and period 6 together with the basin at ∞. We also display the case where $\lambda = -0.5066$ for which there are three different attracting basins of period 4 together with the basin at ∞. In these figures, the black regions represent the basins of the finite attracting cycles while the white regions form the basin of ∞.

1. **Preliminaries.** Consider the degree three family of rational maps of the complex plane given by $F_\lambda(z) = z^2 + \lambda/z$ where λ is a parameter. There are four critical points for F_λ, one at ∞ and the other three of the form $\omega^k c_\lambda$ where $c_\lambda = (\lambda/2)^{1/3}$ is one of the finite critical points and ω is a cube root of unity. So the critical points are arranged with three-fold symmetry about the origin. Similarly, the critical values are arranged symmetrically with respect to ω and are given by $\omega^k v_\lambda$ where

$$v_\lambda = \frac{3}{2^{2/3}} \lambda^{2/3}.$$

There are also three symmetric prepoles given by $(-\lambda)^{1/3}$.

Note that $F_\lambda(\omega z) = \omega^2 F_\lambda(z)$. Hence the orbits of points of the form $\omega^j z$ all behave "symmetrically" under iteration of F_λ. This implies, for example, that if $F_\lambda^i(z) \to \infty$, then $F_\lambda^i(\omega^k z)$ also tends to ∞ for $k = 1, 2$. Similarly, if $F_\lambda^i(z)$ tends to an attracting cycle, then so does $F_\lambda^i(\omega^k z)$. We remark that the cycles involved may be different and indeed, they may even have different periods. They are, however, arranged symmetrically about the origin via $z \mapsto \omega z$. Further note that, when λ is real, $F_\lambda(\overline{z}) = \overline{F_\lambda(z)}$, and therefore the orbits of the points z and $\overline{z}$ also behave symmetrically in this case.

In this paper we shall restrict attention to the case where $\lambda \in \mathbb{R}^-$. For these λ-values there exists a unique critical point in $\mathbb{R}^-$ which we call $c = c(\lambda)$. Since $\mathbb{R}$ is mapped to itself by F_λ, it follows that $F_\lambda^n(c) \in \mathbb{R}$ for all $n \geq 0$. By symmetry there is a critical point on each of the two lines $\omega\mathbb{R}$ and $\omega^2\mathbb{R}$. Note that $F_\lambda : \omega\mathbb{R} \mapsto \omega^2\mathbb{R}$ and vice versa. We call the three lines $\mathbb{R}$, $\omega\mathbb{R}$, and $\omega^2\mathbb{R}$ the *symmetry axes.* While the orbit of c is trapped in $\mathbb{R}$, the other two critical orbits jump between $\omega\mathbb{R}$ and $\omega^2\mathbb{R}$ at each iteration. Therefore, if there is an attracting n-cycle on $\mathbb{R}$, this cycle attracts only c. By symmetry, there must be attracting cycles on $\omega\mathbb{R} \cup \omega^2\mathbb{R}$ that attract the other two critical points. Since $F_\lambda(\overline{z}) = \overline{F_\lambda(z)}$ and $F_\lambda(\omega z) = \omega^2 F_\lambda(z)$, if there is an attracting n-cycle on $\mathbb{R}$, then $\omega\mathbb{R} \cup \omega^2\mathbb{R}$ has either an attracting $2n$-cycle (when n is odd) or a pair of symmetric n-cycles (when n is even). Since there are only three (finite) critical points, it follows that, if there is an attracting n-cycle on $\mathbb{R}$, these are the only other possibilities for attracting cycles in $\mathbb{C}$.

Consider the intervals connecting the critical values to 0 along each of the three symmetry axes. These intervals lie in the rays $\omega^j\mathbb{R}^+$ for $j = 0, 1, 2$. One checks easily that the preimage of the union of these intervals contains a simple closed curve κ that surrounds the origin. All three of the critical points lie in κ as do the three prepoles. See Figure 2. We call κ the *critical curve.* Now consider the three rays given by $\mathbb{R}^-$ and its two symmetric images under $z \mapsto \omega z$. These three rays divide the region inside κ into three sectors which we call the *critical sectors.* We denote by S_0 the critical sector that meets the positive real axis. A straightforward computation shows that F_λ maps S_0 onto the sector $2\pi/3 \leq \operatorname{Arg} z \leq 4\pi/3$ in one-to-one fashion.

Since ∞ is a superattracting fixed point of order two, it is well known that F_λ is conjugate to $z \mapsto z^2$ in a neighborhood of ∞, so we have a basin B_λ at ∞. We denote the boundary of B_λ by ∂B_λ. Since F_λ has a pole of order 1 at 0, there is an open set containing the origin that is mapped one-to-one onto B_λ. If B_λ does not meet this set, then this set is called the *trap door* and we denote it by T_λ. Since the degree of F_λ is 3 and F_λ maps B_λ two-to-one onto itself, all points in the preimage of B_λ lie either in B_λ or in T_λ.

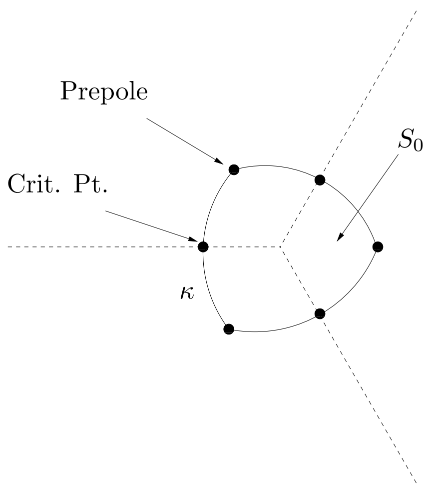

Figure 2. The critical curve and critical sectors.

Proposition. *Both B_λ and T_λ have 3-fold symmetry, i.e., if $z \in B_\lambda$, then $\omega z \in B_\lambda$ as well.*

Proof: Let $U \subset B_\lambda$ be the set of points z in B_λ that have the property that the point ωz also lies in B_λ. U is an open, nonempty set since B_λ contains an open neighborhood around ∞. If $U \neq B_\lambda$, let $z_0 \in \partial U$. Then $z_0 \in B_\lambda$ but $\omega z_0 \notin B_\lambda$. Hence $\omega z_0 \in \partial B_\lambda$. Therefore $F_\lambda^i(z_0) \to \infty$ whereas $F_\lambda^i(\omega z_0) \not\to \infty$. But

$$F_\lambda^i(\omega z_0) = \omega^{2^i} F_\lambda^i(z_0) \to \infty.$$

This gives a contradiction. The case of T_λ is similar. □

By symmetry of B_λ, if one of the critical points lies in B_λ, then all of the critical points do. The same is true if one of the critical points lies in the i^{th} preimage of T_λ, $F_\lambda^{-i}(T_\lambda)$, with $i > 0$. In this case, it is known that each set $F_\lambda^{-i}(T_\lambda)$ has multiple components and the critical points always lie in different components [4].

Recall that the *Julia set* of F_λ is the set of points at which the family of iterates F_λ^n fails to be a normal family in the sense of Montel. We denote the Julia set of F_λ by $J(F_\lambda)$. There are several alternative characterizations of $J(F_\lambda)$, including the fact that $J(F_\lambda)$ is the closure of the set of repelling periodic points of F_λ. In [4], the following result was proved:

Theorem. (The Escape Dichotomy)

1. *If one of the critical values of F_λ lies in B_λ, then $J(F_\lambda)$ is a Cantor set and $F_\lambda \mid J(F_\lambda)$ is conjugate to the shift map on three symbols. In this case T_λ is empty and B_λ is the only component of the full basin of ∞.*
2. *Otherwise, $J(F_\lambda)$ is a connected set and B_λ and T_λ are disjoint, open, simply connected sets. In this case, there infinitely many distinct preimages of T_λ.*
3. *In the special case where one of the critical values lies in a preimage of T_λ, then $J(F_\lambda)$ is a Sierpinski curve.*

We remark that for many of these families of rational maps, there is actually an escape trichotomy. For our degree three family, it can be shown that the critical values never lie in the trap door. However, for higher degree maps, this may happen. Indeed, McMullen [7] has shown that, if $1/n + 1/d < 1$, then provided $|\lambda|$ is sufficiently small, $J(F_\lambda)$ is a Cantor set of quasicircles. As shown in [4], this occurs whenever the critical values lie in T_λ.

In this paper we will further restrict attention to the case where F_λ has an attracting cycle on $\mathbb{R}$, and hence all three critical points are attracted to cycles. Therefore we are in case 2 of the above result and so $J(F_\lambda)$ is connected.

The graph of F_λ on $\mathbb{R}$ shows that B_λ meets $\mathbb{R}$ in the intervals $(p(\lambda), \infty)$ and $(-\infty, q(\lambda))$, where $p(\lambda)$ is the fixed point in $\mathbb{R}^+$ and $q(\lambda)$ is the leftmost preimage of $p(\lambda)$ in $\mathbb{R}^-$. See Figure 3.

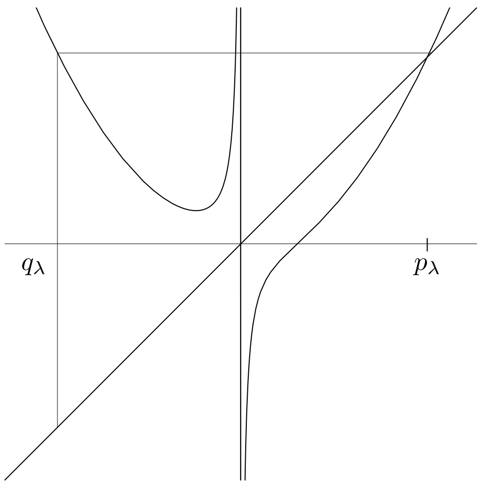

FIGURE 3. The graph of F_λ.

There are no points in $[q(\lambda), p(\lambda)] \cap B_\lambda$ since, by the $\overline{z}$ symmetry, B_λ would then not be simply connected, or equivalently, $J(F_\lambda)$ would not be connected. That would contradict the Escape Dichotomy Theorem.

2. **Existence of Superattracting Cycles.** In this section we will show that there is a sequence $\lambda_n \in \mathbb{R}^-$ with $n = 3, 4, \ldots$ having the property that F_{λ_n} has a superattracting cycle of period n lying in $\mathbb{R}$. We will later prove that $J(F_{\lambda_n})$ is a Sierpinski curve.

Let $\lambda^* = -16/27$. A straightforward calculation shows that F_{λ^*} has a repelling fixed point at $4/3$. The critical point on the real axis for this map is given by $-2/3$ and the critical value is $4/3$, so for λ^* the real critical point of F_{λ^*} maps to the fixed point $p(-16/27) = 4/3$.

We consider nearby λ-values. The critical point on the real axis is given by $c(\lambda) = (\lambda/2)^{1/3}$ and the critical value is given by

$$v(\lambda) = \frac{3}{2^{2/3}} \lambda^{2/3}.$$

Recall that $p(\lambda)$ is the real fixed point for F_λ. The graph of F_λ shows that $p(\lambda) > 1$ when $\lambda < 0$. Now $p(\lambda)$ satisfies the equation

$$(p(\lambda))^3 - (p(\lambda))^2 + \lambda = 0.$$

Using this we derive

$$p'(\lambda) = \frac{-1}{3(p(\lambda))^2 - 2p(\lambda)}$$

so that $p'(\lambda^*) = -3/8$. Using the fact that $p(\lambda) > 1$ it follows that $-1 < p'(\lambda) < 0$ for all negative λ. Also, since

$$v(\lambda) = \frac{3}{2^{2/3}}\lambda^{2/3},$$

we have

$$v'(\lambda) = (2/\lambda)^{1/3} < -1$$

as long as $-2 < \lambda < 0$. Therefore it follows that for $\lambda \in (\lambda^*, 0] = (-16/27, 0]$, $v(\lambda)$ decreases faster than $p(\lambda)$, and so $v(\lambda) < p(\lambda)$ for these values of the parameter (since $p(\lambda^*) = v(\lambda^*)$).

Proposition. *There exists a decreasing sequence λ_n for $n \geq 3$ with $\lambda_n \to \lambda^*$ and having the property that F_{λ_n} has a superattracting cycle of period n given by $x_j(\lambda_n) = F_{\lambda_n}(x_{j-1}(\lambda_n))$ where*

1. *$x_0(\lambda_n) = x_n(\lambda_n) = c(\lambda_n)$, and*
2. *$x_0 < 0 < x_{n-1} < x_{n-2} < \cdots < x_1 = v(\lambda_n) < p(\lambda_n)$.*

Proof: If $0 < x < p(\lambda)$, then the iterates $F_\lambda^j(x)$ decrease with j as long as $F_\lambda^{j-1}(x)$ remains positive. In particular, we may choose λ close enough to λ^* so that the forward orbit of $v(\lambda)$ remains in the interval $(0, p(\lambda))$ for as many iterates as we desire.

We claim that there exists a sequence μ_n with $n \geq 2$ satisfying

1. $F^n_{\mu_n}(c(\mu_n)) = 0$, and
2. $0 < F^j_{\mu_n}(c(\mu_n)) < p(\mu_n)$ for $j = 1, \ldots, n-1$.

To see this, note first that μ_2 may be chosen to be $-4/27$. Define $G_n(\lambda) = F^n_\lambda(c(\lambda))$. So $G_2(\mu_2) = 0$ while $G_2(\lambda^*) = 4/3$. Then G_3 maps the interval (λ^*, μ_2) over the entire half line $(-\infty, 4/3)$, so that there exists μ_3 with $\lambda^* < \mu_3 < \mu_2$ and $G_3(\mu_3) = 0$. Continuing inductively yields the sequence μ_n.

Now consider G_n on the interval $(\mu_n, \mu_{n-1}]$ for $n \geq 3$. G_n maps this interval over at least the negative real axis since $G_n(\mu_n) = 0$ and $G_n(\mu_{n-1}) = F_{\mu_{n-1}}(0) = -\infty$, so there exists a λ_n in this interval with $G_n(\lambda_n) = c(\lambda_n)$. This yields the parameters λ_n.

3. **Buried Basins.** In this section we fix a particular parameter value $\lambda = \lambda_n$ for which F_λ has a superattracting periodic point $x_0 = c(\lambda_n)$ lying in $\mathbb{R}^-$ as described in the previous section. Let A_j denote the immediate basin of attraction in $\mathbb{C}$ of $x_j = F^j_\lambda(x_0)$. So $F^j_\lambda(A_0) = A_j$. Let $C_j = \omega A_j$ and $C_{j+n} = \omega^2 A_j$. The C_i are the basins of the nonreal superattracting cycle(s), but the indexing here does not necessarily correspond to the iteration, i.e., it is not in general true that $F^j_\lambda(C_0) = C_j$. Also, recall from Section 1 that the C_j surround a pair of attracting n-cycles when n is even and a single attracting $2n$-cycle when n is odd.

We say that a basin of attraction of F_λ is *buried* if the boundary of this basin is disjoint from the boundaries of all other basins of attraction (including B_λ). Note that, if the basin of one point on an attracting cycle is buried, then so too are all forward and backward images of this basin, so the entire basin of the cycle is buried. Our goal is to show that all of the basins of F_λ are buried. To accomplish this, it suffices to show that A_0 and B_λ are buried, for if that is the case, then all forward and backward images of A_0 and B_λ are also buried. By symmetry, the basins of the symmetric cycles are also buried since each C_j has the form $\omega^i A_k$ for some i and k.

We begin by showing that ∂A_0 and ∂B_λ are disjoint. Recall that, in Section 1, we showed that the interval $[q(\lambda), p(\lambda)]$ does not meet B_λ, but that $q(\lambda)$ and $p(\lambda)$ lie in ∂B_λ. By symmetry, the corresponding intervals on the other two symmetry axes also do not meet B_λ. We claim that the endpoints of these three intervals are the only points in the intersection of ∂B_λ and the symmetry axes:

Proposition. *The boundary of B_λ meets each of the symmetry axes in exactly two points, namely $p(\lambda)$ and $q(\lambda)$ or their symmetric images.*

Proof: It suffices to consider the case of $\mathbb{R}$. Recall from Section 1 that $B_\lambda \cap [q(\lambda), p(\lambda)]$ is empty. Suppose $y_0 \in \mathbb{R} \cap \partial B_\lambda$ and $y_0 \neq p(\lambda), q(\lambda)$. Then either y_0 or $y_1 = F_\lambda(y_0)$ lies in the interval $(0, p(\lambda))$ since F_λ maps $\mathbb{R}^-$ to $\mathbb{R}^+$. But then, since F_λ is decreasing on $\mathbb{R}^+$, there is a first point $y_n = F_\lambda^n(y_0)$ such that $y_n \in (0, (-\lambda)^{1/3})$ where we recall that $(-\lambda)^{1/3}$ is the prepole in $\mathbb{R}^+$, i.e., $F_\lambda(-\lambda^{1/3}) = 0$. We have that $y_n \in \partial B_\lambda$ since ∂B_λ is invariant.

Now recall that the critical sector S_0 is the region bounded by the rays $\omega^2 t$, ωt, and a third of the critical curve, where $c(\lambda) \leq t \leq 0$. The vertices of this "triangular" region are given by 0 and the two nonreal critical points of F_λ. We claim that B_λ cannot meet the boundary of S_0. To see this, note that the straight line boundaries of S_0 lie strictly inside the symmetric images of $[q(\lambda), p(\lambda)]$ on the nonreal symmetry axes, so B_λ misses them. Also, the portion of the boundary of S_0 on the critical curve is mapped by F_λ onto the intervals between 0 and the critical value along $\omega\mathbb{R}^+$ and $\omega^2\mathbb{R}^+$. But these intervals are contained inside the symmetric copies of $[0, p(\lambda))$ in these rays. Hence there are no points in B_λ on this part of the boundary of S_0 as well.

Now since y_n lies in the interior of S_0 and also on ∂B_λ, it follows that there are points in B_λ inside the set S. But since B_λ is connected and extends to ∞, it follows that there are points in B_λ that also lie on the boundary of S. This contradiction establishes the result. □

In particular, note that the proof of this result implies that ∂B_λ does not meet the the critical curve, for otherwise the image of such a point would lie in one of the symmetric copies of $(q(\lambda), p(\lambda))$, in contradiction to the previous Proposition. The same is true for ∂T_λ. Since the critical circle therefore surrounds T_λ, it follows that $\partial B_\lambda \cap \partial T_\lambda$ is empty. It follows immediately that none of the preimages of ∂B_λ meet ∂B_λ.

Now we show that $\partial B_\lambda \cap \partial A_j$ is empty. We first observe that the basins A_j cannot intersect the nonreal symmetry axes. This follows since any point on these two symmetry axes must remain on the union of these axes for all iterations and hence the orbit of this point cannot tend to a (non-zero) cycle in $\mathbb{R}$. The A_j miss 0 since 0 maps to ∞. Now the point x_{n-1} on the real superattracting cycle lies in the interval $(0, (-\lambda)^{1/3})$ since this is the subinterval of $\mathbb{R}^+$ that is mapped to $\mathbb{R}^-$. Consequently A_{n-1} must intersect the critical sector S_0. But the interior of A_{n-1} cannot meet the boundary of this sector for, as in the previous Proposition, this boundary is mapped to the nonreal symmetry axes. Hence ∂A_{n-1} is contained in the closed set S_0 and therefore must be disjoint from ∂B_λ. Therefore all of the basins A_j have this property and we have proved:

Proposition. *The boundaries of* B_λ *and the* A_j *are disjoint.*

By symmetry, it follows that the boundaries of B_λ and the C_j are also disjoint. Next we have:

Proposition. *The basins* A_j *and all of their preimages have disjoint boundaries.*

Proof: This result follows immediately from the fact that ∂A_{n-1} lies in the closed set S_0 and therefore is contained in the half plane $\operatorname{Re} z > 0$ (note that the origin is not in ∂A_{n-1}). At the same time, ∂A_0 is contained in $\operatorname{Re} z < 0$, for otherwise this basin would meet a nonreal symmetry axes. Hence ∂A_0 is disjoint from ∂A_{n-1} and the result follows. □

To complete the proof that all basins of attraction are buried, we must show that $\partial C_k \cap \partial A_j = \emptyset$ for all k, j. To see this, we first observe that a given C_k cannot intersect both nonreal symmetry axes. If this were to happen, then we would have a pair of points inside C_k whose iterates always lie on different nonreal symmetry axes and so these two orbits could not lie in the same immediate basin of attraction. Now there are $2n-2$ C_k's that lie completely in the "left" sector J_L defined by $\pi/3 < \operatorname{Arg} z < 5\pi/3$ and there are only two C_k's that are completely contained in the "right" sector J_R given by $-2\pi/3 < \operatorname{Arg} z < 2\pi/3$. Recall here that $n \geq 3$, so there are more C_k's in J_L than in J_R. Similarly, there is only one A_j, namely A_0, in J_L, while the remaining $n-1$ A_j's lie in J_R. It follows that if the boundary of some C_k meets ∂A_0, then some subsequent iterate $F_\lambda^i(C_k)$ must lie in J_L whereas $F_\lambda^i(A_0)$ lies in J_R. This uses the fact that $n \geq 3$. But we must have $F_\lambda^i(\partial C_k) \cap F_\lambda^i(\partial A_0) \neq \emptyset$. Therefore the basin $F_\lambda^i(C_k)$ must intersect both of the nonreal symmetry axes. Since this cannot happen, it follows that ∂C_k must be disjoint from ∂A_0 and hence from each ∂A_j for all j and k. This completes the proof of the fact that all of the attracting basins of F_λ are buried.

4. **Sierpinski Curves.** In this section we complete the proof that $J(F_{\lambda_n})$ is a Sierpinski curve for each n. Again we fix n and write $\lambda = \lambda_n$.

We need to show that $J(F_\lambda)$ is compact, connected, locally connected, nowhere dense, and the boundaries of all the Fatou components are disjoint simple closed curves. We remark that, for topologically constructed Sierpinski curves, the difficulty that usually arises in showing that a set is a Sierpinski curve is proving local connectivity or nowhere density. But complex dynamics makes the proofs of these properties easy.

First, $J(F_\lambda)$ is compact and connected since $J(F_\lambda)$ is the complement of the union of countably many open, simply connected basins of attraction and their preimages. Since $J(F_\lambda)$ omits these basins, it follows that $J(F_\lambda)$ is not the entire Riemann sphere and hence contains no interior points implying that it is nowhere dense. Finally, since all critical points lie on attracting cycles, it follows that F_λ is hyperbolic on $J(F_\lambda)$ and so the Julia set is locally connected. See [9] for details. It remains to prove that the boundaries of the basins are simple closed curves, as the previous section guarantees that they are mutually disjoint. This is straightforward for the bounded basins.

Proposition. *The basins of attraction* A_j *and* C_k *have boundaries that are simple closed curves.*

Proof: We prove this for A_0; the other cases follow by symmetry and/or by taking iterates of F_λ. The point $x_0 \in A_0$ is a superattracting fixed point of F_λ^n. Hence there is a conjugacy $\phi_\lambda : \mathbb{D} \to A_0$ satisfying $\phi_\lambda(z^2) = F_\lambda(\phi_\lambda(z))$ where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$. The image of a straight ray in $\mathbb{D}$ given by $te^{i\theta}$ with $0 \leq t < 1$ under ϕ_λ is called an internal ray. Since the boundary of A_0 is locally connected, Carathéodory theory (see [9]) guarantees that each internal ray lands on a single point in ∂A_0, i.e.,

$$\lim_{t \to 1} \phi_\lambda(te^{i\theta})$$

exists for each θ. It then suffices to show that no two internal rays land at the same point. But if two rays did land at a given point $p \in \partial A_0$, then the union of these two internal rays together with p forms a simple closed curve γ that lies entirely inside A_0 (except for p). Let Γ denote the interior of this simple closed curve. Then Γ must contain other points in the boundary of A_0, for otherwise an entire interval of rays would land at p, and this is impossible. But then the union of the forward images of Γ cannot meet points on B_λ, for example, since the images of γ all lie in the union of the $\overline{A}_j$. This contradicts Montel's Theorem which says that the union of these images of Γ must cover all of $\mathbb{C}$ (except for at most one point). □

The fact that the boundary of B_λ is a simple closed curve must be handled differently, for in this case the forward images of the analogue of Γ are no longer bounded. Therefore we proceed differently.

Let W denote the open connected component of $\mathbb{C} - \overline{B}_\lambda$ that contains the origin. As we showed earlier, the interval $(q(\lambda), p(\lambda))$ lies in W as do each of the two symmetric intervals. In particular, the three prepoles lie in W as do all of the critical points and values. Since all three preimages of 0 lie in W, one checks easily that all three preimages of any point in W also lie in W, so $F_\lambda(W) \supset W$.

We claim that W is the only component of $\mathbb{C} - \overline{B}_\lambda$. If there were a second component X in this set, then X could not contain any points in $J(F_\lambda)$, for images of X would then necessarily cover all of $\mathbb{C}$ by Montel's Theorem and thus map over W. But this cannot happen. Therefore X must be a Fatou domain for F_λ. But then one of the images of X must contain a critical point of F_λ, and this too cannot happen since all of the critical points of F_λ lie in W. This shows that W is the unique component of $\mathbb{C} - \overline{B}_\lambda$.

Now we argue as above. Consider the conjugacy ϕ_λ between z^2 and F_λ taking $\mathbb{D}$ to B_λ. Choose the curve γ and the open set Γ as before, where the curve γ now consists of two external (as opposed to internal) rays and the common landing point p. Now we know that the forward images of Γ cannot map onto the interior of W, so just as before, all of the rays associated to ϕ_λ land at unique points and ∂B is a simple closed curve. This completes the proof that $J(F_{\lambda_n})$ is a Sierpinski curve.

By Whyburn's theorem ([13]), any two Sierpinski curves are homeomorphic. Hence $J(F_{\lambda_n})$ is topologically equivalent to $J(F_{\lambda_m})$ for any n and m. However, each of these Julia sets is dynamically distinct from the others.

Theorem. *If $n \neq m$, F_{λ_n} is not topologically conjugate to F_{λ_m} on their Julia sets.*

Proof: A conjugacy between F_{λ_n} and F_{λ_m} on their Julia sets must take the boundaries of attracting basins to boundaries of attracting basins. But the three immediate basins that contain critical points are mapped two-to-one onto their images and these are the only basins that have this property (except for B_λ). Since these basin boundaries are dynamically distinct, they must be mapped to each other by the conjugacy. But the periods of these basins are different, and so they cannot be mapped to one another by a conjugacy. □

In this result we have concentrated on the case where F_{λ_n} has a superattracting cycle. However, the results go over immediately to a neighborhood of each λ_n in the parameter plane. For these nearby parameters, F_λ also has an attracting cycle. While F_λ^n is no longer conjugate to z^2 in the immediate basin of the cycle, quasi-conformal surgery allows us to modify these maps so that they have this property and thereby establish the fact that the Julia set is again a Sierpinski curve. See [1] for more details on this construction.

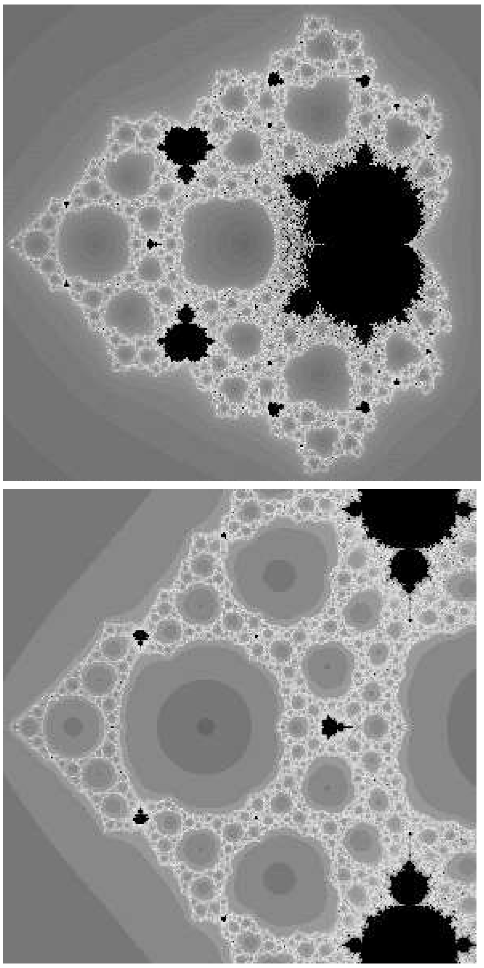

FIGURE 4. The parameter plane for the degree three family of rational maps and a magnification.

5. **Concluding Remarks.**

In this paper we have concentrated on the family

$$F_\lambda(z) = z^2 + \frac{\lambda}{z}.$$

However, all of the results go over immediately to the higher degree families given by

$$F_\lambda(z) = z^{2n} + \frac{\lambda}{z^{2d+1}}.$$

One checks easily that, for $\lambda \in \mathbb{R}^-$, the real axis is again invariant and we have similar symmetries for this map. The proofs therefore go over more or less unchanged.

In Figure 4 we display the parameter plane for the degree three family

$$F_\lambda(z) = z^2 + \frac{\lambda}{z}.$$

together with a magnification of a certain region along the negative real axis.

The holes in this parameter plane correspond to parameter values for which the Julia set is a Sierpinski curve. See [4] for a complete discussion of these Sierpinski curve Julia sets. Note the existence of a small copy of a Mandelbrot set in this image. The parameters λ_n described in this paper are drawn from the centers of the main cardioids of these Mandelbrot sets.

Note that these Mandlebrot sets are somewhat different in appearance from many of the other baby Mandelbrot sets in this picture. The small copies of the Mandelbrot sets whose cusp meets the outer boundary of the parameter plane also seems to touch many of the other holes in the parameter plane. This is quite different from the Mandelbrot sets from which our parameters are drawn: they do not seem to extend to any of the holes. Indeed, we conjecture that these baby Mandelbrot sets are also "buried" in the sense that there are no parameters in these sets that also lie on the boundaries of one of the Sierpinski curve holes in the parameter plane.

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