The formal ball model for $Q$-categories

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We generalize the construction of the formal ball model for metric spaces due to A. Edalat and R. Heckmann to obtain computational models for separated $Q$-categories. We fully describe (a) Yoneda complete and (b) continuous Yoneda complete $Q$-categories via their formal ball models. Our results yield solutions to two open problems in the theory of quasi-metric spaces: we show that (a) a quasi-metric space $X$ is Yoneda complete iff its formal ball model is a dcpo, and (b) a quasi-metric space $X$ is continuous and Yoneda complete iff its formal ball model $B_X$ is a domain that admits a simple characterisation of approximation.

1. Introduction

One of the goals of domain theory (Abramsky and Jung 1994; Gierz et al. 2003) is to identify, abstract and axiomatize computational features of spaces, and to provide the spaces with appropriate computational models. The work of Edalat on analysis and measure theory (Edalat 1995a; Edalat 1995b; Edalat et al. 2003; Edalat and Lieutier 2004) is an excellent example of how this goal is achieved in practice. In domain theory we say that a topological space has a computational model if it is homeomorphic to the space of maximal points of a domain endowed with the subspace Scott topology (Lawson 1997; Lawson 1998). A particularly direct construction of computational models for metric spaces is due to Edalat and Heckmann (Edalat and Heckmann 1998): Let $X$ be a metric space. Then the set $B_X$ of formal balls is defined as:

$$B_X = \{(x, r) \mid x \in X, r \geq 0\} \subseteq X \times [0, \infty)$$

$$(x, r) \leq (y, s) \iff X(x, y) \leq r - s.$$ (†)

Edalat and Heckmann prove that the poset $B_X$ is a computational model of $X$ that is continuous with

$$(x, r) \ll (y, s) \iff X(x, y) < r - s$$ (††)

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and, moreover, $X$ is a complete metric space iff $B_X$ is a dcpo. Furthermore, $X$ is separable iff $B_X$ is $\omega$-continuous. In addition they give an order-theoretic proof of the Banach fixed point theorem and demonstrate that probabilistic powerdomains of $B_X$ provide a computational framework for measure theory on Polish spaces.

Edalat and Heckmann’s model makes it completely transparent that metric and order-theoretic notions, e.g. Cauchy sequences in $X$ vs. $\omega$-chains in $B_X$, limits vs. suprema, are intimately related. On the other hand, for arbitrary $X$, $B_X$ has a fairly general order structure, and so it becomes possible to classify metric spaces according to additional order properties possessed by $B_X$. As a step in this direction, Lawson (Lawson 2008) shows e.g. that for any compact metric space the formal ball model is an FS-domain.

2. Motivation

A guiding force for obtaining results of this paper was a simultaneous admiration and dissatisfaction with the very recent trials to generalise Edalat and Heckmann’s construction to spaces with non-symmetric distances. On the one hand (Ali-Akbari et al. 2009; Romaguera and Valero 2009; Romaguera and Valero) convinced us that symmetry of the distance is not crucial in the construction: they established that for a quasi-metric space $X$: (i) $X$ is sequentially Yoneda complete iff $B_X$ is an $\omega$-ccpo; (ii) if $X$ is Yoneda complete, then $B_X$ is a dcpo; (iii) all four conditions mentioned so far are equivalent when $X$ is $T_1$; (iv) $X$ is Smyth complete iff $B_X$ is a domain with approximation given by $(\dagger\dagger)$; (v) if $X$ is algebraic and Yoneda complete, then $B_X$ is a domain. On the other hand, outside the $T_1$ case, we still have no full characterisation of these quasi-metric spaces which have directed-complete (resp. continuous and directed-complete) formal ball models, and hence the conceptual clarity of the original construction is still not met.

In order to improve, we need to update both implications (ii) and (v) to equivalences, which, on the face of it, looks challenging. In order to prove that “if $B_X$ is a dcpo, then $X$ is complete”, Edalat and Heckmann heavily relied on the fact that in the metric case every directed subset of $B_X$ contains a cofinal $\omega$-chain, which is, in general, impossible in quasi-metric spaces. Thus, a proof of equivalence in (ii) will ask for a different approach.

The problem with (v) is more involved. It seems unlikely that implication (v), in the above form, can be replaced by equivalence, since we would have to match a stronger notion of algebraicity of the space with the continuity of the model. It is reasonable to expect that the class of all quasi-metrics that have continuous models is properly containing algebraic quasi-metric spaces.

Our paper provides both solutions. The desired converse to (ii) is based on a natural idea presented in Lemma 7.8. To solve (v) we use results from (Waszkiewicz 2009), where we worked out a definition of a continuous $Q$-category, and hence of a continuous quasi-metric. Here we show, in Theorem 9.1, that all continuous Yoneda complete quasi-metric spaces admit formal ball models that are domains (and conversely: if the formal ball model $B_X$ is a domain with a simply characterized way-below relation, then $X$ is continuous and Yoneda complete).
3. The scope of our results

In fact we are going to work not with quasi-metric spaces but with separated categories that are enriched in value quantales. There are pros and cons of this approach but we believe that the virtues outnumber the disadvantages. Thanks to abstraction from quasi-metric spaces to \(Q\)-categories we can:

1. understand exactly which features of the real line permit transport of properties between \(X\) and \(B_X\). It is actually a perfect compatibility of + and < that is crucial, while e.g. considerations about speed of convergence of sequences (like the fact that \(2^{-n} = \sum_{k>n} 2^{-k}\)) are utterly unimportant;
2. take advantage from the variety of \(Q\). For example our results apply uniformly to formal balls over real numbers, natural numbers or other quantales of values;
3. enjoy economy of notation: directed-completeness, Yoneda completeness, sequential Yoneda completeness and other forms of completeness can be introduced at once, since they are special cases of \(Q\)-categorical relative cocompleteness as explained in Section 6.

In this paper we follow the approach of (Rutten 1998) that by Fact 6.8 subsumes the quasi-metric case considered in (Ali-Akbari et al. 2009; Romaguera and Valero 2009). Following Rutten, up to duality, for a separated \(Q\)-category \(X\), we define the \(Q\)-category \(B_X\) of formal balls to be a certain subcategory of \(Q\)-functors of type \(X \to Q\); however \(B_X\) still plays its traditional role, since it remains partially ordered, and in favorable circumstances \(X\) fully and faithfully embeds onto the maximal elements of \((B_X, \leq_B X)\).

Our main contribution is then a demonstration that a separated \(Q\)-category \(X\) over a value quantale \(Q\) is Yoneda complete iff \(B_X\) is \(J_3\)-cocomplete iff \((B_X, \leq_B X)\) is directed-complete. Likewise, \(X\) is continuous and Yoneda complete iff \(B_X\) is continuous and \(J_3\)-cocomplete iff \((B_X, \leq_B X)\) is a domain that admits a simple characterisation of approximation.

4. \(Q\)-categories

Since (Lawvere 1973) we know that partial orders and metric spaces are examples of categories enriched in a closed category. More precisely, posets and (generalized) metric spaces are certain \(Q\)-categories, i.e. categories enriched in a unital, commutative quantale \(Q\). There are several major papers about the structure of general and particular \(Q\)-categories: (Rutten 1996; Flagg and Kopperman 1997; Flagg 1997) — on their topologies; (America and Rutten 1989; Wagner 1994; Flagg and Kopperman 1995) — on recursive domain equations; (Bonsangue et al. 1998; Vickers 2005) — on powerspaces; (Zhang and Fan 2005; Waszkiewicz 2009) — on approximation and continuity, etc. Recently we have been witnessing a development of a unified categorical/algebraic description of many familiar elementary structures in mathematics: topology, uniformity, order, metric (Clementino and Hofmann 2003; Clementino and Tholen 2003; Clementino et al 2004; Hofmann 2007) that is based on \(Q\)-categories and their further substantial generalizations.

In this paper we assume some familiarity with categories enriched in a commutative
unital quantale \( Q = (Q, \leq, \otimes, 1) \). Such \( Q \) satisfies conditions (q1), (q4) and (q5a) of Defn. 5.2, and our major example is the quantale \(([0, \infty], \geq, +, 0)\) (note that its order is opposite to the natural one). We refer to (Kelly 1982; Clementino and Hofmann 2009; Lai and Zhang 2007) for a basic theory of \( Q \)-categories and to (Kostanek and Waszkiewicz 2010; Waszkiewicz 2009) for elementary introductions to the basic theory. We recall that a \( Q \)-category \( X \) is a set \( X \) with a function \( X : X \times X \to Q \), called the structure on \( X \), with two properties: (1) \( X(x, x) = 1 \) for all \( x \in X \); (2) \( X(x, y) \otimes X(y, z) \leq X(x, z) \) for all \( x, y, z \in X \). In the case \( Q = [0, \infty) \) the conditions above correspond respectively to the assumption of self-distance being zero and to the triangle inequality. A \( Q \)-category is (a) separated if \( X(x, y) = X(y, x) = 1 \) implies \( x = y \), for all \( x, y \in X \); (b) \( T_1 \) if \( X(x, y) = 1 \) implies \( x = y \), for all \( x, y \in X \). For example a separated \([0, \infty)\)-category is a quasi-metric space, where points can possibly be at infinite distance. In our paper \( Q\text{-Cat} \) denotes the category of \( Q \)-categories, where morphisms, called \( Q \)-functors, are maps \( f : X \to Y \) such that \( X(x, z) \leq Y(fx, fz) \) for all \( x, z \in X \). Any \( Q \)-category \( X \) is preordered by the relation \( x \preceq_X y \) iff \( 1 \leq X(x, y) \), which is antisymmetric iff \( X \) is separated. Clearly, \( Q \)-functors are \( \preceq_X \)-preserving.

For any two \( Q \)-categories \( X, Y \), also \( Y^X \) is a \( Q \)-category if considered with the structure \( Y^X(f, g) := \bigwedge_{x \in X} Y(fx, gx) \). The induced order on \( Y^X \) is pointwise. The quantale \( Q \) is made into a separated \( Q \)-category by \( Q(a, b) := a \to b \) (see Defn. 5.2.(q5a) below). Its induced order coincides with the original order on \( Q \). By \( X^\circ \) we mean a \( Q \)-category dual to \( X \). \( X^\circ \) is defined as \( Q^{X^\circ} \), that is \( X^\circ(f, g) = \bigwedge_{x \in X} (fx \to gx) \). For any \( X \), we have the Yoneda \( Q \)-functor \( y : X \to \mathcal{X} , yx = X(-, x) \). The Yoneda \( Q \)-functor is fully faithful. Furthermore, for all \( x \in X \) and \( f \in \mathcal{X} \), we have \( \mathcal{X}(yx, f) = fx \), and this equality is the statement of the Yoneda Lemma for \( Q \)-categories.

5. Formal balls in \( Q \)-categories?

As we have announced above, we initially build on (Rutten 1998), where a formal ball in a quasi-metric space \( X \) is defined to be a non-expansive map \( F(r, x) = r + X(x, -) \) of type \( X \to [0, \infty] \). But since we prefer to keep the original order of formal balls from (Edalat and Heckmann 1998), we propose a definition that is dual to Rutten’s.

**Definition 5.1.** Let \( X \) be a \( Q \)-category, \( x \in X \) and \( r \in Q \). A formal ball of center \( x \) and radius \( r \) is a map

\[
\langle x, r \rangle(z) := X(z, x) \otimes r
\]

of type \( X^\circ \to Q \). The induced order is characterized by: \( \langle x, r \rangle \leq_X \langle y, s \rangle \) iff \( r \leq X(x, y) \otimes s \), and agrees with the definition (1) in the case \( Q = ([0, \infty], \geq, +, 0) \).

There are several problems with manipulating formal balls over general \( Q \)-categories, i.e. problems that do not appear when \( Q = [0, \infty] \). For example, in the general case, tensor does not have to preserve nonempty infima — this is a property which is very often used
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in proofs about quasi-metric spaces. Therefore we must impose additional properties on
the quantale $\mathcal{Q}$:

**Definition 5.2.** A **value quantale** is a tuple $\mathcal{Q} = (Q, \preceq, \otimes, 1)$ such that:

(q1) $(Q, \preceq)$ is a complete lattice with the bottom element $\bot$ and the top element $1$
(q2) $(Q, \preceq)$ is completely distributive. This means that for any $a \in Q$,
\[ a = \bigvee \{ b \in Q \mid b \prec a \}, \]
where $b \prec a$ if $\forall A \subseteq Q \ (a \preceq \bigvee A \Rightarrow (\exists c \in A \ (b \preceq c)))$
(q3) the set $\{ a \in Q \mid a \prec 1 \}$ is directed and contains a cofinal $\omega$-chain;
(q4) there exists an associative, commutative operation $\otimes: Q \times Q \rightarrow Q$, called tensor, having $1$ as its unit;
(q5) the operation $x \mapsto x \otimes a$ has a right adjoint $x \mapsto a \triangleright x$;
(q6) preserves all nonempty infima;
(q7) if $a, b, c \neq \bot$, then $b \prec c$ implies $a \otimes b \prec a \otimes c$;
(q7) if $a \neq \bot$ and $b \in Q$, then $a \triangleright (a \otimes b) = b$.

The existence of the cofinal $\omega$-chain in (q3) is needed just for Lemma 7.9.

Now observe that (q7) gives that $a \otimes b \preceq a \otimes c$ implies $b \preceq c$ for $a \neq \bot$. Therefore no Heyting algebra, with the exception of $2$, is a value quantale. Moreover, (q3) and (q6) together imply that all sets $\downarrow b := \{ a \in Q \mid a \prec b \}$ for $b \neq \bot$ are directed. This property, however, implies linearity of the order:

**Fact 5.3.** Let $(Q, \preceq)$ be a complete, completely distributive lattice. The following are equivalent:

(i) for all $a \neq \bot$ the set $\downarrow a$ is directed,
(ii) for all $a \neq \bot$, $\downarrow a = \uparrow a$, where $\uparrow a$ is the set of elements way-below $a$,
(iii) for all $a \in Q$, if $a = x \lor y$, then either $a \preceq x$, or $a \preceq y$,
(iv) $(Q, \preceq)$ is a linear order.

**Proof.** (i)$\Rightarrow$(ii). Let $a \neq \bot$. Suppose $b \ll a$. Since $a = \bigvee \downarrow a$ and the set $\downarrow a$ is directed, there is $z \prec a$ with $b \preceq z$. Hence $b \prec a$ and thus $\downarrow a \subseteq \downarrow a$. The other inclusion follows by definition. (ii)$\Rightarrow$(iii). Case $a = \bot$ is trivial. Hence let $a \neq \bot$. It is easy to see that $\downarrow a = \uparrow a = \downarrow x \lor \downarrow y$. Since $(Q, \preceq)$ is continuous, $\downarrow a$ is directed. However, if a directed set is a union of two sets, then one of them is cofinal in the union. Therefore either $\downarrow a \subseteq \downarrow x$, or $\downarrow a \subseteq \downarrow y$. In the former case $a \preceq x$, in the latter — $a \preceq y$. The rest of the proof is obvious.

**Example 5.4.** The two-element lattice $2 = \{ \bot, \top \}$ with infimum as tensor, is a value quantale. Separated $2$-categories are partial orders, $2$-functors are monotone maps.

**Example 5.5.** Nonnegative reals $([0, \infty], \geq, +, 0)$ — warning: the order is opposite to the natural one! — form a value quantale. Separated $[0, \infty]$-categories are quasi-metric
spaces extended by infinity. They are not $T_1$ in general. $[0, \infty]$-functors are called non-expansive maps.

**Example 5.6.** Let $\mathbb{N}_\bot$ be the chain of natural numbers with bottom, in the reverse order: $\bot < \ldots < 3 < 2 < 1 < 0$. Define tensor to be addition extended by $\bot + n = n + \bot = \bot$ for all $n \in \mathbb{N}_\bot$. Then $\mathbb{N}_\bot$ is a value quantale.

**Example 5.7.** The unit interval $([0, 1], \leq, \cdot, 1)$ in the natural order, with multiplication, is a value quantale.

Now we are ready to set the following:

**Definition 5.8.** Let $Q$ be a value quantale. The formal ball model of a $Q$-category $X$ is a set:

$$B_X := \{ (x, r) \mid x \in X, r \in Q \setminus \{\bot\} \}$$

with the structure:

$$B_X((x, r), (y, s)) := r \diamond (s \otimes X(x, y)).$$

Since

$$\hat{X}(\langle x, r \rangle, \langle y, s \rangle) = \bigwedge_{z \in X} ((X(z, x) \otimes r) \diamond (X(z, y) \otimes s))$$

$$= \bigwedge_{z \in X} (r \diamond (X(z, x) \diamond (X(z, y) \otimes s)))$$

$$= r \diamond \bigwedge_{z \in X} (X(z, x) \diamond (X(z, y) \otimes s))$$

$$= r \diamond (X(x, x) \diamond (X(x, y) \otimes s))$$

$$= r \diamond (1 \diamond (X(x, y) \otimes s))$$

$$= r \diamond (s \otimes X(x, y))$$

the formal ball model is well-defined. The induced order is:

$$\langle x, r \rangle \leq_{B_X} \langle y, s \rangle \iff r \leq s \otimes X(x, y).$$

The relation above is indeed antisymmetric, since $\leq_{B_X}$ is defined pointwise using the order on $Q$.

We conclude this section with some elementary observations:

**Fact 5.9.** Let $Q$ be a value quantale. For any $Q$-category $X$ the following are equivalent:

(i) $X$ is separated;

(ii) $\langle x, r \rangle = \langle y, s \rangle$ iff $x = y$ and $r = s$, for all $\langle x, r \rangle, \langle y, s \rangle \in B_X$.

*Proof.* Clearly $\langle x, r \rangle = \langle y, s \rangle$ implies $r = s$. Moreover, $1 = r \diamond (s \otimes X(x, y)) = r \diamond (r \otimes X(x, y)) = X(x, y)$ using (q7). Similarly, $1 = X(y, x)$. By separateness of $X$, $x = y$. 

For the converse, assume \( X(x, y) = X(y, x) = 1 \). This implies that \( \langle x, 1 \rangle = \langle y, 1 \rangle \) and hence \( x = y \).

**Fact 5.10.** Let \( Q \) be a value quantale. Suppose that a \( Q \)-category \( X \) is separated. Then the following are equivalent:

1. \( X \) is \( T_1 \),
2. \( \max(BX) = \{ \langle x, 1 \rangle \mid x \in X \} \).

*Proof.* Suppose \( X \) is \( T_1 \) and let \( \langle x, 1 \rangle \leq \langle y, s \rangle \). This yields \( 1 \leq X(x, y) \otimes s \), i.e. \( s = 1 \) and \( x = y \). This means \( \langle x, 1 \rangle \in \max(BX) \) for every \( x \in X \). On the other hand, if \( \langle x, r \rangle \in \max(BX) \), then since \( \langle x, r \rangle \leq \langle x, 1 \rangle \), we obtain \( \langle x, r \rangle = \langle x, 1 \rangle \) by maximality.

Conversely, suppose \( X(x, y) = 1 \) for some \( x, y \in X \). Hence \( \langle x, 1 \rangle \leq \langle y, 1 \rangle \). Since \( \langle x, 1 \rangle \) is maximal, \( \langle x, 1 \rangle = \langle y, 1 \rangle \). By separateness of \( X \), \( x = y \), i.e. \( X \) is \( T_1 \). \( \square \)

As a corollary of Facts 5.9, 5.10 we obtain Thm. 7.1. of (Rutten 1998): Suppose \( X \) is \( T_1 \). Then \( y \colon X \to \max(BX) \) is a fully faithful bijection. We also get that the map \( \langle -, - \rangle : X \otimes Q \to BX \) is a \( Q \)-functor, which boils down to the fact that for all \( x, y, r, s \):

\[
X(x, y) \otimes Q(r, s) \leq BX((x, r), (y, s)).
\]

Moreover, the restricted Yoneda embedding \( y \colon X \to BX \) factorises through \( \langle -, -, \rangle \), i.e. \( y = \langle -, -, \rangle \circ \iota \), where \( \iota : X \leftrightarrow X \otimes Q \) is the inclusion \( x \mapsto (x, 1) \).

### 6. Relative cocompleteness

The formal ball model of a \( Q \)-category is no longer just a partially ordered set but a \( Q \)-category. Therefore we have a right to ask how completeness of the modelled space \( X \) affects (co)completeness of the \( Q \)-category \( BX \).

Conveniently for our purposes, various forms of cocompleteness for \( Q \)-categories can be introduced all at once using the notion of relative cocompleteness developed in (Kelly 1982; Albert and Kelly 1998; Kock 1995; Kelly and Schmitt 2005; Kelly and Lack 2000; Schmitt 2006) and further explained for specific cases in (Lai and Zhang 2007; Clementino and Hofmann 2009; Kostanek and Waszkiewicz 2010; Waszkiewicz 2009). The idea is that for every \( Q \)-category \( X \) we consider a subset \( JX \subseteq \hat{X} \) that makes \( J \) qualify as a *class of weights* (according to definition, a functor \( J : \textbf{Q-Cat} \to \textbf{Q-Cat} \) is a class of weights if (a) for every \( Q \)-category \( X \), \( JX \subseteq \hat{X} \), (b) \( yx \in JX \) for all \( x \in X \), and (c) for a \( Q \)-functor \( f : X \to Y \), \( J(f) = \overline{f} \), where \( \overline{f} : \hat{X} \to \hat{Y} \) is given by \( \overline{f}(\phi)(y) = \bigvee_{z \in X} (\phi z \otimes Y(y, f z)) \)).

Some well-known examples of classes of weights are:

**Example 6.1.** For any quantale \( Q \), and for any \( Q \)-category \( X \), we can choose \( J_0X = \hat{X} \).

**Example 6.2.** For any value quantale \( Q \) and any \( Q \)-category \( X \), a net \( (x_i)_{i \in I} \subseteq X \) is forward Cauchy if \( \forall \varepsilon \prec 1 \ \exists n \forall i, j \geq n \ (\varepsilon \prec X(x_j, x_i)) \). Define \( \phi \in J_1X \) iff \( \phi = \bigvee_i \bigwedge_{j \geq i} X(-, x_j) \) for some forward Cauchy net \( (x_i)_{i \in I} \) on \( X \).

**Example 6.3.** For any value quantale \( Q \) and any \( Q \)-category \( X \), define \( \phi \in J_2X \) iff \( \phi = \bigvee_n \bigwedge_{j \geq n} X(-, x_j) \) for some forward Cauchy sequence \( (x_n)_{n \in \omega} \) in \( X \).
We further add:

**Example 6.4.** Let $Q$ be a value quantale and let $X$ be a $Q$-category. Consider the formal ball model on $X$ and define $\Phi \in J_3(BX)$ iff $\Phi = \bigvee_{i \in I} \bigwedge_{j \geq i} BX(-, \langle x_j, r_j \rangle)$ for a forward Cauchy net $(x_i)_{i \in I}$ on $X$ and a monotone net $(r_i)_{i \in I}$ in $Q$.

**Example 6.5.** Let $Q$ be a value quantale and let $X$ be a $Q$-category. Consider the formal ball model on $X$ and define $\Phi \in J_4(BX)$ iff $\Phi = \bigvee_{n \in \omega} \bigwedge_{j \geq n} BX(-, \langle x_j, r_j \rangle)$ for a forward Cauchy sequence $(x_n)_{n \in \omega}$ on $X$ and an $\omega$-chain $(r_n)_{n \in \omega}$ in $Q$.

**Definition 6.6.** We will say that $\phi$ is a $J$-ideal on $X$ if $\phi \in JX$.

**Definition 6.7.** A $Q$-category $X$ is $J$-cocomplete iff there exists a $Q$-functor $S : JX \to X$ such that $X(S \phi, x) = \hat{X}(\phi, yx)$ for any $\phi \in JX$ and $x \in X$, i.e. iff $S \dashv y$ in $Q$-Cat. In this case we say that $S \phi$ is the supremum of the $J$-ideal $\phi$.

For example $Q$ itself is a $J$-cocomplete $Q$-category, for any choice of $J$. Indeed, if $\phi \in JQ$, then $S \phi = \bigvee_{b \in Q} (\phi b \otimes b)$. Furthermore:

**Fact 6.8.** A quasi-metric is Yoneda complete iff it is $J_1$-cocomplete, and is sequentially Yoneda complete iff it is $J_2$-cocomplete.

**Proof.** $J_1$-cocompleteness amounts to existence of a map $S : J_1X \to X$ such that $X(S \phi, x) = \hat{X}(\phi, yx)$ for any $\phi \in J_1X$ and $x \in X$. But since $\phi = \inf_{i \in I} \sup_{j \geq i} X(-, x_j)$ for some net $(x_i)_{i \in I}$, and by Lemma 46 of (Flagg et al. 1996) we have that $\hat{X}(\phi, yx) = \inf_{i \in I} \sup_{j \geq i} X(x_i, x)$, it follows that each $S \phi$ is the Yoneda limit of the net $(x_i)_{i \in I}$. Hence a quasi-metric $X$ is Yoneda complete iff it is $J_1$-complete. The proof for the class $J_2$ is analogous.

This is the reason why we wish to propose the following nomenclature for $Q$-categories:

**Definition 6.9.** Let $Q$ be a value quantale. A $Q$-category $X$ is:

(a) Yoneda complete if it is $J_1$-cocomplete;
(b) sequentially Yoneda complete if it is $J_2$-cocomplete.

7. **Main result on completeness of $X$ and $BX$**

**Theorem 7.1.** Let $Q$ be a value quantale and let $X$ be a separated $Q$-category. Then the following are equivalent:

(i) $X$ is Yoneda complete;
(ii) $BX$ is $J_3$-cocomplete;
(iii) $(BX, \leq_{BX})$ is a dcpo.

Also the following are equivalent:

(iv) $X$ is sequentially Yoneda complete;
(v) $BX$ is $J_4$-cocomplete;
(vi) $(BX, \leq_{BX})$ is an $\omega$-ccpo (i.e. has sups of $\omega$-chains).
Remark 7.2. By Prop. 2 of (Romaguera and Valero), if $X$ is a $T_1$ quasi-metric, then all six conditions (i)-(vi) are equivalent.

Proof. The proof consists of several technical lemmata gathered below. We may not have chosen the most economical proof structure, but we wished to highlight the interplay between Yoneda completeness of $X$ and the order structure of $(BX, \leq_{BX})$. Concretely, the equivalences of (i)-(ii) and (iv)-(v) follow from Lemmata 7.5, 7.6; and the equivalence (ii)-(iii) is a consequence of Lemmata 7.7, 7.8. The equivalence (iv)-(vi) follows from Lemma 7.9.

The lemmata that we present below relate ideals on $X$ to ideals on $BX$, and to directed subsets of $BX$. The proofs are in most cases elementary, in the sense that they can be found by unwinding the definitions and a smart use of the following rules:

- $d \prec a$ implies $d \prec \varepsilon \otimes c$, for some $\varepsilon \prec 1$ and $c \prec a$,
- if $d \prec a$ implies $d \leq b$, for all $d \in Q$, then $a \leq b$,
- $d \prec a$ implies $d \prec b \otimes c \prec a$, for some $b,c \in Q$,

that are consequences of complete distributivity of $Q$, together with the (non-trivial) fact that:

$$\tilde{X}(\phi, yz) = \bigvee_{i \in I} \bigwedge_{j \geq i} X(x_j, z), \text{ for all } z \in X, \text{ and } \phi = \bigvee_{i \in I} \bigwedge_{j \geq i} X(\varepsilon, x_j), \text{ where}$$

that can be distilled from Lemma 46 of (Flagg et al. 1996).

For the rest of this section we assume that $Q$ is a value quantale and that $X$ is a separated $Q$-category.

Lemma 7.3. If $\langle x_i, r_i \rangle_{i \in I}$ is a forward Cauchy net (resp. sequence) in $BX$ such that $(r_i)_{i \in I}$ is a monotone net, then $(x_i)$ is a forward Cauchy net (resp. sequence) in $X$.

Proof. The proof works verbatim for nets and sequences. Let $\varepsilon \prec 1$. Choose $\eta, \delta \prec 1$ such that $\varepsilon \prec \eta \otimes \delta \prec 1$. Since $\delta \otimes r \prec r$, for $r = \bigvee r_i$, there exists $k \in I$ such that $\delta \otimes r \leq r_k$. Choose $N \in I$ such that $N \leq i \leq j$ implies $\eta \prec BX((x_i, r_i), (x_j, r_j)) = r_i \rightarrow (X(x_i, x_j) \otimes r_j)$, whence $\eta \otimes r_i \leq X(x_i, x_j) \otimes r_j$. Choose $s, n, m \in I$ with $N, k \leq s \leq n \leq m$. Then $\eta \otimes \delta \otimes r \leq \eta \otimes r_k \leq \eta \otimes r_n \leq X(x_n, x_m) \otimes r_m \leq X(x_n, x_m) \otimes r$. Therefore $\varepsilon \prec \eta \otimes \delta \leq X(x_n, x_m)$ for all $s \leq n \leq m$, as required.

Lemma 7.4. If $\langle x_i, r_i \rangle_{i \in I}$ is a directed subset (resp. an $\omega$-chain) of $(BX, \leq_{BX})$, then $(x_i)_{i \in I}$ is a forward Cauchy net (resp. a forward Cauchy sequence) in $X$ and $(r_i)_{i \in I}$ is a monotone net (resp. $\omega$-chain) of $Q$.

Proof. If $i \leq j$, i.e. $\langle x_i, r_i \rangle \leq_{BX} \langle x_j, r_j \rangle$, then $r_i \leq r_j$. Thus $(r_i)$ is a monotone net. Since the directed set $\langle x_i, r_i \rangle_{i \in I}$ is clearly a forward Cauchy net, we can apply Lemma 7.3 to demonstrate that $(x_i)_{i \in I}$ is a forward Cauchy net on $X$. The proof for the countable case works verbatim.

Lemma 7.5. If $X$ is Yoneda (resp. sequentially Yoneda) complete, then $BX$ is $J_3$-cocomplete (resp. $J_4$-cocomplete).
Proof. We start with the non-sequential case. Let $\Phi$ be an ideal on $\mathbf{B}X$, say $\Phi = \bigvee_i \wedge_{j \geq i} \mathbf{B}X(\cdot, x, r_j)$. Since by Lemma 7.3, $\phi = \bigvee_i \wedge_{j \geq i} X(\cdot, x, r_j)$ is an ideal on $X$, by assumption we can take $x = S\phi$. We claim that $S\Phi = \langle x, r \rangle$ for $r = \bigvee_i r_i$.

Perhaps that most delicate part is the proof of an auxiliary claim: $\Phi \leq y(x, r)$. For, take any $(y, s) \in \mathbf{B}X$. Then $s \otimes \Phi((y, s)) = \bigvee_i \bigwedge_{j \geq i} s \otimes (s \rightarrow (X(y, x) \otimes r_j)) \leq \bigvee_i \bigwedge_{j \geq i} X(y, x) \otimes r_j$. Now, let $\varepsilon \prec s \otimes \Phi((y, s))$. We can thus find $\delta, \eta \prec 1$ and $i \in I$ such that

$$\forall j \geq i \quad \varepsilon \prec X(y, x) \otimes r_j \otimes \delta \otimes \eta.$$  \hfill (1)

Since $x = S\phi$ we have $1 = X(S\phi, x) = \hat{X}(\phi, yx) = \bigvee_{k \in I} \bigwedge_{l \geq k} X(x, l)$, and hence

$$\exists_{k \in I} \forall_{l \geq k} \quad \delta \prec X(x, l).$$ \hfill (2)

Since $(x_n)_{n \in I}$ is a forward Cauchy net,

$$\exists_{N \in I} \forall_{m \geq N} \quad \eta \otimes r_n \leq X(x_m, x_m) \otimes r_m.$$ \hfill (3)

Hence if one chooses $m \geq n \geq i$, $N, k$, then by (1)-(3): $\varepsilon \leq X(y, x_n) \otimes r_n \otimes \delta \otimes \eta \leq X(y, x_n) \otimes X(x_n, x_m) \otimes X(x_m, x) \otimes r_m \leq X(y, x) \otimes r$. Since $\varepsilon \prec 1$ was an arbitrary element totally below $s \otimes \Phi((y, s))$, we have $s \otimes \Phi((y, s)) \leq X(y, x) \otimes r$, which gives $\Phi((y, s)) \leq s \rightarrow (X(y, x) \otimes r) = y(x, r)((y, s))$. We have now proved that:

$$\Phi \leq y(x, r).$$ \hfill (4)

Therefore $1 = \mathbf{B}X(\Phi, y(x, r))$ and thus

$$\mathbf{B}X((x, r), (z, t)) \leq \mathbf{B}X(\Phi, y(x, r)) \otimes \mathbf{B}X((y, s), (z, t)) \leq \mathbf{B}X(\Phi, y(z, t)).$$ \hfill (5)

On the other hand, if $\varepsilon \prec \mathbf{B}X(\Phi, y(z, t)) = \bigvee_{i \in I} \bigwedge_{j \geq i} \mathbf{B}X((x, j), (z, t))$, then there exists $i \in I$ with $\varepsilon \prec \bigwedge_{j \geq i} (r_j \rightarrow (X(x, j, z) \otimes t))$. Then for all $j \geq i$, $\varepsilon \rightarrow r_j \rightarrow (X(x, j, z) \otimes t)$, and so $\varepsilon \otimes r_j \leq X(x, j, z) \otimes t$. Hence $\varepsilon \otimes r_i = \bigwedge_{j \geq i} (\varepsilon \otimes r_j) \leq \bigwedge_{j \geq i} X(x, j, z) \otimes t$, and consequently $\varepsilon \otimes r = \bigwedge_{j \geq i} (\varepsilon \otimes r_j) \leq \bigwedge_{j \geq i} X(x, j, z) \otimes t$. Since $\varepsilon$ was arbitrary,

$$\mathbf{B}X(\Phi, y(z, t)) \leq r \rightarrow (X(x, z) \otimes t) = \mathbf{B}X((x, r), (z, t)).$$ \hfill (6)

By (5) and (6) and Definition 6.7 we get $S\Phi = \langle x, r \rangle$.

The sequential version of the above proof follows the same steps.

\begin{lemma}
If $\mathbf{B}X$ is $J_\omega$-complete (resp. $J_\omega$-cocomplete), then $X$ is Yoneda (resp. sequentially Yoneda) complete.
\end{lemma}

\begin{proof}
Suppose $\phi = \bigvee_i \bigwedge_{j \geq i} X(\cdot, x, r_j)$ is an ideal on $X$. By assumption we know that the ideal $\Phi = \bigvee_{i \in I} \bigwedge_{j \geq i} \mathbf{y}((x, j), 1)$ has a supremum $S\Phi = \langle x, r \rangle$. By definition of supremum, $1 = \mathbf{B}X(\Phi, y(x, r)) \leq r$. Hence $r = 1$, and consequently, for any $z \in X$, $X(x, z) = \mathbf{B}X((x, r), (z, 1)) = \mathbf{B}X(\Phi, y(z, 1)) = \bigvee_i \bigwedge_{j \geq i} X(x, j, z)$, which yields $x = S\phi$.
\end{proof}

\begin{lemma}
If $X$ is Yoneda (resp. sequentially Yoneda) complete, then $\mathbf{B}X \leq \mathbf{B}X$ is a dcpo (resp. $\omega$-ccpo).
\end{lemma}
The formal ball model for \(Q\)-categories

**Proof.** We start with the non-sequential case. Let \(\{(x_i, r_i)\}_{i \in I}\) be a directed subset of \(BX\). By Lemma 7.4, \(\phi = \bigvee_i \bigwedge_{j \geq i} X(\neg, x_j)\) is an ideal on \(X\). Let \(x = S\phi\) and \(r = \bigvee r_i\). We will show that \(\bigvee\{x_i, r_i\} = \langle x, r \rangle\).

Firstly, observe that \(1 = X(\phi, x) = \bar{X}(\phi, yr) = \bigvee_i \bigwedge_{j \geq i} X(x_j, x)\). Therefore:

\[
\forall_{i \in I} \exists_{k \in I} \forall_{l \geq k} \delta \prec X(x_i, x).
\] (7)

Let \(i \in I\) and \(\varepsilon \prec r_i\). By directedness, for all \(j \geq i\) we have \(\varepsilon \prec r_i \leq X(x_i, x_j) \otimes r_j\). Hence for a fixed \(j \geq i\) we can find \(\delta \prec 1\) such that \(\varepsilon \prec r_i \leq X(x_i, x_j) \otimes \delta \otimes r_j\). By (7) we obtain:

\[
\exists_{k \in I} \forall_{l \geq k, j} \varepsilon \prec X(x_i, x_j) \otimes X(x_l, x) \otimes r_j.
\] (8)

However since \(l \geq j\),

\[
\varepsilon \leq X(x_j, x_l) \otimes X(x_l, x) \otimes r_i \leq X(x_i, x) \otimes r.
\] (9)

Since \(\varepsilon \prec r_i\) was arbitrary we conclude that \(r_i \leq X(x_i, x) \otimes r\), that is \(\langle x, r \rangle\) is an upper bound for \(\{(x_i, r_i)\}_{i \in I}\) in \(BX\).

Suppose now that \(\langle x_i, r_i \rangle \leq_{BX} \langle y, s \rangle\) for some \(y \in X\) and \(s \in Q\). Fix \(i \in I\). Let \(j \geq i\). Then \(r_i \leq r_j \leq X(x_j, y) \otimes s\) and so \(r_i \leq \bigwedge_{j \geq i} X(x_j, y) \otimes s\). Consequently \(r = \bigvee_i r_i \leq \bigvee_i \bigwedge_{j \geq i} X(x_j, y) \otimes s = X(x, y) \otimes s\). That is, \(\langle x, r \rangle \leq_{BX} \langle y, s \rangle\).

The sequential version of the above proof follows the same steps. 

Now the distinction between directed-sets and \(\omega\)-chains becomes important, hence the next two lemmata, albeit being analogous statements, must be proved separately.

**Lemma 7.8.** If \((BX, \leq_{BX})\) is a dcpo, then \(X\) is Yoneda complete.

**Proof.** Let \(\{x_i\}_{i \in I}\) be a forward Cauchy net in \(X\). Set \(a_i := \bigwedge_{j \geq i} X(x_i, x_j)\) for all \(i \in I\). First, we will prove that \(D := \bigcup_{i \in I} \{x_i, \delta_i\} \mid \delta \prec a_i\) is a directed subset of \(BX\). It is clearly nonempty. Suppose that \(\{x_{i_1}, \delta_1\}, \{x_{i_2}, \delta_2\} \in D\). Then for some \(\varepsilon \prec 1\), \(\delta_1 \prec \bigwedge_{j \geq i_1} X(x_i, x_j) \otimes \varepsilon\) and \(\delta_2 \prec \bigwedge_{j \geq i_2} X(x_{i_2}, x_j) \otimes \varepsilon\). By Cauchyness, there is \(k \geq i_1, i_2\) with \(\varepsilon \prec \bigwedge_{j \geq k} X(x_{i_1}, x_j)\). Then \(\{x_{i_1}, \delta_1\}, \{x_{i_2}, \delta_2\} \leq_{BX} \langle x, \varepsilon \rangle\).

By assumption, \(D\) has a supremum, say \(\langle x, r \rangle\). Clearly, \(r = 1\). We show that \(x\) is the supremum of \(\phi := \bigvee_i \bigwedge_{j \geq i} X(\neg, x_j)\).

Fix \(y \in X\), \(i \in I\). Then for any \(k \in I\), \(a_i \otimes \bigwedge_{j \geq k} X(x_j, y) \leq X(x_i, x_j) \otimes X(x_j, y) \leq X(x_i, y)\) where \(i, k \leq j\). Therefore for all \(\delta \prec a_i\), \(\delta \otimes \bigvee_{k \in I} \bigwedge_{j \geq k} X(x_j, y) \leq X(x_i, y)\). Set \(s := \bigvee_k \bigwedge_{j \geq k} X(x_j, y)\), so we can write compactly \(\delta \otimes s \leq X(x_i, y)\). Therefore, \(\{x_i, s \otimes \delta\} \leq_{BX} \langle y, 1 \rangle\) for all \(i \in I\).

Observe that for a fixed \(y\), \(s\) is also a fixed element of \(Q\), and thus with a slight change of perspective we can use it to get a function: \((s \otimes -) : BX \to BX\) given by \(\langle z, t \rangle \mapsto \langle z, s \otimes t \rangle\). It is easy to see that this function is monotone, so images of directed sets are directed. In particular \(s \otimes D := \bigcup_{i \in I} \{x_i, s \otimes a_i\} = \bigwedge_{i \in I} X(x_i, y)\) is directed.

Moreover, since \(BX\) is directed-complete, the map \((s \otimes -)\) is Scott-continuous: firstly, note that \(\bigvee(s \otimes D)\) exists, since the formal ball model is directed-complete. Secondly, by
the proof of Lemma 7.7, \( (s \otimes D) = (x, r) \), where \( r = \bigvee \{ s \otimes \delta \mid \delta \prec a_i \} = s \otimes \bigvee \{ \delta \mid \delta \prec a_i \} = s \otimes s = s \otimes \bigvee D \).

Coming back to the main line of the proof, we thus have \( (x, s) = (s \otimes D) \leq_{\text{BX}} (y, 1) \), and this yields: \( \bar{X}(\phi, yy) = \bigvee_k \bigwedge_{j \geq k} X(x_j, y) \leq X(x, y) \).

The proof will be finished if we supply the opposite inequality, namely: \( X(x, y) \leq \bar{X}(\phi, yy) \). But since \( (x, 1) \) is a supremum of \( D \), for all \( i \in I \) we have \( \bigwedge_{j \geq i} X(x_i, x_j) \leq X(x_i, x) \). In particular, for a fixed \( N \in I \), \( \bigwedge_{j \geq N} X(x_i, x_j) \leq \bigwedge_{i \geq N} X(x_i, x) \).

Now we use Cauchyness to conclude \( 1 = \bigwedge_{N \in I} \bigwedge_{j \geq N} X(x_i, x_j) \leq \bigwedge_{N \in I} \bigwedge_{i \geq N} X(x_i, x) = \bar{X}(\phi, yx) \).

Hence \( X(x, y) = \bar{X}(yx, yy) = \bar{X}(\phi, xy) \otimes \bar{X}(yx, yy) \leq \bar{X}(\phi, yy) \), as required.

\( \square \)

\textbf{Lemma 7.9.} If \( (\text{BX}, \leq_{\text{BX}}) \) is an \( \omega \)-ecpo, then \( X \) is sequentially Yoneda complete.

\textbf{Proof.} Let \( (x_n) \) be a forward Cauchy sequence in \( X \). Let \( \varepsilon_0 < \varepsilon_1 < \ldots < 1 \) be a sequence in \( Q \) with \( \bigvee \varepsilon_n = 1 \), which exists by \( (q3) \).

Set \( \delta_0 := \varepsilon_0 \). Then one has:

\[ \exists \delta_1 < 1 \exists n_0 \in I \forall j \geq n_0 \ (\delta_0 \prec \delta_1 \land \delta_0 \prec X(x_{n_0}, x_j) \otimes \delta_1), \]

\[ \exists \delta_2 < 1 \exists n_1 \geq n_0 \forall j \geq n_1 \ (\varepsilon_1 \prec \delta_2 \land \delta_1 \prec X(x_{n_1}, x_j) \otimes \delta_2), \]

and consequently, for all \( k \in \omega \):

\[ \exists \delta_{k+2} < 1 \exists n_{k+1} \geq n_k \forall j \geq n_{k+1} \ (\varepsilon_{k+1} \prec \delta_{k+1} \land \delta_k \prec X(x_{n_{k+1}}, x_j) \otimes \delta_{k+2}). \]

In this way we constructed an \( \omega \)-chain in \( \text{BX} \):

\[ \langle x_{n_0}, n_0 \rangle \leq \langle x_{n_1}, n_1 \rangle \leq \langle x_{n_2}, n_2 \rangle \leq \langle x_{n_3}, n_3 \rangle \leq \ldots \leq \langle x_{n_k}, n_k \rangle \leq \ldots \]

By assumption, this chain has a supremum \( (x, r) \). By construction, \( r = 1 \). We must show that \( x \) is the least upper bound of \( \phi := \bigvee_i \bigwedge_{j \geq i} X(-, x_j) \).

Fix \( y \in X \) and \( i \in \omega \). Similarly to the previous lemma, we can show that \( \delta_i \otimes s \leq X(x_n, y) \), where \( s := \bigvee_{k \in \omega} \bigwedge_{j \geq k} X(x_n, y) \). This inequality means that \( \langle x_n, \delta_i \otimes s \rangle \leq (y, 1) \). Therefore, \( \langle x, s \rangle = s \otimes (x, 1) \).

On the other hand, \( X(x, y) = \bigvee_k \bigwedge_{j \geq k} X(x_n, x) \otimes X(x_n, y) = \bigvee_k \bigwedge_{j \geq k} (X(x_n, x) \otimes X(x_n, y)) \leq \bigvee_k \bigwedge_{j \geq k} X(x_n, y) \).

Now, in the final step, we turn to the original sequence \( (x_n)_{n \in \omega} \). By the conclusion of the previous paragraph and since \( (x_n)_{n \in \omega} \) is forward Cauchy, \( \bigwedge_{n \geq k} X(x_n, y) = 1 \).

Thus

\[ \bigvee_k \bigwedge_{n \geq k} X(x_n, y) \geq \bigvee_k \bigwedge_{n \geq k} (X(x_n, x) \otimes X(x_n, y)) = \bigvee_k \bigwedge_{n \geq k} X(x_n, x) \otimes X(x, y) = X(x, y), \]

however the converse follows from \( X(x, y) = \bigvee_k \bigwedge_{j \geq k} X(x_n, y) \geq \bigvee_k \bigwedge_{n \geq k} X(x_n, y) \).

Therefore \( X(x, y) = \bigvee_k \bigwedge_{n \geq k} X(x_n, y) = \bar{X}(\phi, yy) \), which yields \( x = S\phi \).

\( \square \)

\textbf{8. When is \( (\text{BX}, \leq_{\text{BX}}) \) a domain? — an algebraic subcase}

For any metric space \( (X, d) \), its formal ball model is a continuous poset and admits a neat characterisation of approximation relation, namely given by \( (\dagger \dagger) \) in the Introduction. In
the asymmetric case, the situation is more complicated: the natural quasi-metric on the Sorgenfrey line has a continuous formal ball model, where (††) does not hold. This is a paradigmatic counterexample since (Romaguera and Valero) proves that $BX$ is a continuous dcpo (a domain) with approximation characterised by (††) iff $X$ is Smyth-complete. In what follows we obtain an analogous characterisation for general $Q$-categories, although we avoid speaking about Smyth-completeness. This is possible, since Smyth-completeness of a quasi-metric $X$ boils down exactly to Yoneda completeness of $X$ plus the fact that all $x \in X$ are finite (Ali-Akbari et al. 2009; Romaguera and Valero).

Let $Q$ be a value quantale and let $X$ be a separated $Q$-category. Define:

$$
\langle x, r \rangle \triangleleft (y, s) \iff r \prec X(x,y) \otimes s
$$

(10)

It is clear that $\triangleleft$ is an auxiliary relation on $BX$, i.e. $\triangleleft$ is contained in $\leq_{BX}$, and $\triangleleft$ equals the composition: $\leq_{BX} \circ \triangleleft \circ \leq_{BX}$. Moreover:

**Lemma 8.1.** If $\langle x, r \rangle \triangleleft (y, s)$, then there exist $\eta \prec s$ such that $\langle x, r \rangle \triangleleft (y, \eta) \triangleleft (y, s)$.

**Proof.** Since $r \prec X(x,y) \otimes s$, then there exists $\eta \prec s$ with $r \prec X(x,y) \otimes \eta$. Hence $\langle x, r \rangle \triangleleft (y, \eta)$ and $(y, \eta) \triangleleft (y, s)$. \hfill \Box

**Lemma 8.2.** The relation $\triangleleft$ is approximating.

**Proof.** Let $\langle x, r \rangle \in BX$. We have to show that $A = \{(y, s) \mid (y, s) \triangleleft \langle x, r \rangle\}$ is directed with supremum $\langle x, r \rangle$. Directedness follows from the same argument as in the preceding lemma: if $\langle z, t \rangle, (y, s) \triangleleft \langle x, r \rangle$, then $\langle z, t \rangle, (y, s) \triangleleft (y, \eta) \triangleleft (y, s)$ for some $\eta < r$. Since the set $B = \{(x, \eta) \mid \eta < r\}$ is cofinal in $A$, we have $\bigvee B = \bigvee A$, provided at least one supremum exists. But clearly, $\bigvee B = \langle x, r \rangle$. \hfill \Box

Now we are ready to prove:

**Theorem 8.3.** Let $Q$ be a value quantale and let $X$ be a Yoneda complete separated $Q$-category. Then the following are equivalent:

(i) $\langle x, r \rangle \in BX$, is a domain with the approximation $\leq_{BX} = \triangleleft$.

(ii) for any $\langle x, r \rangle \in BX$ the set $\uparrow \langle x, r \rangle = \{(y, s) \mid (x, r) \triangleleft (y, s)\}$ is Scott-open.

(iii) any $x \in X$ is finite, that is: $X(x, S\emptyset) \leq \emptyset x$ for any $\phi \in JX$.

**Proof.** It is a domain-theoretic fact that $\triangleleft$ (as well as any other approximating relation) coincides with way-below iff $\uparrow \langle x, r \rangle$ is Scott-open for any $\langle x, r \rangle \in BX$. Now assume (ii).

Let $x \in X$ be fixed and let $\phi = \vee_{i \in I} \bigwedge_{j \geq i} X(-, x_j)$ be a $J_1$-ideal. By the proof of Lemma 7.8, there exists a directed subset $D$ of $BX$ such that by the proof of Lemma 7.7, $\bigvee D = \langle S\emptyset, 1 \rangle$. Take any $\varepsilon \prec X(x, S\emptyset)$, or, equivalently $\langle x, \varepsilon \rangle \triangleleft \langle S\emptyset, 1 \rangle$. By (ii), there exists $i \in I$ and $\delta \triangleleft \bigwedge_{j \geq i} X(x_i, x_j)$ with $\langle x, \varepsilon \rangle \triangleleft \langle x_i, \delta \rangle$, or equivalently: $\varepsilon \prec X(x_i, x_j).$

Consequently, $\varepsilon \prec \bigvee_{i \in I} X(x_i, x_j) \leq \bigvee_{i \in I} \bigwedge_{j \geq i} X(x_i, x_j) \leq \bigvee_{i \in I} X(x, x_j) = \emptyset x$. This proves (iii). Now assume (iii) and let $\langle x, r \rangle \in BX$ be arbitrary. Let $\langle z_i, t_i \rangle_{i \in I}$ be directed with supremum $\langle z, t \rangle \in \uparrow \langle x, r \rangle$. By Lemma 7.7, $z = S\emptyset$, where $\phi$ is the ideal constructed from the forward Cauchy net $(z_i)_{i \in I}$. Moreover, $(t_i)_{i \in I}$ is directed and $t = \vee_{k \in I} t_k$. By (iii), $X(x, z) = X(x, S\emptyset) \leq \emptyset x = \bigvee_{i \in I} X(x, z_i)$. Thus $r \prec X(x, z) \otimes t$ implies $r \prec \bigvee_{i \in I} \bigwedge_{j \geq i} X(x, z_j) \otimes \bigvee_{k \in I} t_k$. Consequently, there are $i \in I$
and \( k \in I \) with \( r < \bigwedge_{j \geq 1} X(x, z_j) \otimes t_k \). Then any choice of \( j \geq i, k \) yields \( r < X(x, z_j) \otimes t_j \). This means that \( \langle z_j, t_j \rangle \in \mathcal{G}(x, r) \), so \( \mathcal{G}(x, r) \) is Scott-open, as required.

Of course, as a corollary we obtain the characterisation of (Romaguera and Valero) of Smyth-complete quasi-metrics. But there is another conclusion that may be interesting to draw here:

**Corollary 8.4.** For a symmetric \( \mathbb{Q} \)-category \( X \), \( X \) is Yoneda complete iff \( (BX, \leq_{BX}) \) is a continuous dcpo with approximation characterized by the equivalence (10).

*Proof.* Let \( \phi \) be a \( J_1 \)-ideal on \( X, x \in X \). Since it is always true that \( \hat{X}(\phi, x) = \bigvee_i \bigwedge_{j \geq i} X(x, j) = \bigvee_i \bigwedge_{j \geq i} X(x_j) = \phi x \), then assuming \( X(S\phi, x) = \hat{X}(\phi, x) \) we get that every element of \( X \) is finite, and, conversely, the assumption \( \phi x = X(S\phi, x) \) yields Yoneda completeness. \( \square \)

**Example 8.5.** The value quantale \( \mathbb{N}_\perp \) is an algebraic \( \mathbb{N}_\perp \)-category. In fact, all its elements are finite: Indeed, since the unit of \( \mathbb{N}_\perp \), which is 0, satisfies \( 0 < 0 \), for any forward Cauchy net \( (x_i)_{i \in I} \), there is \( N \in I \) such that for all \( j \geq i \geq N \), \( 0 = \mathbb{N}_\perp(x_i, x_j) \). So the net \( (x_i)_{i \geq N} \) is just an eventually increasing chain. However, for any \( J_1 \)-ideal \( \phi \) on \( \mathbb{N}_\perp \) we have \( \phi = \bigvee_{i \in I} \bigwedge_{j \geq i} X(\cdot, x_j) = \bigvee_{i \geq N} \bigwedge_{j \geq i} X(\cdot, x_j) \), hence \( \phi \) must be of the form \( y \in \mathbb{N}_\perp \) for some \( k \geq N \). This means that any \( n \in \mathbb{N}_\perp \) is finite and so \( B\mathbb{N}_\perp \) is a continuous dcpo with approximation characterised by equivalence (10).

9. **When is \( (BX, \leq_{BX}) \) a domain? — full characterisation**

We now show that continuous formal ball models that admit a simple characterisation of the way-below relation arise exclusively from so called continuous \( \mathbb{Q} \)-categories that have been introduced in (Waszkiewicz 2009). We give the necessary definitions in the most concise form (but encourage the reader to consult (Waszkiewicz 2009) for the full account): a map of type \( X \times X \to \mathbb{Q} \) is called a \( \mathbb{Q} \)-relation on \( X \). (Observe that \( \mathbb{2} \)-relations can be identified with relations.) A \( \mathbb{Q} \)-relation \( v: X \times X \to \mathbb{Q} \) is auxiliary if: (i) \( v(x, y) \leq X(x, y) \), and: (ii) \( X(x, y) \otimes v(y, z) \otimes X(z, t) \leq v(x, t) \) for all \( x, y, z, t \in X \). If \( v \) is auxiliary and \( v(\cdot, x) \in J_1(X) \) and \( Sv(\cdot, x) = x \), for all \( x \in X \), then we say that \( v \) is approximating. A \( \mathbb{Q} \)-relation is Scott-continuous if it preserves suprema of \( J_1 \)-ideals wrt the second coordinate, i.e. \( v(x, S\phi) = \bigvee_{z \in X} (\phi z \otimes v(x, z)) \). We define the so called way-below \( \mathbb{Q} \)-relation by:

\[
w(x, y) := \bigwedge_{\{\phi \in J_1(X) | S\phi \text{ exists}\}} (X(y, S\phi) - \phi x).
\]

Clearly, the way-below \( \mathbb{2} \)-relation is just the way-below relation on the poset \( X \).

By definition a separated \( \mathbb{Q} \)-category \( X \) is continuous if its way-below \( \mathbb{Q} \)-relation is approximating. (Thus continuously separated \( \mathbb{2} \)-categories are exactly the continuous posets.) Lemma 3.2 of (Waszkiewicz 2009) gives a further description of this situation: any approximating and Scott-continuous \( \mathbb{Q} \)-relation on \( X \) must coincide with the way-below \( \mathbb{Q} \)-relation. Therefore, \( X \) is continuous iff it admits an approximating, Scott-continuous \( \mathbb{Q} \)-relation. Now we are ready to prove:
Theorem 9.1. Let $\mathcal{Q}$ be a value quantale. If $X$ is a separated, continuous and Yoneda complete $\mathcal{Q}$-category, then $(\mathbf{B}X, \leq_{\mathbf{B}X})$ is a continuous dcpo with the way-below relation characterized by:

$$
\langle y, s \rangle \preceq_{\mathbf{B}X} \langle x, r \rangle \iff s \prec w(y, x) \otimes r,
$$

where $w: X \times X \to \mathcal{Q}$ is the way-below $\mathcal{Q}$-relation on $X$. Conversely, if $(\mathbf{B}X, \leq_{\mathbf{B}X})$ is a continuous dcpo with approximation relation characterized by:

$$
\langle y, s \rangle \preceq_{\mathbf{B}X} \langle x, r \rangle \iff s \prec v(y, x) \otimes r,
$$

for some $\mathcal{Q}$-relation $v: X \times X \to \mathcal{Q}$, then $X$ is a separated continuous and Yoneda complete $\mathcal{Q}$-category and $v$ is in fact the way-below $\mathcal{Q}$-relation on $X$.

Proof. For any $\langle x, r \rangle \in \mathbf{B}X$ consider a set:

$$
B_{(x,r)} = \{ \langle y, s \rangle | s \prec w(y, x) \otimes r \}.
$$

It is nonempty; pick any $s \prec r$ and interpolate to get $s \prec \delta \prec r$. Since $w(-, x)$ is a $J_1$-ideal on $X$, there exists $y \in X$ with $s \prec \delta \preceq w(y, x) \otimes r$.

Let us show that $B_{(x,r)}$ is directed: suppose $\langle y_1, s_1 \rangle, \langle y_2, s_2 \rangle \in B_{(x,r)}$. Then for $i = 1, 2$, by Lemma and Definition 3.3. of (Waszkiewicz 2009), there are $y'_i \in X$ and $\delta \prec 1$ with $s_i \prec w(y_i, y'_i) \otimes w(y'_i, x) \otimes r \otimes \delta$. Then there are $\varepsilon_i \prec 1$ so that $s_i \prec w(y_i, y'_i) \otimes \varepsilon_i \otimes r \otimes \delta$ and $\varepsilon_i \prec w(y'_i, x)$. Now we have to call Lemma 46 of (Flagg et al. 1996) to find $z \in X$ such that $\delta \preceq w(z, x)$ and $\varepsilon_i \preceq X(y'_i, z)$. Consequently, $s_i \prec w(y_i, y'_i) \otimes X(y'_i, z) \otimes r \otimes \delta \preceq w(y_i, z) \otimes r \otimes \delta$, i.e. $\langle y_i, s_i \rangle \leq_{\mathbf{B}X} \langle z, r \otimes \delta \rangle$. Moreover, from $\delta \preceq w(z, x)$, we infer $r \otimes \delta \preceq w(z, x) \otimes r$, i.e. $\langle z, r \otimes \delta \rangle \in B_{(x,r)}$.

Next, we head to prove $V B_{(x,r)} = \langle x, r \rangle$. Clearly, $B_{(x,r)} \preceq_{\mathbf{B}X} \langle x, r \rangle$. By the proof of Lemma 7.7 we know that the supremum of $B_{(x,r)}$ exists and that its radius equals $r$; let it be $\langle z, r \rangle$. Note that $B_{(x,r)} \preceq_{\mathbf{B}X} \langle z, r \rangle$ can be read as:

$$
\forall y \in X \forall s \in \mathcal{Q} \quad (s \prec w(y, x) \otimes r \Rightarrow s \preceq X(y, z) \otimes r).
$$

This is, however, equivalent to say that $w(y, x) \otimes r \subseteq X(y, z) \otimes r$, or that $w(y, x) \subseteq X(y, z)$, for all $y \in X$. The last inequality can be rewritten as $1 = \hat{X}(w(-, x), yz)$, but $\hat{X}(w(-, x), yz) = X(Sw(-, x), z) = X(z, x)$. Therefore $\langle x, r \rangle \leq_{\mathbf{B}X} \langle z, r \rangle$ and thus $\bigvee B_{(x,r)} = \langle x, r \rangle$.

So far we have discussed the most difficult part of a proof that a binary relation $\bullet$, defined on $\mathbf{B}X$ as:

$$
\langle y, s \rangle \bullet (x, r) \iff \langle y, s \rangle \in B_{(x,r)}
$$

is approximating (that $\bullet$ is auxiliary is easy). In order to show continuity of the poset $(\mathbf{B}X, \leq_{\mathbf{B}X})$ it is enough to show that $\bullet$ is the way-below relation on $\mathbf{B}X$, for which, in turn, it is enough to see that the upper $\bullet$-cone is inaccessible by directed suprema. For, suppose that $\langle y, s \rangle \bullet \bigvee A$ for some directed subset $A = \{ \langle i, t_i \rangle | i \in I \}$ of $\mathbf{B}X$.

By Lemma 7.7, we have $s \prec w(y, \phi) \otimes \bigvee t_i$, where $\phi = \bigvee_{i \in I} \bigwedge_{j \geq i} X(-, x_j)$. Using Lemma and Definition 3.2 of (Waszkiewicz 2009), $s \prec \bigvee_{z \in X} (\phi z \otimes w(y, z)) \otimes \bigvee t_i = \bigvee_{z \in X} \bigwedge_{j \geq i} (X(z, x_j) \otimes w(y, z)) \otimes \bigvee t_i \leq \bigwedge_{j \geq i} w(y, x_j) \otimes \bigvee t_i$. Now we easily find
a big enough $k \in I$ that satisfies $s \prec w(y, x_k) \otimes t_k$, i.e. $(y, s) \downarrow (x_k, t_k)$.

For the converse assume that $(BX, \leq_{BX})$ is a continuous dcpo with approximation relation characterized by

$$
(x, r) \ll_{BX} (y, s) \iff s \prec v(y, x) \otimes r.
$$

It is enough to show that $v$ is approximating and Scott-continuous.

Firstly, suppose that $\varepsilon \prec v(x, y)$ for $x, y \in X$ and $\varepsilon \prec 1$. Then $(x, \varepsilon) \ll_{BX} (y, 1)$, hence $(x, \varepsilon) \ll_{BX} (y, 1)$, hence $\varepsilon \leq X(x, y)$. Since $\varepsilon \prec v(x, y)$ was arbitrary, $v(x, y) \leq X(x, y)$.

It is equally easy to show that $X(x, y) \otimes v(y, z) \leq v(x, z)$ for all $x, y, z \in X$. For let $\varepsilon, \delta \prec 1$ be such that $\delta \prec v(y, z)$ and $\varepsilon \prec X(x, y) \otimes \delta$. This means that $(x, \varepsilon) \ll_{BX} (y, \delta)$, hence $(x, \varepsilon) \ll_{BX} (z, 1)$, i.e. $\varepsilon \prec v(x, z)$. Since $\varepsilon \prec X(x, y) \otimes \delta$ was arbitrary, $X(x, y) \otimes v(y, z) \leq v(x, z)$.

Suppose now that $\varepsilon \prec v(x, y) \otimes X(y, z)$ for $x, y, z \in X$. Then $(x, \varepsilon) \ll_{BX} (y, X(y, z))$. However, $(y, X(y, z)) \ll_{BX} (z, 1)$, hence $(x, \varepsilon) \ll_{BX} (z, 1)$, i.e. $\varepsilon \prec v(x, z)$. We have shown $v(x, y) \otimes X(y, z) \leq v(x, z)$ for all $x, y, z \in X$.

Now, using the fact that $BX$ is a continuous poset, we easily prove that $v$ is approximating.

For Scott-continuity of $v$, we need to show that for any $J_1$-ideal $\phi = \bigvee_{i \in I} \bigwedge_{j \geq i} X(-, x_j)$, $v(x, S\phi) \leq \bigvee_{i \in I} (v(x, z) \otimes \phi(z))$ for any $x \in X$. Hence suppose that $\varepsilon \prec v(x, S\phi)$ for some $\varepsilon \in Q$. This means $(x, \varepsilon) \ll_{BX} (S\phi, 1)$. Fix $k \in I$. Using the proof of Lemma 7.8, we can deduce that there are $i \in I$ and $\delta \prec \bigwedge_{j \geq i} X(x_i, x_j)$ with $(x, \varepsilon) \ll_{BX} (x_i, \delta) \ll_{BX} (S\phi, 1)$.

Without loss of generality we can assume that $i \geq k$. Then $\varepsilon \prec v(x, x_i) \otimes \delta \leq v(x, x_i) \otimes \bigwedge_{j \geq i} X(x_i, x_j)$. To make the next steps transparent we set $z := x_i$ and so we have $\varepsilon \prec v(x, z) \otimes \bigvee_{i \geq k} \bigwedge_{j \geq i} X(z, x_j)$. But $\bigvee_{i \geq k} \bigwedge_{j \geq i} X(z, x_j) = \bigvee_{i \in I} \bigwedge_{j \geq i} X(z, x_j) = \phi(z)$.

Therefore $\varepsilon \leq \bigvee_{i \geq k} (v(x, z) \otimes \phi(z))$ and since $\varepsilon \prec v(x, S\phi)$ was arbitrary, $v(x, S\phi) \leq \bigvee_{i \geq k} (v(x, z) \otimes \phi(z))$.

In summary, we have proved that $v$ coincides with the way-below $Q$-relation on $X$ and hence $X$ is continuous. It is Yoneda complete by Lemma 7.8. This concludes the proof.

10. $(BX, \leq_{BX})$ as a computational model

In this section let $Q$ be a value quantale and let $X$ be a $T_1$, Yoneda complete and continuous $Q$-category. We would like to conclude the paper with a confirmation that in such a case $(BX, \leq_{BX})$ is a computational model of $X$ in the domain-theoretic sense. Recall from the end of Section 5 that having $T_1$ and separation guarantees that $y: X \to \text{max}(BX)$ is a bijection. We will now show that this map is in fact a homeomorphism from the generalized Scott topology on $X$ to the subspace Scott topology on $\text{max}(BX)$.

Recall that for any $Q$-category $X$ we have the so called generalized Scott topology (Flagg et al. 1996; Bonsangue et al. 1998): we declare a set $U \subseteq X$ open (we write $U \in \sigma(X)$) iff for all $\phi \in JX$, $S\phi \in U$ implies that there exist $\delta \prec 1$ and $z \in X$ such that $\delta \prec \phi z$ and $B(z, \delta) \subseteq U$, where the open ball is defined as $B(z, \delta) = \{y \in X \mid \delta \prec X(z, y)\}$. 

\[\text{\square}\]
Lemma 10.1. The sets $O(x,r) = \{y \in X \mid r \prec w(x,y)\}$, where $x \in X$ and $r \prec 1$, form a basis for the generalized Scott topology on $X$.

Proof. Firstly, we prove that for all $x \in X$ and $r \prec 1$, $O(x,r)$ is open in $X$. Suppose $S \phi \in O(x,r)$. Then there is $\delta \prec 1$ such that $r \prec \delta \otimes w(x,S \phi)$. By Lemma 3.3 of (Waszkiewicz 2009), there is $z \in X$ with $r \prec \delta \otimes w(x,z) \otimes w(z,S \phi)$ and $z \prec w(z,S \phi)$. Note that $z \prec w(z,S \phi) \leq \phi z$. Suppose $\delta \prec X(z,y)$. Then $r \prec \delta \otimes w(x,z) \otimes w(z,S \phi) \leq X(z,y) \otimes w(x,z) \leq w(x,y)$. This means that $y \in O(x,r)$. Hence $O(x,r)$ is open.

Secondly, suppose that $x \in U$, where $U$ is open. Since $X$ is continuous, $x = S w(-,x)$. Therefore there exist $\delta \prec 1$ and $z \in X$ with $\delta \prec w(z,x)$ and $B(z,\delta) \subseteq U$. Thus $x \in O(z,\delta)$ and $O(z,\delta) \subseteq B(z,\delta) \subseteq U$.

Proposition 10.2. The Yoneda embedding $y: X \to \text{max}(BX)$ is a homeomorphism between the generalized Scott topology on $X$ and the subspace Scott topology on the maximal elements of $(BX, \leq BX)$.

Proof. By Theorem 9.1, $(BX, \leq BX)$ is a continuous dcpo; hence sets of the form $\uparrow\uparrow \langle x, r \rangle$ constitute a basis for the Scott topology on formal balls. On the other hand sets $O(x,r)$ are a basis for $\sigma(X)$ on $X$. However $y^{-1}(\uparrow\uparrow \langle x, r \rangle) = \{z \mid \langle x, r \rangle \leq BX \langle z, 1 \rangle\} = O(x,r)$. Hence the map $y: X \to \text{max}(BX)$ is a homeomorphism.

11. Further work

There are some immediate research problems that deserve further attention.

Firstly, one can ask whether for $T_1$ separated $\mathcal{Q}$-categories Yoneda completeness and sequential Yoneda completeness coincide.

Secondly, we still do not know much about continuous quasi-metric spaces alone. Is their induced order continuous? When is their topology compact? Which subcategories are cartesian closed? What kind of ordered topological spaces are they in the symmetric topology? Which domains of data types are continuous quasi-metrics? Is Lawson duality still working on the level of continuous Yoneda-complete quasi-metrics?

Lastly, we have proved that if $X$ is $T_1$, continuous and Yoneda-complete, then $BX$ is its computational model. This opens a possibility to study $\mathcal{Q}$-categories order-theoretically, in the spirit of Edalat and Heckmann’s paper. This approach promises to be a source of many further research questions.

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References


The formal ball model for $Q$-categories