

The Method of an Exact Linearization of n -order Ordinary Differential Equations

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Abstract

Necessary and sufficient conditions are found that the n -order nonlinear and nonautonomous ordinary differential equation could be transformed into a linear equation with constant coefficients with the help, generally speaking, nonlocal transformation of dependent and independent variables. These conditions are expressed in terms of factorization through first order nonlinear differential operators. Examples are considered also.

”Two subjects that are theoretical physics and integration of differential equations, are quite impossible one without another, were always developing together, and the success of one of them influenced another”

(V.P. Ermakov)

1 Introduction

While seeking for the reason of progress and failures in ODE integration, the author concluded that the key to understanding the integrability problem is in a couple of words: **factorization and transformations**.

Today is already understood the necessity to use the corresponding deep analogies between algebraic and differential equations, and, especially, those connected with the possibilities of factorization. But the greatest importance is in understanding the necessity of mutual using of the factorization and transformation methods. Being used each one alone, they are not so effective. But using both of them together gives the maximal profit because the sum effect is greater than of each one alone.

In relation with going now **de-linearization** of **Science**, in general, and **Physics**, especially it looks actual to develop the already known and creating new mathematical methods of **linearization** of the differential equations.

The main result. Theorem 1.1. *In order the equation*

$$F(x, y, y', \dots, y^{(n)}) = 0 \tag{1.1}$$

to admit a transformation

$$y = v(x, y)z, \quad dt = u_1(x, y)dx + u_2(x, y)dy, \quad (1.2)$$

(where v, u_1 and u_2 are sufficiently differentiable functions in some domain $\mathbf{G}(\mathbf{x}, \mathbf{y})$, which are not annuled in it), to a linear autonomous form

$$z^{(n)}(t) + b_{n-1}z^{(n-1)}(t) + \dots + b_1z'(t) + b_0z(t) = 0, \quad b_k = \text{const}, \quad (1.3)$$

it is necessary and sufficient that (1.1) allow a noncommutative factorization

$$F \sim \prod_{k=n}^1 \left[D - \frac{v_x + v_y y'}{v} - (k-1) \frac{D(u_1 + u_2 y')}{u_1 + u_2 y'} - r_k(u_1 + u_2 y') \right] y = 0, \quad (1.4)$$

or a commutative factorization

$$F \sim \prod_{k=1}^n \left[\frac{1}{u_1 + u_2 y'} D - \frac{v_x + v_y y'}{v(u_1 + u_2 y')} - r_k \right] y = 0, \quad (1.5)$$

through the nonlinear differential first order operators with $D = d/dx$, and r_k being roots of the characteristic equation

$$r^n + b_{n-1}r^{n-1} + \dots + b_1r + b_0 = 0. \quad (1.6)$$

See the special cases of Theor.1.1 in [1, 2]; by using it the Halphen problem about the reducible linear n -th order differential equation was solved [1]. It was used also for linearization of autonomous equations [2]. Other possible applications of this theorem are a research of integrable cases of the Newton equation,

$$y'' = f(x, y, y'), \quad (1.7)$$

other nonlinear equations, and, particularly, autonomization of non-autonomous equations.

Relations (1.2) include the following important transformations: Kummer-Liouville transformation

$$y = v(x)z, \quad dt = u(x)dx, \quad v, u \in \mathbf{C}^n(\mathbf{I}), \quad \mathbf{uv} \neq \mathbf{0}, \quad \forall \mathbf{x} \in \mathbf{I} = \{\mathbf{x} \mid \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\}; \quad (1.8)$$

general point linearization

$$X = f(x, y), \quad Y = \varphi(x, y), \quad \frac{\partial(X, Y)}{\partial(x, y)} = X_x Y_y - X_y Y_x \neq 0, \quad (1.9)$$

that corresponds (1.2) for $u_1 y = u_2 x$; fibre-preserving point linearization

$$X = f(x), \quad Y = \varphi(x, y); \quad (1.10)$$

linearization of nonlinear autonomous equations

$$y = v(y)z, \quad dt = u(y)dx, \quad u(y(x))v(y(x)) \neq 0, \quad \forall x \in \mathbf{I} = \{\mathbf{x} \mid \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\}; \quad (1.11)$$

i.e., linearization in the restricted sense; linearization

$$y = v(x, y)z, \quad dt = u(x, y)dx, \quad (1.12)$$

which corresponds to a point Lie symmetry with the generator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}; \quad (1.13)$$

and general non-point linearization (1.2), i.e., linearization in the broad sense.

Let us note that if one performs factorization according to (1.4) or (1.5), then the class of linearizable by (1.2) equations keeps its form under different transformations (1.2). This allows us to formulate for the equations (1.1) the problem of transformation to themselves or to some given form with another structure, and of reduction to canonical forms as well.

By this way, the ODE integrability problem leads us to the necessity of reseaching such a general problem as equivalence of different classes. The corresponding problem for a n -th order LODE was solved in [1, 3, 4] with usage of the KL-transformation.

2 Linearization of the second-order autonomous equations

The linearization with the help of transformation of a desired function was applied in [5] and of an independent variable in [6]. In [2], a general class of nonlinear autonomous equations of the second order is constructed, dependent on two arbitrary functions, which being linearized under the condition of combined using transformations of both kinds.

Theorem 2.1 [2]. *In order for the equation*

$$y'' + f(y)y'^2 + b_1\varphi(y)y' + \psi(y) = 0, \quad y' = \frac{dy}{dx}, \quad (2.1)$$

to be linearized by transformation (1.11), i.e., reduce to the form

$$\ddot{z} + b_1\dot{z} + b_0z + c = 0, \quad \dot{z} = \frac{dz}{dt}, \quad (2.2)$$

$b_1, b_0, c = \text{const}$, *it is necessary and sufficient that it should be presented in one of the following forms*

$$y'' + fy'^2 + b_1\varphi y' + \varphi \exp\left(-\int f(y)dy\right) \left[b_0 \int \varphi \exp\left(\int f(y)dy\right)dy + \frac{c}{\beta}\right] = 0, \quad (2.3^1)$$

$$y'' - \left(\frac{2a}{ay+b} + \frac{\varphi^*}{\varphi}\right)y'^2 + b_1\varphi y' + \frac{b_0}{b}\varphi^2 y(ay+b) + \frac{c}{b}\varphi^2(ay+b)^2 = 0, \quad \varphi^* = \frac{d}{dy}, \quad (2.3^2)$$

$\beta = \text{const}$ *is a normalizing factor. Moreover, equations (2.3) reduce to the form (2.2) by the transformations*

$$z = \beta \int \varphi \exp\left(\int f dy\right) dy, \quad dt = \varphi(y)dx, \quad (2.4^1)$$

$$z = \frac{y}{ay+b}, \quad dt = \varphi(y)dx, \quad (2.4^2)$$

Example 1. We shall find conditions of integrability in finite terms for the equation

$$y'' + yy' + ky^3 = 0, \quad k = \text{const}, \quad (2.5)$$

which, besides the theory of univalent functions being in integrals of second-order nonlinear ordinary differential equations (NODE-2) [7], arises also in a number of other theoretical and applied questions [8].

Theorem 2.2. 1) Equation (2.5) is resolved in quadratures for any k , since by a substitution

$$z = y^2, \quad dt = ydx, \quad (2.6)$$

it is linearized: $\ddot{z} + \dot{z} + 2kz = 0$.

2) In order for equation (2.5) to be integrated in finite terms through elementary functions, it is necessary and sufficient that parameter k admits the values

$$k = \frac{l(l+1)}{2(2l+1)^2}, \quad l \in \mathbf{Z}, \quad \mathbf{k} \in [0, 1/8]. \quad (2.7)$$

3) For $k \neq 1/9$ equation (2.5) admits a two-dimensional Lie algebra with the generators

$$G_1 = \frac{\partial}{\partial x}, \quad G_2 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad [G_1, G_2] = G_1. \quad (2.8)$$

4) For $k = 1/9$ equation (2.5) admits a Lie algebra, isomorphic to $sl(3, \mathbf{R})$.

For integration of equation (2.5), the method of exact linearization Chebyshev [9] for integration of differential binomials is applied.

REMARK 1. The similar results were obtained in [8] due to a combination of a standard method of point Lie symmetries with the so-called method of **hidden** Lie symmetries [10] as well as with the test of Painleve-Kovalevsky. Generalization of the equation (2.5) is included in a Ex.4.

REMARK 2. Let the dynamic system be described by equations of the form

$$\dot{y}_1 = P(y_1, y_2), \quad \dot{y}_2 = Q(y_1, y_2). \quad (2.9)$$

Some special cases of (2.9), including a classical system of Lotka - Volterra, are reduced by exception of a variable either to equations of the form (2.3¹) or to equations, equivalent (2.3¹).

Example 2. A linearization of Liouvillian systems.

It is known that kinetic and potential energies corresponding to these systems have the form

$$T = \frac{1}{2}b(q) \sum_{i=1}^n a_i(q_i)q_i^2, \quad U = \frac{1}{b(q)} \sum_{i=1}^n d_i(q_i), \quad b(q) = \sum_{i=1}^n b_i(q_i). \quad (2.10)$$

Using the Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i}, \quad i = 1, \dots, n, \quad (2.11)$$

and the expression for the integral of energy $T - U = h$ obtained from (2.10), we shall come to the following system of second-order differential equations

$$\ddot{q}_i + \left(\frac{1}{2} \frac{a_i^*}{a_i} \dot{q}_i + \frac{\dot{b}}{b} \right) \dot{q}_i - \frac{1}{b^2 a_i} \frac{d}{dq_i} [b_i(h - d_i)] = 0. \tag{2.12}$$

Theorem 2.3. *Liouvillian system submitted in Lagrange form (2.12) is linearized by the transformation*

$$Q_i = \sqrt{2(hb_i - d_i)}, \quad ds_i = -a_i^{-1/2} b^{-1} \frac{d}{dq_i} \sqrt{2(hb_i - d_i)} dt \tag{2.13}$$

to the form

$$Q_i''(s_i) - Q_i(s_i) = 0. \tag{2.14}$$

3 A linearization of third-order autonomous equations

Consider the equation

$$y''' + f_5(y)y'y'' + f_4(y)y'' + f_3(y)y'^3 + f_2(y)y'^2 + f_1(y)y' + f_0(y) = 0. \tag{3.1}$$

By transformation of the type (1.11), it is reduced to the form

$$\ddot{z} + b_2 \dot{z} + b_1 z + b_0 z + c = 0, \quad b_2, \quad b_1, \quad b_0, \quad c = \text{const.} \tag{3.2}$$

The separate examples of the type (3.1) were considered in [11].

Theorem 3.1. *In order for the equation (3.1) to be linearized by transformation (1.11), it is sufficient that (3.1) be presented in one of the following forms:*

$$y''' + f(y)y'y'' + \frac{1}{9} \left(3 \frac{\varphi^{**}}{\varphi} - 5 \frac{\varphi^{*2}}{\varphi^2} - f \frac{\varphi^*}{\varphi} + f^2 + 3f^* \right) y'^3 + b_2 \varphi y'' + \frac{1}{3} b_2 \varphi \left(f + \frac{\varphi^*}{\varphi} \right) y'^2 +$$

$$b_1 \varphi^2 y' + \varphi^{5/3} \left(b_0 \int \varphi^{4/3} \exp\left(\frac{1}{3} \int f dy\right) dy + \frac{c}{\beta} \right) \exp\left(-\frac{1}{3} \int f dy\right) = 0; \tag{3.3}$$

$$y''' - \left(\frac{6a}{ay + b} + 4 \frac{\varphi^*}{\varphi} \right) y'y'' + b_2 \varphi y' +$$

$$\left[6 \frac{a^2}{(ay + b)^2} + 6 \frac{a}{ay + b} \frac{\varphi^*}{\varphi} + 3 \frac{\varphi^{*2}}{\varphi^2} - \frac{\varphi^{**}}{\varphi} \right] y'^3 -$$

$$b_2 \varphi \left(\frac{\varphi^*}{\varphi} + 2 \frac{a}{ay + b} \right) y'^2 + b_1 \varphi^2 y' + \frac{b_0}{b} \varphi^3 y(ay + b) + \frac{c}{b} (ay + b)^2 \varphi^3 = 0. \tag{3.4}$$

The equations (3.3) and (3.4) are resulted to (3.2) by transformations

$$z_1 = \int \varphi^{4/3} \exp\left(\frac{1}{3} \int f dy\right) dy, \quad dt = \varphi(y) dx; \tag{3.5}$$

$$z_2 = \frac{y}{ay + b}, \quad b \neq 0, \quad dt = \varphi(y) dx. \tag{3.6}$$

Example 3. Case of Euler-Poinsot in a problem of a rotation of a rigid body around a fixed point.

As is known, this problem is described by the system

$$A\dot{p} - (B - C)qr = 0, \quad B\dot{q} - (C - A)rp = 0, \quad C\dot{r} - (A - B)pq = 0, \quad A, B, C = \text{const.} \quad (3.7)$$

The system (3.7) admits a separation of variables. If we eliminate variables q, r , we'll come to the next equation of the third order

$$y''' - \frac{1}{y}y'y'' - \frac{4(A - B)(C - A)}{BC}y'y^2 = 0, \quad (3.8)$$

where $t \rightarrow x$, $p \rightarrow y$ $(\cdot) \rightarrow (\prime)$.

The equation (3.8) belongs to a special case of class (3.3). Namely, if in (3.3) we put $f = -\frac{\varphi^*}{\varphi}$, we shall come to the equation

$$y''' - \frac{\varphi^*}{\varphi}y'y'' + b_2\varphi y'' + b_1\varphi^2 y' + \varphi^2 \left(b_0 \int \varphi dy + \frac{c}{b} \right) = 0. \quad (3.9)$$

We shall make further simplifications. If now in (3.9) we set $\varphi = y$, $b_2 = b_0 = c = 0$, we shall come to equation (3.8). Equation (3.8) is linearized by transformation (2.6):

$$y''' + by' = 0, \quad b = \frac{4(A - B)(C - A)}{BC} > 0. \quad (3.10)$$

4 Point linearization of nonautonomous equations of the second-order and general method of an exact linearization

4.1. Point linearization

It was S. Lie who described a class ([12], see also [13, 14]) of ODE (1.7) linearized by a point change of variables (1.9), namely, reduced to the form

$$\frac{d^2 Y}{dX^2} = 0. \quad (4.1)$$

Theorem 4.1. *General form of the second-order nonlinear equation reduced to the linear form*

$$\frac{d^2 Y}{dX^2} + b_1 \frac{dY}{dX} + b_0 Y + c = 0, \quad b_1, b_0, c = \text{const}, \quad (4.2)$$

by point transformation (1.9) is following:

$$\begin{aligned} & (f_x \varphi_y - \varphi_x f_y) y'' + \left[(f_y \varphi_{yy}) + b_1 \varphi_y f_y^2 + (b_0 \varphi + c) f_y^3 \right] y'^3 + \\ & \left[f_x \varphi_{yy} - \varphi_x f_{yy} + 2(f_y \varphi_{xy} - \varphi_y f_{xy}) + b_1 (\varphi_x f_y^2 + 2f_x f_y \varphi_y) + 3(b_0 \varphi + c) f_x f_y^2 \right] y'^2 + \\ & \left[f_y \varphi_{xx} - \varphi_y f_{xx} + 2(f_x \varphi_{xy} - \varphi_x f_{xy}) + b_1 (2f_x f_y \varphi_x + f_x^2 \varphi_y) + 3(b_0 \varphi + c) f_x^2 f_y \right] y' + \\ & (f_x \varphi_{xx} - \varphi_x f_{xx}) + b_1 \varphi_x f_x^2 + (b_0 \varphi + c) f_x^3 = 0. \end{aligned} \quad (4.3)$$

Example 4 (the generalization of Ex.1).

$$y'' + ayy' + \frac{1}{2}by^3 = 0. \quad (4.4)$$

Equation (4.4) for $b = \frac{2}{9}a^2$ is linearized and reduced to the form (4.3) by the substitution

$$X = \frac{1}{3}ax - \frac{1}{y}, \quad Y = \frac{1}{6}ax^2 - \frac{x}{y}. \quad (4.5)$$

For comparison we shall note that (4.4) with any a and b is linearized by the substitution (1.11) to the form $\ddot{z} + a\dot{z} + bz = 0$. Equations of form (4.4) occur also in theoretical and mathematical physics (see [15]).

A special case of transformation (1.9) is the fibre-preserving point linearization (1.10).

Theorem 4.2 [16]. *For the second-order ODE (1.7) to admit a six-dimensional Lie group of fibre-preserving point symmetries, it is necessary and sufficient that it have the form*

$$y'' = \frac{1}{2}M_y y'^2 + M_x y' + N, \quad (4.6)$$

where $M(x, y)$ and $N(x, y)$ satisfy to the underdetermined equation

$$M_{xxy} + (NM_y)_y - M_{xy}M_x - 2N_{yy} = 0. \quad (4.7)$$

Theorem 4.3. *In order for equation (1.7) to be reduced to (4.3) by transformation (1.10), it is necessary and sufficient that the following conditions be executed:*

$$F = f_2(x, y)y'^2 + f_1(x, y)y' + f_0(x, y), \quad (4.8)$$

where

$$\frac{\partial f_1}{\partial y} = 2 \frac{\partial f_2}{\partial x}, \quad (4.9)$$

$$f_0 = \frac{1}{2}e^{\int f_2 \partial y} \int e^{-\int f_2 \partial y} \left(\frac{\partial f_1}{\partial x} - \frac{1}{2}f_1^2 \right) \partial y. \quad (4.10)$$

Theorem 4.4. *In order for the equation (1.7) to be linearized to a form (4.1) by transformation (1.10), it is necessary that a new independent variable X be a linear-fractional function of the old independent variable, i.e.,*

$$X = f(x) = \frac{ax + b}{cx + d}, \quad ad - bc = 1. \quad (4.11)$$

4.2. General method of an exact linearization

The method an exact linearization of **autonomous** equations considered in sections 2,4 is **particular** (linearization in the restricted sense) in relation to the following **general** method of a linearization of **nonautonomous** equations (linearization in the broad sense). The transformation used has a form (1.2), where v , u_1 and u_2 are at least twice differentiable functions of both arguments at any point (x, y) of some domain \mathbf{G} , $v(u_1 + u_2 y') \neq 0$, $\forall (x, y) \in \mathbf{G}$.

As the base of the method can served the following theorem, which is the specialization of Theor.1.1 for order $n = 2$.

Theorem 4.5. *In order for the equation of Newton (1.7) to reduce to the linear form (4.2) by transformation (1.2), it is necessary that it admits the commutative factorization:*

$$\left[\frac{1}{u_1 + u_2 y'} D - \frac{v_x + v_y y'}{v(u_1 + u_2 y')} - r_2 \right] \left[\frac{1}{u_1 + u_2 y'} D - \frac{v_x + v_y y'}{v(u_1 + u_2 y')} - r_1 \right] y + cv = 0, \quad (4.12)$$

or noncommutative factorization:

$$\left[D - \frac{D(u_1 + u_2 y')}{u_1 + u_2 y'} - \frac{v_x + v_y y'}{v} - r_2(u_1 + u_2 y') \right] \left[D - \frac{v_x + v_y y'}{v} - r_1(u_1 + u_2 y') \right] y + c(u_1 + u_2 y')^2 v = 0, \quad D = \frac{d}{dx}, \quad (4.13)$$

where r_k , $k = 1, 2$, satisfy the characteristic equation

$$r^2 + b_1 r + b_0 = 0. \quad (4.14)$$

5 Semilinear equations, corresponding to nonlinear evolution equations of diffusion type

A search for self-similar, in particular, invariant solutions of travelling-wave a type is one of main sources of appearance of nonlinear ODEs.

An important class of nonlinear equations is the so-called class of **semilinear** equations representable in the form of a sum of linear differential expressions with constant coefficients and nonlinear terms.

The Kolmogorov-Petrovskii-Piskunov equation (KPP) is a nonlinear diffusion equation of the form

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + F(u), \quad k = \text{const}, \quad (5.1)$$

where a nonlinear function $F(u)$ satisfies the conditions

$$F(0) = F(1) = 0, \quad F'(0) = \alpha > 0, \quad F'(u) < \alpha, \quad 0 < u < 1, \quad (') = d/du. \quad (5.2)$$

It was considered in [17] in connection with a problem of finding invariant solutions of travelling wave a type $u(x, t) = u(\tau)$, $\tau = ax + bt$.

5.1. Group-theoretical properties of the semilinear KPP equation

We shall find point symmetries of a semilinear ODE corresponding (5.1), which we present in the form

$$y'' + b_1 y' + \Phi(y) = 0. \quad (5.3)$$

Theorem 5.1. *In order for equation (5.3) to admit a point symmetry (one-parameter Lie group) with generator (1.13), where $X \neq \frac{\partial}{\partial \tau}$, it is necessary and sufficient that $\Phi(y)$ and X should take one of the following respective forms:*

$$1) \quad \Phi(y) = b_0 F(y) = r_1 r_2 \left[y^* + \frac{s}{r_1 r_2} y^{*(2r_2 - r_1)/r_1} \right],$$

$$X = e^{(r_1-r_2)x} \left[\frac{\partial}{\partial x} + r_1 y^* \frac{\partial}{\partial y} \right];$$

$$2) \quad \Phi(y) = b_0 F(y) = r_1 r_2 \left[y^* + \frac{s}{r_1 r_2} y^{*(2r_1-r_2)/r_2} \right],$$

$$X = e^{(r_2-r_1)x} \left[\frac{\partial}{\partial x} + r_2 y^* \frac{\partial}{\partial y} \right],$$

where $y^* = y + q/(r_1 r_2)$

$$3) \quad \Phi(y) = q + s \exp(2b_1^2 y/q), \quad q \neq 0$$

$$X = e^{b_1 x} \left(\frac{\partial}{\partial x} - \frac{q}{b_1} \frac{\partial}{\partial y} \right), \quad b_0 = 0; \quad b_1 \neq 0;$$

$$4) \quad \Phi(y) = s(y+q)^{-1}, \quad X = e^{-b_1 x} \left(\frac{\partial}{\partial x} - b_1(y+q) \frac{\partial}{\partial y} \right);$$

$$5) \quad \Phi(y) = b_0 F(y) = b_0 \left[\left(y + \frac{q}{b_0} \right) + \frac{s}{b_0} \left(y + \frac{q}{b_0} \right)^{-3} \right], \quad b_1 = 0, \quad b_0 \neq 0;$$

$$X_{1,2} = \exp(\mp 2\sqrt{-b_0}x) \left[\frac{\partial}{\partial x} + \left(\mp \frac{q}{\sqrt{-b_0}} \mp \sqrt{-b_0}y \right) \frac{\partial}{\partial y} \right].$$

$$6) \quad \Phi(y) = b_0 y + \frac{1}{4k} \left(b_0^2 - \frac{36}{625} b_1^4 \right) + ky^2,$$

$$X = \exp\left(\frac{b_1}{5}x\right) \left\{ \frac{\partial}{\partial x} - \left[\frac{2}{5}b_1 y - \frac{1}{5k}b_1 \left(b_0 - \frac{6}{25}b_1^2 \right) \right] \frac{\partial}{\partial y} \right\},$$

where r_1 and r_2 satisfy the equation (4.14).

Note, that in cases 1)–4) and 6), if we denote a generator of translation through $X_2 = \frac{\partial}{\partial x}$, and other admitting generator-through X_1 , then we obtain two-dimensional Lie algebras, respectively, with commutators:

$$1) [X_1, X_2] = (r_2 - r_1)X_1; \quad 2) [X_1, X_2] = (r_1 - r_2)X_1; \quad 3) [X_1, X_2] = -b_1 X_1;$$

$$4) [X_1, X_2] = b_1 X_1; \quad 6) [X_1, X_2] = -\frac{1}{5}b_1 X_1.$$

In case 5) which reduces to a special case of the Ermakov equation (see [18]), the Lie algebra of symmetries is three-dimensional. Let us denote the generator of translation through X_3 , and other two generators admitting symmetries through X_1, X_2 . Then nonzero commutators as follows

$$5) [X_1, X_3] = -2\sqrt{-b_0}X_1; \quad [X_2, X_3] = -2\sqrt{-b_0}X_2; \quad [X_1, X_2] = -2\sqrt{-b_0}(X_1 + X_2).$$

Note that one case 6) were passed in [19,20].

The lack of a sufficient place does not allow us to consider in detail the method of factorization [20, 21] of constructing exact solutions for the found semilinear equations.

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