Method versus calculus in Newton's criticisms of Descartes and Leibniz

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Abstract. In my talk I will consider Newton's views on mathematical method. Newton never wrote extensively about this issue. However, in his polemic writings addressed against Descartes and Leibniz he expressed the idea that his method was superior to the ones proposed by the French and the German. Considering these writings can help us in understanding the role attributed to algebra and calculus in Newton's mathematical thought.

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1. Newton's memorandum on his early discoveries

Newton blossomed as a creative mathematician in 1665–1666, the so-called *anni mirabiles*, about four years after matriculating at Cambridge.[1] A Newtonian memorandum, written about fifty years later, gives an account that has been basically confirmed by manuscript evidence:

> In the beginning of the year 1665 I found the Method of approximating series & the Rule for reducing any dignity of any Binomial into such a series. The same year in May I found the method of Tangents of Gregory & Slusius, & in November had the direct method of fluxions & the next year in January had the theory of Colours & in May following I had entrance into y^{e} inverse method of fluxions. And the same year I began to think of gravity extending to y^{e} orb of the Moon […] All this was in the two plague years of 1665–1666. For in those days I was in the prime of my age for invention & minded Mathematicks & Philosophy more than any time since. ([1])

There would be much to say in order to decipher and place into context Newton's discourse. For instance, the task of commenting on the meaning of the term 'philosophy' would require space and scholarship not at my disposal [2].

[1] Readers interested in Newton's mathematics should read Tom Whiteside's introductions and commentaries in [9].

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Figure 1. Newton's home at Woolsthorpe where – he claimed – he made his early discoveries in mathematics and natural philosophy when Cambridge University was evacuated because of the plague during the biennium 1665–1666. As a matter of fact, he did important work in mathematics during periods in which he returned to the University. Further, his juvenile insights – particularly those concerning gravitation – had to be elaborated during the next decades. Source: [1], 54.

Let me note three things about the above memorandum. The 'Method of approximating series' is the method of series expansion via long division and root extraction (as well as other methods which were later subsumed under more general techniques usually attributed to Puiseux) that allowed Newton to go beyond the limitation of what he termed 'common analysis' – where 'finite equations' were deployed – and express certain curves locally in terms of infinite fractional power series, which Newton called 'infinite equations'. The 'Rule for reducing any dignity of any Binomial' is what we call the 'binomial theorem'. Such methods of series expansion were crucial for attaining two goals: the calculation of areas of curvilinear surfaces and the rectification of curves (see Figure 2). Notice that Newton does not talk about a theorem, but rather about 'methods' and a 'rule'. This last fact is of utmost importance and deserves our commentary in Sections 2, 3, and 4, before turning in Section 5 to the direct and inverse methods of fluxions which are the Newtonian equivalent of the Leibnizian differential and integral calculus.[2]

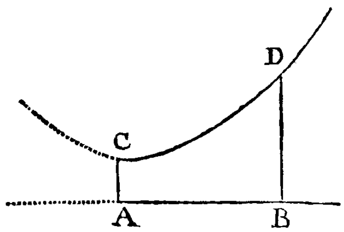
[2] For a recent evaluation of Newton's early mathematical researches see [3].

Examples, where the Square Root must be extracted.

15. If it be $\sqrt{aa+xx}=y$, I extract the Root thus:

$$aa+xx\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\ \&c.\right.$$

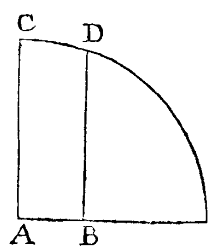
$$\begin{array}{l} aa \\ \hline 0+x^2 \\ \quad x^2+\frac{x^4}{4a^2} \\ \hline \quad 0-\frac{x^4}{4a^2} \\ \quad -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\ \hline \quad\quad 0+\frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\ \quad\quad +\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}} \\ \hline \quad\quad\quad 0-\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}} \\ \quad\quad\quad \&c. \end{array}$$



Whence for the Equation $\sqrt{aa+xx}=y$, a new one is produced, *viz.* $y=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$ *&c.* And (by the second Rule) You will have the Area sought ABDC $=ax+\frac{x^3}{6a}-\frac{x^5}{40a^3}+\frac{x^7}{112a^5}-\frac{5x^9}{1152a^7}$ *&c.*

And this is the Quadrature of the Hyperbola.

16. After the same Manner if it be $\sqrt{aa-xx}=y$, it's Root will be $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$ *&c.* and therefore the Area sought ABDC will be equal to $ax-\frac{x^3}{6a}-\frac{x^5}{40a^3}-\frac{x^7}{112a^5}-\frac{5x^9}{1152a^7}$ *&c.* And this is the Quadrature of the Circle.



17.

Figure 2. Calculation of areas of hyperbolic and circular surfaces via extraction of root of $\sqrt{aa+xx}=y$ and $\sqrt{aa-xx}=y$. This technique of series expansion and termwise integration was basic in Newton's early mathematical work and was displayed in a tract entitled *On the analysis by means of infinite equations* (written in 1669, but printed only in 1711), an extension of 'common analysis' which proceeds via 'finite equations' only. Source: [8], vol. 1, 8.

2. Pappus on the method of analysis and synthesis

Newton belonged to a mathematical community in which the distinction between theorems and problems was articulated according to criteria sanctioned by the venerated Greek tradition. Most notably in the work of the late Hellenistic compiler Pappus entitled *Mathematical Collection* which appeared in 1588 in Latin translation Newton – who avidly read this dusty work – could find a distinction between 'theorematic and problematic analysis'.

In the 7th book of the *Collection* there was a description of works (mostly lost and no longer available to early modern mathematicians) which – according to Pappus – had to do with a heuristic method followed by the ancient geometers. The opening of the seventh book is often quoted. It is an obscure passage whose decoding was top in the agenda of early modern European mathematicians, convinced as they were that here lay hidden the key to the method of discovery of the ancients. Given the importance this passage had for Newton, it is worth quoting at length:

> That which is called the *Domain of Analysis*, my son Hermodorus, is, taken as a whole, a special resource that was prepared, after the composition of the *Common Elements*, for those who want to acquire a power in geometry that is capable of solving problems set to them; and it is useful for this alone. It was written by three men: Euclid the Elementarist, Apollonius of Perge, and Aristaeus the elder, and its approach is by analysis and synthesis.
>
> Now analysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what come before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call this kind of method 'analysis', as if to say *anapalin lysis* (reduction backward). In synthesis, by reversal, we assume what was obtained last in the analysis to have been achieved already, and, setting now in natural order, as precedents, what before were following, and fitting them to each other, we attain the end of the construction of what was sought.
>
> There are two kinds of analysis: one of them seeks after the truth, and is called 'theorematic': while the other tries to find what was demanded, and is called 'problematic'. In the case of the theorematic kind, we assume what is sought as a fact and true, then advancing through its consequences, as if they are true facts according to the hypothesis, to something established, if this thing that has been established is a truth, then that which was sought will also be true, and its proof the reverse of the analysis; but if we should meet with something established to be false, then the thing that was sought too will be false. In the case of the problematic kind, we assume the proposition as something we know, then, proceeding through its consequences, as if true, to something

> established, if the established thing is possible and obtainable, which is what mathematicians call 'given', the required thing will also be possible, and again the proof will be the reverse of the analysis; but should we meet with something established to be impossible, then the problem too will be impossible. ([4])

Pappus here made a distinction between analysis and synthesis. Analysis ('resolutio' in Latin) was often conceived of as a method of discovery, or a method of problem solving, which, working step by step backwards from what is sought as if it had already been achieved, eventually arrives at what is known. Synthesis ('compositio' or 'constructio') goes the other way round: it starts from what is known and, working through the consequences, arrives at what is sought. On the basis of Pappus' authority it was often stated that synthesis 'reverses' the steps of analysis. It was synthesis which provided the rigorous proof. Thus the belief – widespread in early modern Europe – that the ancients had kept the method of analysis hidden and had published only the rigorous synthesis, either because they considered the former not wholly demonstrative, or because they wanted to hide the method of discovery.

Another distinction which was of momentous importance for early modern mathematicians is that between problems and theorems. A problem asks a construction for its solution. It starts from certain elements considered as already constructed either by postulate or by previously established constructions. Such elements are the 'givens' (in Latin the 'data') of the problem. A problem ends with a 'Q.E.I.' or with a 'Q.E.F.' ('quod erat inveniendum' – 'what was to be discovered'–, and 'quod erat faciendum' – 'what was to be done'–, respectively). A theorem asks for a deductive proof, a sequence of propositions each following from the previous one by allowed inference rules. The starting point of the deductive chain can be either axioms or previously proved theorems. A theorem ends with 'Q.E.D.' ('quod erat demonstrandum'– 'what was to be demonstrated'). According to Pappus, therefore, there are two kinds of analysis: the former referred to problems, the latter to theorems. But it is clear from classical sources that the most important, or at least the most practiced kind, was problematic analysis: and indeed early modern European mathematicians were mainly concerned with the analysis of geometrical problems.

Another powerful idea that began to circulate in Europe at the end of the seventeenth century was that the analysis of the Greeks was not geometrical but rather symbolical: i.e. the Greeks were supposed to have had algebra and to have applied it to geometrical problem solving. The evidence that symbolic algebra was within the reach of the ancients was provided by a far from philological reading of the work of Diophantus and of parts of Euclid's *Elements*. The approach of Renaissance culture towards the classics, in sculpture, architecture, music, philosophy, and so on, was characterized by admiration united to a desire to restore the forgotten conquests of the ancients. This approach often confined with worship, a conviction of the occurrence of a decay from a glorious, golden past. The works of Euclid, Apollonius, Archimedes were considered unsurpassable models by many Renaissance mathematicians. The question that often emerged was: how could the Greeks have achieved such a wealth of results? In the decades following the publication of the *Collection* the belief in the existence of a lost, or hidden, 'Treasure of analysis' promoted many efforts aimed at 'restoring' the ancients' method of discovery. Not everybody trod in the steps of the classicists. Typically, many promoters of the new symbolic algebra were proud to define themselves as innovators, rather than as restorers. It was common, however, even among creative algebraists such as François Viète, John Wallis and Isaac Newton, to relate symbolic algebra to the ancient analysis, to the hidden problem solving techniques of the ancients.

3. Descartes' method of problem solving and problem construction

Newton was deeply embedded in the conceptual space defined by Pappus and by his readers, interpreters and critics. Mainly he referred his views on mathematical method to Descartes' *Géométrie* (1637), an early source of inspiration for him and soon a target of his fierce criticisms ([5]). From this tradition Newton derived the idea that a problem, once analyzed (resolved), must be synthesized (composed or constructed).

How did Descartes define his canon of problem solving and the role of algebra in the analysis and synthesis of geometrical problems? The historian who has done most to clarify this issue is Henk Bos. It is to his work that we now turn for advice ([6]).

In book 1 of the *Géométrie* Descartes explained how one could translate a geometric problem into an equation. Descartes was able to do so by a revolutionary departure from tradition. In fact he interpreted algebraic operations as closed operations on segments. For instance, if a and b represent lengths of segments the product ab is not conceived by Descartes as representing an area but rather another length. As he wrote: 'it must be observed that by a^2, b^3, and similar expressions, I ordinarily mean any simple lines', while before the *Géométrie* such expressions represented an area and a volume respectively (see Figure 3).

Descartes' interpretation of algebraic operations was indeed a gigantic innovation, but he proceeded wholly in line with Pappus' method of analysis and synthesis, to which he explicitly referred. In fact, according to Descartes, one has – following Pappus' prescriptions– to 'start by assuming that the problem was solved and consider a figure incorporating the solution'.[3] The segments in the figure are then denoted by letters, $a, b, c, \dots,$ for segments which are given, $x, y, z, \dots,$ for segments which are unknown. Geometrical relationships holding between the segments are then translated into corresponding equations. It is thus that one obtains a system of equations which symbolically express the assumption that the problem is solved. In fact, here we are at the very beginning of the analytic process: the unknown segments are treated as if they were known and manipulated in the equations on a par with the givens of the problem. The resolution of the equation allows the expression of the unknown x in terms of given segments. We have thus moved from the assumption that the problem is solved

[3] [6] on p. 303.

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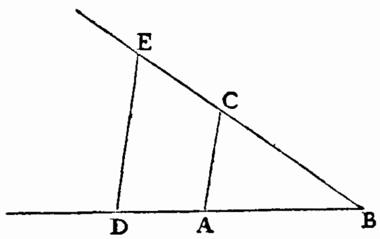
eſt a l'autre, ce qui eſt le meſme que la Diuiſion; ou enfin trouuer vne, ou deux, ou pluſieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne; ce qui eſt le meſme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligibile.

La Multiplication.

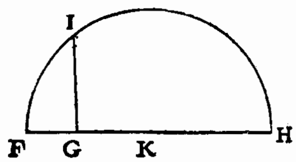


Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E eſt le produit de cete Multiplication.

La Diuiſion.

Oubien s'il faut diuiſer B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C eſt le produit de cete diuiſion.

l'Extraction de la racine quarrée.



Ou s'il faut tirer la racine quarrée de G H, ie luy adiouſte en ligne droite F G, qui eſt l'vnité, & diuiſant F H en deux parties eſgales au point K, du centre K ie tire le cercle F I H, puis eſleuant du point G vne ligne droite iuſques à I, à angles droits ſur F H, c'eſt G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cauſe que i'en parleray plus commodement cy aprés.

Commēt on peut

Mais ſouuent on n'a pas beſoin de tracer ainſi ces ligne

4

Figure 3. Descartes' geometric interpretation of algebraic operations. He writes: 'For example, let AB be taken as unity, and let it be required to multiply BD by BC. I have only to join the points A and C, and draw DE parallel to CA; and then BE is the product of BD and BC'. So, given a unit segment, the product of two segments is represented by another segment, not by a surface. The second diagram is the construction of the square root of GH. Given GH and a unit segment FG, one draws the circle of diameter $FG + GH$ and erects GI, the required root. Source: [5], 4.

(the first step of the analysis) to a reduction of the unknown, sought magnitude to the givens. This is why Descartes, and the other early-modern promoters of algebra, associated algebra with the method of analysis.

The *resolution* of the equation is not, however, the *solution* of the problem. In fact, the solution of the problem must be a geometrical construction of the sought magnitude in terms of legitimate geometrical operations performed on the givens (‘Q.E.F.’!). We now have to move from algebra back to geometry again. Descartes understood this process from algebra to geometry as follows: the real roots of the equation (for him if there are no real roots, then the problem admits no solution) must be geometrically constructed. After Descartes, this process was known as the ‘construction of the equation’. This is where the synthetic, compositive part of the whole process begins.

Descartes accepted from tradition the idea that such constructions must be performed by intersection of curves. That is to say, the real roots are geometrically represented by segments, and such segments are to be constructed by intersection of curves. As a matter of fact, the construction of the equation presented the geometer with a *new* problem: not always an easy one. One had to choose two curves, position and scale them, such that their intersections determine points from which segments – whose lengths geometrically represent the roots of the equation – can be drawn (see Figure 4).

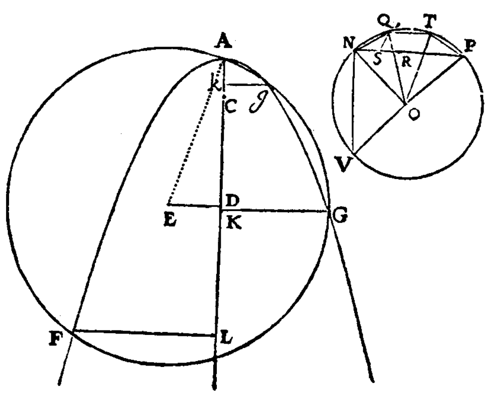


Figure 4. Construction of a third-degree equation in Descartes’ *Géométrie*. The problem of trisecting angle NOP is *resolved* (‘resolutio’ is the Latin translation of the Greek ‘analysis’) by a third-degree equation. Descartes *constructs* the roots (‘constructio’ or ‘compositio’ translate ‘synthesis’) via intersection of circle and parabola. The segments kg, KG and LF represent two positive and one negative root. The smaller of the two positive roots kg must be ‘taken as the length of the required line NQ’. KG is equal to NV, ‘the chord subtended by one-third the arc NVP’. Source: [5], 208.

The synthetic part of Descartes’ process of problem-solving gave rise to two questions: which curves are admissible in the construction of equations? which curves, among the admissible, are to be preferred in terms of simplicity? In asking himself these questions Descartes was continuing a long debate concerning the role and classification of curves in the solution of problems. A tradition that, once again, stems from Pappus, and the interpretations of Pappus given by mathematicians such as Viète, Ghetaldi, Kepler, and Fermat. His answer was that only 'geometrical curves' (we would say 'algebraic curves') are admissible in the construction of the roots of equations and that one has to choose the curves of the lowest possible degree as these are the simplest. Descartes instead excluded 'mechanical curves' (we would say transcendental curves) as legitimate tools of construction.

Notice that Descartes presented his canon of problem resolution and construction in aggressively anti-classicist terms. His algebraic method, he claimed, was superior to the ones followed by the ancients. He gave pride of place to a problem discussed in Pappus' *Mathematical Collection* that – according to Descartes – neither Euclid nor Apollonius could solve. He proudly showed to the readers of the slim *Géométrie* that, by applying algebra to geometry, he could easily achieve a solution not included in the ponderous Pappusian tomes.[4]

4. Newton versus Descartes

Newton sharply criticized Descartes' canon of problematic analysis and construction.[5] Newton's point was that geometrical constructions have to be carried on in terms independent from algebra. Newton elaborated his criticism to Descartes in his Lucasian Lectures on Algebra which were held before 1684 and which, in somewhat modified form, appeared in 1707 as *Arithmetica Universalis* ([8], vol. 2, 3–135). The *Arithmetica Universalis* ends with an Appendix devoted to the 'construction of equations' which abounds with oft-quoted statements in favour of pure geometry and against the 'Moderns' (read Descartes) who have lost the 'Elegance' of geometry:

> Geometry was invented that we might expeditiously avoid, by drawing Lines, the Tediousness of Computation. Therefore these two sciences [Geometry and Arithmetical Computation] ought not be confounded. The Ancients did so industriously distinguish them from one another, that they never introduced Arithmetical Terms into Geometry. And the Moderns, by confounding both, have lost the Simplicity in which all the Elegance of Geometry consists.[6]

[4]Briefly said, Pappus problem requires the determination of the locus of points P such that their distances d_i $(i = 1, 2, 3, 4)$ from four lines given in position are such that $d_1 d_2 = k(d_3 d_4)$. In the *Géométrie* Descartes introduces a system of oblique coordinates, and notices that the distance of a point from a line is given by an expression of the form $ax + by + c$. Therefore Pappus 4-lines locus has a second-degree defining equation: namely it will be a conic section. The algebraic approach immediately allowed Descartes to generalize Pappus problem for any number of lines.

[5]Further information on Newton's criticisms to Descartes can be gained from [7].

[6][8], vol. 2, 228.

Such statements have often puzzled commentators since they occur in a work devoted to algebra and in which the advantage of algebraic analysis is displayed in a long section on the resolution of geometrical problems. Why was Newton turning his back to 'arithmetic'[7] now saying that algebra and geometry should be kept apart? In order to understand this seemingly paradoxical position we have to briefly recall that according to Descartes the demarcation between admissible and inadmissible curves as means of construction was that between geometrical and mechanical curves. Ultimately, Descartes was forced to make recourse to algebraic criteria of demarcation and simplicity: in fact, algebraic curves coincided for him with the loci of polynomial equations, and the degree of the equation allowed him to rank curves in terms of their simplicity.

As far as demarcation is concerned, in the *Arithmetica Universalis* Newton maintained that it would be wrong to think that a curve can be accepted or rejected in terms of its defining equation. He wrote:

> It is not the Equation, but the Description that makes the Curve to be a Geometrical one. The Circle is a Geometrical Line, not because it may be expressed by an Equation, but because its Description is a Postulate.[8]

Further, Descartes' classification of curves in function of the degree of the equation – Newton claimed – is not relevant for the geometrician, who will choose curves in function of the simplicity of their description. Newton, for instance, observed that the equation of a parabola is simpler than the equation of the circle. However, it is the circle which is simpler and to be preferred in the construction of problems:

> It is not the simplicity of its equation, but the ease of its description, which primarily indicates that a line is to be admitted into the construction of problems. [...] On the simplicity, indeed, of a construction the algebraic representation has no bearing. Here the descriptions of curves alone come into the reckoning.[9]

Newton observed that from this point of view, the conchoid, a fourth degree curve, is quite simple. Independently of considerations about its equation, its mechanical description – he claimed – is one of the simplest and most elegant; only the circle is simpler. Descartes' algebraic criterion of simplicity is thus deemed alien to the constructive, synthetical, stage of problem solving. The weakness of Newton's position is that the concepts of simplicity of tracing, or of elegance, to which he continuously refers are qualitative and subjective. One should be aware that no compelling reason is given in support of Newton's evaluations on the simplicity of his preferred constructions: his are largely aesthetic criteria. Considering them is however crucial for our understanding of Newton's views concerning mathematical method.

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[7]Notice that Newton employed the term 'universal arithmetic' for algebra, since it is concerned with the doctrine of operations, not applied to numbers, but to general symbols.

[8][8], vol. 2, 226.

[9][9], vol. 5, 425–7.

As a matter of fact, Newton – this master of algebraic manipulations – in the mid 1670s developed a deep distaste for symbolism and distanced himself from the mathematics of the 'moderns'. He wrote:

> The Modern Geometers are too fond of the Speculation of Equations. The Simplicity of these is of an Analytick Consideration. [in the Appendix to the *Arithmetica Universalis*] [w]e treat of Composition, and Laws are not given to Composition from Analysis. Analysis does lead to Composition: but it is not true Composition before it is freed from Analysis. If there be never so little Analysis in Composition, that Composition is not yet real. Composition in it self is perfect, and far from a Mixture of Analytick Speculations.[10]

This position, let me restate it, does not exclude the use of algebra in the analysis; it does, however, rule out algebraic criteria of demarcation and simplicity from the synthesis. As Newton was to affirm in a manuscript dating from the early 1690s:

> if a question be answered [...] that question is resolved by the discovery of the equation and composed by its construction, but it is not solved before the construction's enunciation and its complete demonstration is, with the equation now neglected, composed.[11]

But, around 1680, Newton moved a step forward in his opposition to the method proposed in the *Géométrie*: not only Cartesian synthesis, but also Cartesian analysis fell under his fierce attack. He developed a deep admiration for the ancient Greek mathematicians, while he criticized in bitter terms the symbolical analysis pursued by the moderns. He began to doubt that the analysis of the Greeks was algebraical, he rather suspected that Euclid and Apollonius possessed a more powerful *geometrical analysis* displayed in the three lost books on *Porisms* attributed to Euclid and described in Book 7 of the *Mathematical Collection*. So not only the composition (the synthesis) had to be freed from algebra, the algebraic calculus had to be avoided also in the process of resolution (the analysis). His target was often Descartes. For instance in the late 1670s, commenting on Descartes' solution of Pappus problem, he stated with vehemence:

> To be sure, their [the Ancients'] method is more elegant by far than the Cartesian one. For he [Descartes] achieved the result by an algebraic calculus which, when transposed into words (following the practice of the Ancients in their writings), would prove to be so tedious and entangled as to provoke nausea, nor might it be understood. But they accomplished it by certain simple propositions, judging that nothing written in a different style was worthy to be read, and in consequence concealing the analysis by which they found their constructions.[12]

[10] [8], vol. 2, 250.
[11] [9], vol. 7, 307.
[12] [9], vol. 4, 277.

Newton was not alone in his battle against the algebraists. Similar statements can be found in the polemic works of Thomas Hobbes. But probably the deepest influence on Newton in this matter was played by his mentor Isaac Barrow. Newton's quest for the ancient, non-algebraical, porismatic analysis led him to develop an interest in projective geometry (see Figure 5).

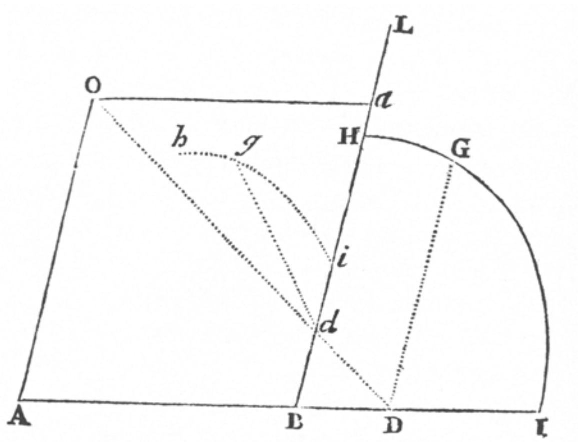


Figure 5. Newton was interested in using projective transformations as a heuristic analytic tool. Here we reproduce the diagram for Lemma 22, Book 1, of the *Principia*. In this Lemma we are taught how 'To change figures into other figures of the same class' (namely, algebraic curves of the same degree). The figure to be transmuted is the curve HGI. Draw the straight parallel lines AO and BL cutting any given third line AB in A and B. Then from some point O in the line AO draw the straight line OD. From the point d erect the ordinate dg (you can choose any angle between the 'new ordinate' dg and the 'new abscissa' ad). The new ordinate and abscissa have to satisfy the following conditions: $AD = (AO \times AB)/ad$ and $DG = (AO \times dg)/ad$. These transformations are exactly those occurring between figures projected from one plane into another. Now suppose that point G 'be running through all the points in the first figure [*HGI*] with a continual motion; then point g – also with a continual motion – will run through all the points in the new figure [*hgi*]'. Source: [11], 162.]

He convinced himself that the ancients had used projective properties of conic sections in order to achieve their results. Moving along these lines he classified cubics into five projective classes.[13]

[13] From his work on cubics ([8], vol. 2, 137–161) Newton derived two lessons. First, Descartes' classification of curves by degree is an algebraic criterion which has little to do with simplicity. Indeed, cubics have rather complex shapes compared to mechanical (transcendental) curves such as the Archimedean spiral. Second, it is by making recourse to projective classification that one achieves order and generality.

5. Newton's new analysis

Now that we know more about Newton's views concerning the role of algebraic symbolism in the method of problem solving, we are in the position to step back to Newton's memorandum on his early mathematical discoveries that I quoted in Section 1. There he mentions the direct and the inverse methods of fluxions. The direct method allowed the determination of tangents (and curvature) to plane curves. Newton approached this problem by conceiving curves as generated by the continuous 'flow' of a point. He called the geometric magnitudes generated by motion 'fluents', while 'fluxions' are the instantaneous rates of flow. In the 1690s he denoted fluxions with overdots, so that the fluxion of x is $\dot{x}$. He deployed a variety of strategies in order to determine tangents. Some of them are algorithmic, but in many cases Newton made recourse to kinematic methods. In Newton's mathematical writings the algorithm is indeed deeply intertwined with geometrical speculations.

By resolving motion into rectilinear components Newton could determine the tangent by composition of motions, even in the case of mechanical lines (see Figure 6). Indeed, the possibility to deal with transcendental curves (as the spiral and the cycloid) was top in Newton's agenda. Or one could focus attention on the 'moment of the arc' generated in a very short interval of time (Newton termed the infinitesimal increment acquired in an infinitesimal interval of time a 'moment') and establish a proportion between the moment of the abscissa and the moment of the ordinate and other finite lines embedded in the figure. When the curve was expressed symbolically via an equation Newton had 'rules' which allowed him to calculate the tangent (see Figure 7). One recognizes here rules which are 'equivalent' to those of the differential calculus; but the reader should be reminded that this equivalence was, and still is, object of debate.

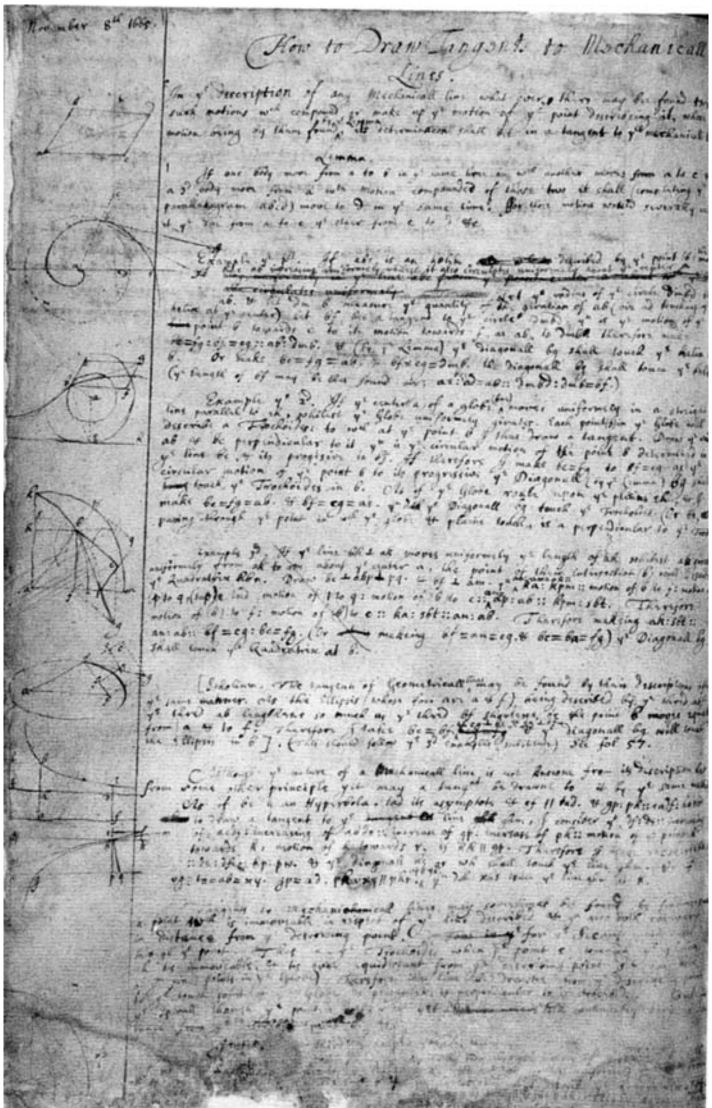
The inverse method of fluxions was Newton's masterpiece. It is this method that allowed him to approach the problem of 'squaring curves'. By conceiving a surface t as generated by the flow of the ordinate y which slides at a right angle over the abscissa z, he understood that the rate of flow of the surface's area is equal to the ordinate (he stated $\dot{t}/\dot{z} = y/1$). This is how the idea of integration as anti-differentiation was born in Newton's mind. His approach consisted in applying the direct method to 'equations at will [which] define the relationship of t to z'. One thus obtains an equation for $\dot{t}$ and $\dot{z}$, and so 'two equations will be had, the latter of which will define the curve, the former its area'.[14] Following this strategy Newton constructed a 'Catalogue of curves' which can be squared by means of 'finite equations' (see Figure 8). In Leibnizian terms, he built the first integral tables in the history of mathematics.

Newton attached much importance to the inverse method. With almost visionary mathematical understanding of what is truly revolutionary, while still in his early years, he wrote:

> If two Bodys A & B, by their velocitys p & q describe y^{e} lines x & y.

[14] [9], vol. 3, 197.

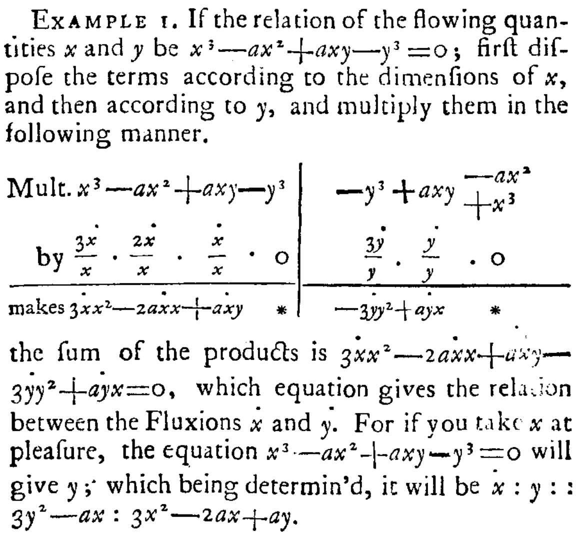
& an Equation bee given expressing y^e relation twixt one of y^e lines x, & ye ratio q/p of their motions q & p; To find y^e other line y. Could this ever bee done all problems whatever might bee resolved.[15]


November 8th 1665.

How to Draw Tangents to Mechanicall Lines.

Figure 6. Newton's early work (November 1666) on tangents to 'mechanicall lines' (i.e. transcendental plane curves). His technique consisted in conceiving curves as generated by motion and resolving motion into components. Source: [9], vol. 1, 378.

[15] [9], vol. 1, 403.



EXAMPLE I. If the relation of the flowing quantities x and y be $x^3 - ax^2 + axy - y^3 = 0$; firſt diſpoſe the terms according to the dimenſions of x, and then according to y, and multiply them in the following manner.

Mult. $x^3 - ax^2 + axy - y^3$	$-y^3 + axy \begin{matrix}-ax^2\\+x^3\end{matrix}$
by $\frac{3\dot{x}}{x} \cdot \frac{2\dot{x}}{x} \cdot \frac{\dot{x}}{x} \cdot 0$	$\frac{3\dot{y}}{y} \cdot \frac{\dot{y}}{y} \cdot 0$
makes $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y$ *	$-3\dot{y}y^2 + a\dot{y}x$ *

the ſum of the products is $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, which equation gives the relation between the Fluxions $\dot{x}$ and $\dot{y}$. For if you take x at pleaſure, the equation $x^3 - ax^2 + axy - y^3 = 0$ will give y; which being determin'd, it will be $\dot{x} : \dot{y} :: 3y^2 - ax : 3x^2 - 2ax + ay$.

Figure 7. Newton's algorithm for the direct method of fluxions. In this example he calculates the relation between fluxions (instantaneous speeds) $\dot{x}$ and $\dot{y}$ of fluent quantities (magnitudes changing continuously in time) x and y related by the equation $x^3 - ax^2 + axy - y^3 = 0$. Source: [8], vol. 1, 50.

In this context Newton developed techniques equivalent to integration by parts and substitution.

Newton labelled the techniques of series expansion, tangent determination and squaring of curves as the 'method of series and fluxions'. This was, he proudly stated, a 'new analysis' which extended itself to objects that Descartes had banished from his 'common analysis' – such as mechanical curves – thanks to the use of infinite series:

> And whatever common analysis performs by equations made up of a finite number of terms (whenever it may be possible), this method may always perform by infinite equations: in consequence, I have never hesitated to bestow on it also the name of analysis.[16]

According to Newton, the 'limits of analysis are enlarged by [...] infinite equations: [...] by their help analysis reaches to all problems'.[17]

[16] [9], vol. 2, 241.
[17] [10]

A TABLE *of ſome Curves related to Rectilinear Figures, conſtructed by* PROBLEM VII.

		Order of Curves.	Values of the Areas.
I		$dz^{\eta-1}=y$	$\frac{d}{\eta}z^{\eta}=t$
II		$\frac{dz^{\eta-1}}{ee+2efz^{\eta}+ffz^{2\eta}}=y$	$\frac{dz^{\eta}}{\eta e^{2}+\eta efz^{\eta}}=t$
III	1	$dz^{\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{2d}{3\eta f}R^{3}=t$
	2	$dz^{2\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{-4e+6fz^{\eta}}{15\eta ff}dR^{3}=t$
	3	$dz^{3\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{16ee-24efz^{\eta}+30ffz^{2\eta}}{105\eta f^{3}}dR^{3}=t$
	4	$dz^{4\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{-96e^{3}+144e^{2}fz^{\eta}-180ef^{2}z^{2\eta}+210f^{3}z^{3\eta}}{945\eta f^{4}}dR^{3}=t$
IV	1	$\frac{dz^{\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{2d}{\eta f}R=t$
	2	$\frac{dz^{2\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{-4e+2fz^{\eta}}{3\eta ff}dR=t$
	3	$\frac{dz^{3\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{16e^{2}-8efz^{\eta}+6ffz^{2\eta}}{15\eta f^{3}}dR=t$
	4	$\frac{dz^{4\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{-96e^{3}+48e^{2}fz^{\eta}-36ef^{2}z^{2\eta}+30f^{3}z^{3\eta}}{105\eta f^{4}}dR=t$

pag. 137.

Figure 8. The beginning of Newton's table of curves (an integral table, in Leibnizian terms), obtained thanks to understanding of what we call the 'fundamental theorem of calculus'. Here Newton lists the first four 'orders'. z is the abscissa, y the ordinate, t the area. In Newton's notation $\dot{t}/\dot{z} = y/1$. Notice that d, e, f, g, h are constants (d is a constant!), η is integer or fractional, and R stands for $\sqrt{e + fz^{\eta}}$ or $\sqrt{e + fz^{\eta} + gz^{2\eta}}$. Source: [8], vol. 1, 105.

6. Newton's synthetical method

One should recall that the 'new analysis' occupied in Newton's agenda a place which, according to the Pappusian canon, was subsidiary to the synthesis or construction, and that the construction had to be carried on in terms independent of algebraic criteria. For instance, as to the squaring of curves (in Leibnizian terms, integration) he wrote:

> After the area of some curve has thus been found, careful considerations should be given to fabricating a demonstration of the construction which

> as far as permissible has no algebraic calculation, so that the theorem embellished with it may turn out worthy of public utterance.[18]

Newton therefore devoted great efforts to providing geometrical demonstrations, somewhat reminiscent of Archimedean exhaustion techniques, of his 'analytical' quadratures. Only such demonstrations were deemed by him 'worthy of public utterance'.

It is in this context that Newton in the 1670s began reworking his early discoveries in 'new analysis' in terms that he conceived concordant with the constructive geometrical methods of the ancients. He termed this more rigorous approach the 'synthetical method of fluxions' and codified it around 1680 in a treatise entitled *Geometria curvilinea* ([9], vol. 4, 420–521). In this method no infinitesimals, or 'moments', occurred and no algebraic symbols were deployed. Everything was based upon geometric limit procedures that Newton termed the 'method of first ratios of nascent quantities and last ratios of vanishing quantities'. It is this method that was widely deployed in the *Principia* (1687) (see Figure 9). It is somewhat astonishing to see one of the most

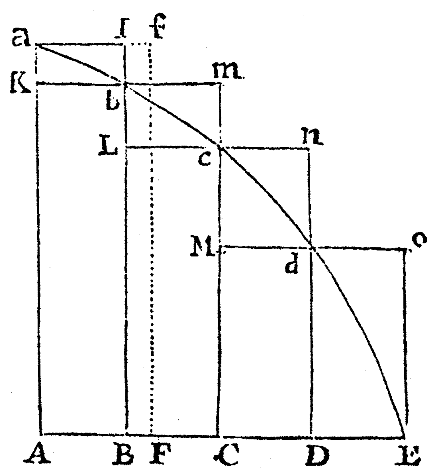


Figure 9. In Section 1, Book 1 of the *Principia* Newton lays down his 'method of first and last ratios', a geometric limit procedure that allows him to avoid infinitesimals. In Lemma 2 Newton shows that a curvilinear area $AabcdE$ can be approached as the limit of inscribed $AKbLcMdD$ or circumscribed $AalbmcndoE$ rectilinear areas. Each rectilinear surface is composed of a finite number of rectangles with equal bases AB, BC, CD, etc. The proof is magisterial in its simplicity. Its structure is still retained in present day calculus textbooks in the definition of the Riemann integral. It consists in showing that the difference between the areas of the circumscribed and the inscribed figures tends to zero, as the number of rectangles is 'increased *in infinitum*'. In fact this difference is equal to the area of rectangle $ABla$ which, 'because its width AB is diminished *in infinitum*, becomes less than any given rectangle'. In Newton's terms AB is a 'vanishing quantity'. Source: [11], 74.

[18] [9], vol. 3, 279.

creative algebraists of the history of mathematics spend so much time and effort in reformulating his analytical results in geometric terms, but Newton had compelling reasons to do so.

First, Newton in his programme of reformation of natural philosophy attributed an important role to mathematics as a source of certainty. From the early 1670s he expressed his distaste for the probabilism and hypotheticism that was characteristic of the natural philosophy[19] practiced at the Royal Society by people like Robert Hooke and Robert Boyle. His recipe was to inject mathematics into natural philosophy. As he stated:

> by the help of philosophical geometers and geometrical philosophers, instead of the conjectures and probabilities that are being blazoned about everywhere, we shall finally achieve a science of nature supported by the highest evidence. ([12])

But if mathematics has to provide certainty to natural philosophy her methods must be above dispute, and Newton was keenly aware of the fact that the new analysis was far from being rigorous.

Second, Newton soon developed a deep anti-Cartesianism associated with a conviction of the superiority of the ancients over the moderns. From his point of view Descartes was the champion of an impious mechanistic philosophy which, conceiving nature as an autonomous mechanism, denied any role to God's providence. Newton conceived himself as a restorer of an ancient, forgotten philosophy according to which nature is always open to the providential intervention of God. Indeed, he thought that, according to the theory of gravitation – which he was convinced the ancient Hebrews possessed–, the quantity of motion in the universe was bound to decline if divine intervention had not prevented the 'corruption of the heavens'. The modern philosophers were dangerous from a theological point of view and had to be opposed on all grounds. Therefore, also in mathematics Newton looked with admiration to ancient exemplars and conceived himself as a restorer of their glory. It goes without saying that the above reasons led Newton into a condition of strain, since his philosophical values were at odds with his mathematical practice, which was innovative, symbolical, and – pace Newton – deeply Cartesian.

Several hitherto unexplained aspects of Newton's mathematical work are related to this condition of stress and strain that characterizes his thoughts on mathematical method. Why did Newton fail to print his method of series and fluxions before the inception of the priority dispute with Leibniz? Why did he hide his competence in quadratures when writing the *Principia*, which are written mostly in geometrical style? Even though there is no single answer to these vexed questions, I believe that Newton's conviction that the analytical symbolical method is only a heuristic tool,

[19] For Newton the aim of 'natural philosophy' is to deduce the forces from phenomena established by experiment, and – once established the forces – to deduce new phenomena from them. Nowadays we would call this enterprise 'physics'.

not 'worthy of public utterance', can in part explain a policy of publication which was to have momentous consequences in the polemic with Leibniz.

7. Leibniz's views

When the war with Leibniz exploded in 1710 Newton had to confront an opponent who not only advanced mathematical results equivalent to his, but was promoting a different view concerning mathematics.[20]

The rhetoric on the novelty of the calculus pervades Leibniz's writings. Reference to the ancient mathematicians generally took the rather abused form of a tribute to Archimedes' 'method of exhaustion'. Leibniz in most of his declarations concerning the calculus wished to highlight the novelty and the revolutionary character of his algorithm, rather than continuity with ancient exemplars. This approach is quite at odds with Newton's 'classicism'. Furthermore, Leibniz often referred to the heuristic character of the calculus understood as an algorithm independent from geometrical interpretation. It is exactly this independence that would render the calculus so efficacious in the process of discovery. The calculus, according to Leibniz, should also be seen as an *ars inveniendi* (an art of discovery): as such it should be valued by its fruitfulness, rather than by its referential content. We can calculate, according to Leibniz, with symbols devoid of referential content (for instance, with $\sqrt{-1}$), provided the calculus is structured in such a way as to lead to correct results.[21]

Writing to Christiaan Huygens in September 1691, Leibniz affirmed with pride:

> It is true, Sir, as you correctly believe, that what is better and more useful in my new calculus is that it yields truths by means of a kind of analysis, and without any effort of the imagination, which often works as by chance. ([13])

[20] The circumstances surrounding the controversy between Newton and Leibniz have been analysed in detail by Rupert Hall [15] and Tom Whiteside [9], vol.8. In broad outlines let me recall a few bare facts. Newton formulated his method of series and fluxions between 1665 and 1669. Leibniz had worked out the differential and integral calculus around 1675 and printed it in a series of papers from 1684. It is clear from manuscript evidence that he arrived at his results independently from Newton. It is only in part in Wallis' *Algebra* in 1685 and *Works* in 1693 and 1699, and in full in an appendix to the *Opticks* in 1704, however, that Newton printed his method. In 1710 a British mathematician, John Keill, stated in the *Philosophical Transactions of the Royal Society* that Leibniz had plagiarized Newton. After Leibniz's protest a committee of the Royal Society secretly guided by its President, Isaac Newton, produced a publication – the so-called *Commercium epistolicum* (1713) – in which it was maintained that Newton was the 'first inventor' and that '[Leibniz's] Differential Method is one and the same with [Newton's] Method of Fluxions'. It was also suggested that Leibniz, after his visits to London in 1673 and 1676, and after receiving letters from Newton's friends, and from Newton himself (in fact Newton addressed two letters to Leibniz in 1676) had gained sufficient information about Newton's method to allow him to publish the calculus as his own discovery, after changing the symbols. It is only after the work of historians such as Fleckenstein, Hofmann, Hall and Whiteside that we have the proof that this accusation was unjust. Newton and Leibniz arrived at equivalent results independently and following different paths of discovery.

[21] Complex numbers received a geometric interpretation only around 1800 thanks to Jean Robert Argand, Carl Friedrich Gauss, and Caspar Wessel.

Leibniz was thus praising the calculus as a *cogitatio caeca* and promoted the 'blind use of reasoning' among his disciples. Nobody, according to Leibniz, could follow a long reasoning without freeing the mind from the 'effort of imagination'.[22]

Leibniz conceived of himself as the promoter of new methods of reasoning, rather than 'just' a mathematician. The calculus was just one successful example of the power of algorithmic thinking. The German diplomat was interested in promoting in Europe the formation of a group of intellectuals who could extend a universal knowledge achieved thanks to a new algorithm that he termed *universal characteristic*. He thus helped to form a school of mathematicians who distinguished themselves by their ability in handling the differentials and the integrals and by their innovative publication strategy. Thanks to Leibniz's recommendation, they colonized chairs of mathematics all over Europe. The efficacy of this new algorithm was affirmed to be independent from metaphysical or cosmological questions. The persons who practised it had to be professional mathematicians, rather than 'geometrical philosophers', able to teach and propagate knowledge of calculus.

A typical Leibnizian attitude emerges in the context of the vexed question of the existence of infinitesimals. The new calculus was often attacked, since – it was maintained – it employed symbols devoid of meaning, such as differentials ordered into a bewildering hierarchy of orders. Newton, as we know, was particularly sensitive to such criticisms, and tried in his synthetical method to dispense with infinitely small quantities. Leibniz, on the other hand, repeated many times that for him the question of the existence of infinitesimals had to be distinguished from that of their usefulness as algorithmic devices. While he was leaning, for philosophical reasons, towards a denial of the existence of infinitesimals, he also wanted to stress that this ontological question was somewhat extraneous to mathematics. A typical statement, written in the early years of the eighteenth century, is the following:

> We have to make an effort in order to keep pure mathematics chaste from metaphysical controversies. This we will achieve if, without worrying whether the infinites and infinitely smalls in quantities, numbers and lines are real, we use infinites and infinitely smalls as an appropriate expression for abbreviating reasonings. ([14])

Leibniz was thus leaving to his disciples the choice of maintaining, *philosophically speaking*, different approaches to the ontological question on the existence of infinitesimals. What he wished to defend was their utility as symbols in mathematical calculation.

8. The war against Leibniz: methodological aspects

When Newton had to confront Leibniz in the squabble over priority he was concerned in building up a forensic and historical document whose purpose was to prove

[22] [14], 205.

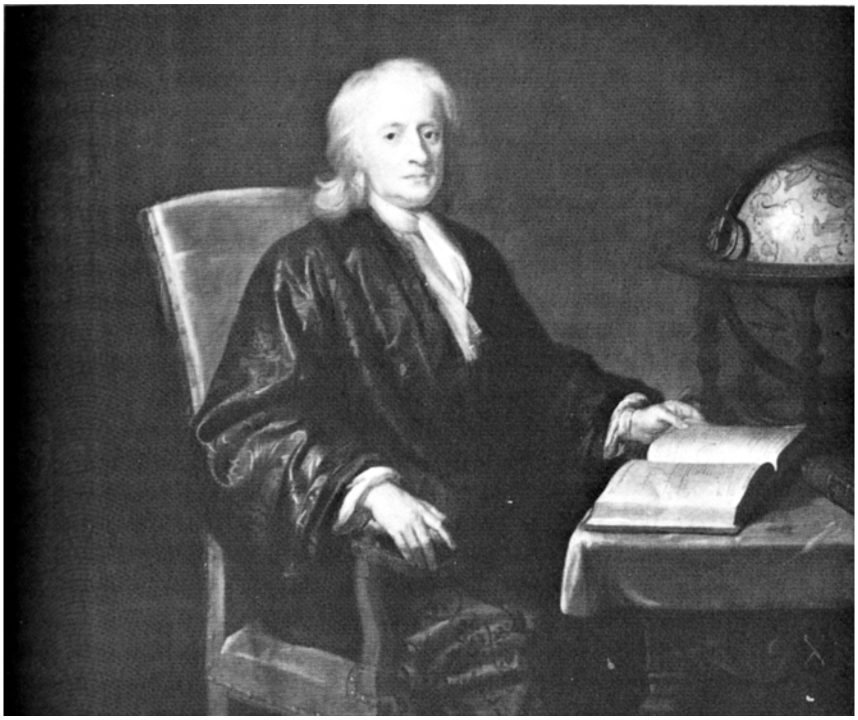


Figure 10. A portrait of Newton in old age (Source: [1], 831). He proudly opens the *Principia* at a page devoted to the attraction of extended bodies. In dealing with this problem Newton made recourse to his 'inverse method of fluxions' (the equivalent of Leibniz's integral calculus) which allowed him to 'square curves'. As a matter of fact, only by making recourse to his tables of curves ('integral tables'), see Figure 8, could Newton solve several problems in the *Principia*. Such analytic methods were not, however, made explicit to the reader. In the polemic with the Leibnizians – who claimed that absence of calculus from the *Principia* was proof positive of Newton's ignorance of quadrature techniques prior to 1687 – Newton was forced to maintain, with some exaggeration, that 'By the help of this new Analysis Mr Newton found out most of the Propositions in his *Principia Philosophiae*. But because the Ancients for making things certain admitted nothing into Geometry before it was demonstrated synthetically, he demonstrated the Propositions synthetically that the systeme of the heavens might be founded upon good Geometry. And this makes it now difficult for unskillful men to see the Analysis by w^{ch} those Propositions were found out.' ([9], vol. 8, 599). On the issue of Newton's use of analytic methods in the *Principia* see [16].

Leibniz's plagiarism. But he did not do only this, he also wished to highlight the superiority of his *method* over Leibniz's *calculus*. The mathematical programme that Leibniz was promoting with so much success was at odds with Newton's deeply felt values.

There is not only mathematics in this story, of course. Leibniz had to be opposed for a series of reasons that have to do with the Hannoverian succession. The German, in fact, who was employed by the Hannover family, wished to move to London as Royal Historian. The idea of having in England such a towering intellectual who was defending a philosophical view which contradicted Newton's voluntarist theology and who was promoting the unification of the Christian Churches was anathema for Newton and his supporters.

For our purposes, it is interesting to turn to some passages that Newton penned in 1715 contained in an anonymous 'Account' to a collection of letters, the *Commercium epistolicum*, that the Royal Society produced in order to demonstrate Leibniz's plagiarism.

In the 'Account', speaking of himself in the third person, Newton made it clear that Leibniz had only approached the analytical, heuristic part of the problem-solving method. He wrote:

> Mr. Newton's Method is also of greater Use and Certainty, being adapted either to the ready finding out of a Proposition by such Approximations as will create no Error in the Conclusion, or to the demonstrating it exactly; Mr. Leibniz's is only for finding it out.[23]

So according to Newton, Leibniz had achieved only the first stage of the Pappusian method and had not attained the rigorous, constructive demonstration. This, as we know, had to be carried on in purely geometric terms.

Further, Newton insisted on the fact that the emphasis with which Leibniz praised the power of his symbolism was excessive. Algorithm is certainly important for Newton, but it has to be viewed only as a component of the method:

> Mr Newton — he wrote – doth not place his Method in Forms of Symbols, nor confine himself to any particular Sort of Symbols.[24]

Finally, Newton noticed that in his method of first and last ratios no infinitesimals occur, everything being performed according to limiting procedures. From Newton's point of view the avoidance of infinitesimals and the possibility of interpreting algebraic symbols as geometric magnitudes had the double advantage of rendering his method endowed with referential content and consonant with ancient mathematics:

> We have no ideas of infinitely little quantities & therefore Mr Newton introduced fluxions into his method that it might proceed by finite quantities as much as possible. It is more natural & geometrical because founded on *primae quantitatum nascentium rationes* [first ratios of nascent quantities] w^ch have a being in Geometry, whilst *indivisibles* upon which the Differential method is founded have no being either in Geometry or in nature. [...] Nature generates quantities by continual flux or increase, & the ancient Geometers admitted such a generation of areas & solids [...]. But the summing up of indivisibles to compose an area or solid was never yet admitted into Geometry.[25]

[23] Cited in [15], 296.
[24] Cited in [15], 294.

Nature and geometry are the two key concepts: they allow Newton to defend his method because of its continuity with ancient tradition as well as its ontological content.

In his polemic writings against Leibniz Newton engineered an attack which was aimed at proving the German's plagiarism. One of Newton's priorities was to assemble evidence which proved Leibniz guilty, and he did so with means that show his ability to employ archival sources as well as his prejudice and egotism. However, Newton also defended positions concerning mathematical method that have deep roots in his protracted opposition against Descartes and the 'modern mathematicians' who, by confounding geometry and algebra, 'have lost the Simplicity in which all the Elegance of Geometry consists'.

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