Incomplete Preference relations: An upper bound condition

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Abstract. In decision making, consistency in fuzzy preference relations is associated with the study of transitivity property. While using additive consistency property to complete incomplete preference relations, the preference values found may lie outside the interval [0, 1] or the resultant relation may itself be inconsistent. This paper proposes a method that avoids inconsistency and completes an incomplete preference relation using an upper bound condition. Additionally, the paper extends the upper bound condition for multiplicative reciprocal preference relations. The proposed methods ensure that if \((n - 1)\) preference values are provided by an expert, such that they satisfy the upper bound condition, then the preference relation is completed such that the estimated values lie inside the unit interval [0, 1] in the case of preference relations and \([1/9, 9]\) in the case of multiplicative preference relation. Moreover, the resultant preference relation obtained using the proposed method is transitive.

Keywords: preference aggregation, multiplicative preference relations, incomplete preference relations, additive consistency

1. Introduction

Decision Making is a routine activity and most decision making processes are based on preference relations. In case of fuzzy preferences and multiplicative fuzzy preferences, transitivity is a traditional requirement to characterize Saaty’s consistency ([8][9]). A consistent fuzzy preference relation should at least satisfy restricted max-max transitivity. Some of the other transitivity properties are max-min transitivity, restricted max-max transitivity and additive transitivity, also discussed in ([10]) and ([12]).

It is not reasonable to expect every decision maker to be certain about the degree of intensity of each alternative over others. An expert may be ambiguous about the problem at hand or may not have sufficient knowledge to discriminate the degree to which some alternatives are better than others.

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when \((n-1)\) preference values \(\{p_{12}, p_{23}, ..., p_{(n-1)n}\}\) are provided by the expert. A more general condition which includes the case where a complete row or column is given, is provided in ([7]). Estimated preference values that surpassed the unit interval were taken care of with a transformation function defined by Herrera in ([7],[5]). These transformation functions result in a complete preference relation with preference values inside the interval \([0, 1]\) but the consistency of the resultant relation is not assured. Moreover, this can void the originality of preference values given by the experts.

We restrict ourselves to the study of incomplete fuzzy preference and multiplicative fuzzy preference relations. It needs to be noticed that although the proposed methods are successful in completing the incomplete fuzzy preference and multiplicative fuzzy preference relations, some of these methods use transformation functions to take care of the surpassed preference values but they are silent about consistency of the resultant completed relation. To bring meaning to the resultant matrices, they need to satisfy some criteria of transitivity because lack of consistency may lead to inconsistent solutions.

In this paper, we propose upper bound conditions to deal with incomplete fuzzy preference and multiplicative fuzzy preference relations. Additive consistency along with these upper bound conditions ensure that the missing preference intensities do not surpass their respective codomain. So construction of translation functions, which may void originality of preference values provided by experts, is not required. Moreover, the fuzzy preference and multiplicative fuzzy preference relations completed using the proposed method posses the property of additive consistency.

The paper is organized as follows. Section 2 contains basic definition required in the sequel. In section 3 the notion of expressible preference values is defined and a method based on additive consistency and an upper bound condition is proposed. The advantage of this method is that the estimated missing values produced are always expressible and the resultant complete fuzzy preference relation attained satisfies additive transitivity. Section 4 focuses on a similar method based on Saaty’s consistency and an upper bound condition for incomplete multiplicative fuzzy preference relations. The estimated preference values using this method are termed as multiplicative expressible since they do not exceed \([1/9, 9]\). Moreover, the resultant multiplicative fuzzy preference relation satisfies Saaty’s consistency. This section also includes the drawbacks of the upper bounds defined for the respective preference relations. Section 5 draws the conclusions and gives insight to possible future work.

2. Preliminaries

Throughout this section the non empty set \(X = \{x_1, x_2, ..., x_n\}\) represents the set of alternatives.

Definition 1: ([6],[10],[11]) A fuzzy preference relation \(P\) on \(X\) is characterized by a function \(\mu_P : X \times X \rightarrow [0, 1]\) where \(\mu(x_i, x_j) = p_{ij}\) indicates the preference intensity or the degree of confidence with which alternative \(x_i\) is preferred over \(x_j\).

\(p_{ij} = \frac{1}{3}\) indicates indifference between the alternatives \(x_i\) and \(x_j\).

\(p_{ij} = 0\) indicates that alternative \(x_i\) is absolutely preferred to \(x_j\).

\(p_{ij} = 1\) indicates that alternative \(x_i\) is absolutely preferred to \(x_j\).

(1). \(P\) is defined to be additive reciprocal if it satisfies \(p_{ij} + p_{jk} = 1, \forall i, j, k\).

(2). \(P\) is additive transitive if \(p_{ij} = p_{ik} + p_{kj} - 0.5, \forall i, j, k\).

where \(i, j, k \in \{1, 2, ..., n\}, i \neq j \neq k\).

Definition 2: ([8],[9]) Let \(A \subset X \times X\) denote a multiplicative preference relation, the intensity of preference, \(a_{ij}\) is measured using a ratio scale, particularly, a \(1 - 9\) scale.

\(a_{ij} = 1\) indicates indifference between \(x_i\) and \(x_j\).

\(a_{ij} = 9\) indicates that \(x_i\) is absolutely preferred to \(x_j\).

(3). \(A\) is multiplicative reciprocal if \(a_{ij}, a_{ji} = 1, \forall i, j\).

(4). A reciprocal multiplicative preference relation is consistent if it satisfies Saaty’s consistency. That is, if \(a_{ij}, a_{ik} = a_{ik}, \forall i, j, k\).

where \(i, j, k \in \{1, 2, ..., 9\}, i \neq j \neq k\).

Definition 3: ([7]) A function \(f : X \rightarrow Y\) is partial when not every element in the set \(X\) necessarily maps onto an element in the non empty set \(Y\). When every element from the set \(X\) maps onto one element of the set \(Y\), then we have a total function.

The incomplete fuzzy preference relation \(P\) and incomplete multiplicative fuzzy preference relation \(A\) on \(X\) is a fuzzy set on the product set \(X \times X\) that is characterized by a partial membership function.

3. Upper bound condition for incomplete preference relations

Methods given in literature have been successful in estimating missing values in fuzzy preference and
multiplicative fuzzy preference relations. However, we highlight that these methods seem unconcerned with the consistency of the resultant completed relations. These methods propose functions to take care of the surpassed estimated values that they produce but these functions may void the originality of the values provided by experts.

We define a method based on additive consistency and an upper bound condition. The aim is to attain two purposes. Firstly, this method should produce expressible preference values, we refer to an estimated preference value as expressible if it does not surpass [0, 1]. Secondly, the resultant fuzzy preference relation must be consistent.

In this section, we refer to an estimated preference value as crucial if it can be found using the least and the greatest preference values provided by the expert. Such value is called crucial because it may not always be expressible. It should be noticed that if the crucial values are expressible then without estimating other missing values we can be certain about the expressibility of other missing values.

**Example 1:** Consider the $4 \times 4$ incomplete fuzzy preference relation where preference intensities of alternative $x_3$ over $X = \{x_1, x_2, x_3, x_4\}$ are stated.

$$P = \begin{bmatrix}
0.5 & - & - & - \\
-0.5 & - & - & - \\
0.8 & 0.9 & 0.5 & 0.2 \\
- & - & - & 0.5
\end{bmatrix} \quad (1)$$

The crucial value $p_{42}$ in this case is the amalgamation of $p_{32}$ and $p_{42}$ estimated using additive transitivity as $p_{42} = p_{43} + p_{32} - 0.5 = (1 - p_{34}) + p_{32} - 0.5 = 0.8 + 0.9 - 0.5 = 1.2$. The crucial value found is inexpressible. Transformation functions proposed in the literature would drag the inexpressible values back in the unit interval but the resultant relation will not necessarily be consistent.

We now propose a method to find a complete and consistent preference relation containing expressible preference intensities.

**Theorem 1:** If $(n-1)$ preference values $p_{kj}, i,j \in \{1,2,3,...,n\}$ are provided by an expert in an $n \times n$ fuzzy preference relation then this incomplete preference relation can be completed with expressible preference degrees only if the greatest value provided $\delta$ satisfies the upper bound $\delta \leq 0.5 + \epsilon$ where $\epsilon$ is the least preference value provided by the expert.

**Proof:** Suppose that an expert provides two preference values $p_{ki}$ and $p_{kj}, i,j \in \{1,2,3\}, i \neq j \neq k$ over the set of alternatives $X = \{x_1, x_2, x_3\}$. Let $0 \leq p_{ki} = \epsilon < 0.5$ and $p_{kj}$ be the least and greatest given preference values respectively such that $p_{kj}$ satisfies the upper bound condition.

$$0 \leq p_{kj} \leq 0.5 + \epsilon \quad (2)$$

We claim that if crucial values are expressible then so are other missing preference values. We identify the crucial value $p_{ij}$ and test its expressibility. Using additive consistency we state that $p_{ij} = p_{ik} + p_{kj} - 0.5 = (1 - p_{ki}) + p_{kj} - 0.5 = p_{kj} - p_{ki} + 0.5$, which is expressible since $p_{kj} - p_{ki} \leq 0.5$ according to (2). This implies that $p_{ij} \leq 1$. So, $p_{ij}$ and hence $p_{ji} \in [0, 1]$. Therefore the missing entries are expressible.

Suppose that $X = \{1,2,...,q\}$ and expert provides preference degrees of $k^{th}$ alternative over others $\{p_{ki}, p_{ki}, ..., p_{kj}, ..., p_{kj}\}$. Where $p_{ki}$ and $p_{kj}$ is the least and greatest respective preference value provided such that they satisfy the upper bound condition $p_{kj} \leq 0.5 + p_{ki}$.

Then the crucial value $p_{ij} = p_{ik} + p_{kj} - 0.5 = (1 - p_{ki}) + p_{kj} - 0.5 \leq (1 - p_{ki}) + (0.5 + p_{ki}) - 0.5 = 1$ is expressible.

We prove the claim that if crucial values are expressible, then other missing preference values are also expressible.

Let $p_{sj}, s \neq j, s,j \in \{1,2,...,n\}$ be a missing preference value other than the crucial values. Using upper bound condition we prove that this is also expressible.

$$p_{sj} = p_{sk} + p_{kj} - 0.5 \leq p_{sk} + (0.5 + p_{ki}) - 0.5 \quad (3)$$

We know that, $p_{ki} \leq p_{ks} \leq p_{kj}$ which implies $1 - p_{ki} \geq p_{sk} \geq 1 - p_{kj}$. Using this in (3) provides $p_{sj} \leq p_{sk} + (0.5 + p_{ki}) - 0.5 \leq p_{sk} + p_{ki} \leq (1 - p_{ki}) + p_{ki} = 1$. Therefore, if crucial values are expressible then so are other missing preference values.

**Corollary 1:** When missing preference values are estimated using additive consistency along with the upper bound condition then the resultant fuzzy preference relation satisfies additive consistency.

**Proof:** The proof directly follows from proof of Theorem 1. We prove that the largest preference value of each row of a complete preference relation obeys the upper bound condition. This is proved in the following corollary. Each row has a least preference value, the expression $0.5+$ least preference value exhibits preference values that are more than the intensity of indifference in each row, where $+$ is the addition operation...
for real numbers.

Corollary 2: The largest preference value of any row of a complete additive transitive relation is less than 0.5 + least preference value of that row.

Proof: Suppose on the contrary that for \( k \in \{1, 2, ..., n\}, k \neq j \neq i \)

\[ p_{kj} > p_{ki} + 0.5 \tag{4} \]

where \( p_{kj} \) and \( p_{ki} < 0.5 \) are the greatest and least preference values of the \( k^{th} \) row. Then (4) implies \( p_{kj} - p_{ki} - 0.5 > 0. \) That is \( p_{kj} - (1 - p_{ik}) - 0.5 > 0. \) So \( p_{ik} + p_{kj} - 0.5 - 1 > 0. \) Using additive consistency \( p_{ij} - 1 > 0. \) Which implies that \( p_{ij} > 1 \) is not expressible. Therefore the greatest preference value in each row of an additive transitive preference relation obeys the upper bound condition.

Example 2: For instance, in the following complete preference relation, notice that the relation \( \delta \leq 0.5 + \epsilon \) holds true for each row.

\[
P = \begin{bmatrix}
0.5 & 0.7 & 0.8 & 0.8 & 0.3 \\
0.3 & 0.5 & 0.6 & 0.6 & 0.1 \\
0.2 & 0.4 & 0.5 & 0.5 & 0 \\
0.2 & 0.4 & 0.5 & 0.5 & 0 \\
0.7 & 0.9 & 1 & 1 & 0 \end{bmatrix}
\tag{5}
\]

4. Upper bound condition for incomplete multiplicative preference relations

Given a reciprocal multiplicative preference relation \( A = (a_{ij}), a_{ij} \in [\frac{1}{9}, 9] \). Chiclana et.al in ([3]) proposed a function \( f(a_{ij}) = \frac{1}{2}(1 + \log(a_{ij})) \) to evaluate fuzzy reciprocal preference relations corresponding to their respective multiplicative reciprocal preference relations. Since this function is bijective, we can formulate an inverse function

\[ a_{ij} = g(p_{ij}) = g^{2p_{ij}-1} \tag{6} \]

to find reciprocal multiplicative preference relation corresponding to each reciprocal fuzzy preference relation. We will use this idea to construct an upper bound condition for incomplete multiplicative preference relations.

Example 3: Assume that a \( 4 \times 4 \) incomplete multiplicative preference relation is provided by an expert such that \( a_{12} = \frac{1}{5}, a_{14} = 9, a_{14} = 1. \)

\[
A = \begin{bmatrix}
1 & 1/8 & 9 & 1 \\
8 & 1 & - & - \\
1/9 & 1 & - & - \\
1 & - & - & 1 \end{bmatrix}
\tag{7}
\]

Without any condition on the least and greatest multiplicative preference values \( a_{ij} \) and \( a_{kl} \) respectively, we use Saaty’s consistency to find the crucial preference value \( a_{23} = a_{21}a_{13} = 72 \) which is not expressible since it does not belong to \([\frac{1}{9}, 9]\). We use equation (6) to formulate an upper bound condition for multiplicative reciprocal preference relations. This condition would confirm the expressibility of missing multiplicative preference values.

Theorem 2: If \( (n - 1) \) preference values \( p_{kj}, j \in \{1, 2, 3, ..., n\} \) are provided in an \( n \times n \) multiplicative preference relation, then the missing preference intensities are expressible only if the relation \( g^{mult} \leq 9e^{mult} \) is satisfied by the least \( e^{mult} \) and greatest \( g^{mult} \) preference values given by expert.

Proof: Trivial using equation (6) and proof of Theorem 1.

Corollary 3: If an incomplete multiplicative preference relation is completed using upper bound condition of Theorem 2 and Saaty’s consistency then the resultant relation satisfies Saaty’s consistency.

Proof: Directly follows from Theorem 2.

Example 4: Suppose that \( X = \{x_1, x_2, x_3, x_4\} \) and some knowledge about the preferences is provided such that the situation is modeled by the preference values \( a_{12} = 4, a_{13} = \frac{1}{2}, a_{14} = 3. \) Since \( a_{12} \leq 9.a_{13}, \) Therefore, according to the Theorem 2, the incomplete multiplicative preference relation can be completed with expressible preference values.

\[
a_{23} = a_{21}.a_{13} = (1/4).(1/2) = 1/8; a_{24} = a_{21}.a_{14} = (1/4).(3) = 3/4; a_{34} = a_{31}.a_{14} = (2).(3) = 6; a_{41} = 1/a_{13} = 2; a_{41} = 1/a_{14} = 1/3; a_{42} = 1/a_{24} = 4/3; a_{43} = 1/a_{14} = 1/6. \]

\[
A = \begin{bmatrix}
1 & 4 & 1/2 & 3 \\
1/4 & 1 & 1/8 & 3/4 \\
2 & 8 & 1 & 6 \\
1/3 & 4/3 & 1/6 & 1 \end{bmatrix}
\tag{8}
\]

The completed relation satisfies Saaty’s consistency. Following is a complete fuzzy preference relation corresponding to the consistent multiplicative preference relation in Example 4. It needs to be noticed that a
fuzzy preference relation corresponding to a consistent multiplicative preference relation satisfies additive transitivity.

\[
P = \begin{bmatrix} 0.5 & 0.4 & 0.6 & 0.8 \\ 0.6 & 0.5 & 0.7 & 0.9 \\ 0.4 & 0.8 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0.3 & 0.5 \end{bmatrix}
\]

(9)

Remark 1: If a multiplicative preference relation satisfies Saaty’s consistency then the corresponding fuzzy preference relation constructed \( f(a_{ij}) = \frac{1}{2}(1 + \log a_{ij}) \) satisfies additive transitivity property.

Remark 2: For an additive transitive preference relation with non-zero preferences, the corresponding multiplicative preference relation satisfies Saaty’s consistency. Although, the upper bound condition addresses and resolves the issue of incomplete preferences but it needs to be noted that the interval of the values of the incomplete relation is smaller than the interval of predicted values using additive or multiplicative consistency properties. Because of the implication of upper bound conditions, experts may experience difficulty in expressing preference intensities within the subintervals of \([0, 1]\) or \([1, 1/9]\) in the cases of preference relations and multiplicative preference relations respectively.

5. Conclusions and Future Work

The paper focuses on incomplete preference and multiplicative preference relations. Methods proposed in literature to complete such relations are silent on consistency of the resultant relation. The purpose of completing an incomplete relation is to make it useful in the process of decision making. If the resultant relations are not consistent then the purpose of completing incomplete relations at the first place needs attention. While estimating missing values, Herrera ([7][5]) used transformation functions to bring surpassed values back in the interval \([0, 1]\). In this way the originality of preference values given by the expert may be voided. Moreover, the missing values attained using such transformation functions do not promise consistency of the resultant relation.

This paper constructs an upper bound condition for incomplete fuzzy preference and multiplicative fuzzy preference relations which ensures the expressibility of missing preference values. Moreover, this method provides a resultant relation that satisfies additive transitivity and is therefore consistent.

It is brought to notice that corresponding to each consistent multiplicative preference relation the fuzzy preference relation found using function proposed by Chiclana in ([3]) is additive transitive (and vice versa). For future work, more relaxed bounds may be defined to complete preference relations but which do not restrict the experts to express preferences in the dictated subinterval. Also, an upper bound condition for fuzzy preference and multiplicative fuzzy preference relations may be constructed in the case when \((n - 1)\) diagonal preference intensities are provided. Moreover, a parallel model in fuzzy linguistic preference relations setting could be formulated.

References

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