

Fisher information as a generalized measure of coherence in classical and quantum optics

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Abstract: We show that metrological resolution in the detection of small phase shifts provides a suitable generalization of the degrees of coherence and polarization. Resolution is estimated via Fisher information. Besides the standard two-beam Gaussian case, this approach provides also good results for multiple field components and nonGaussian statistics. This works equally well in quantum and classical optics.

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1. Introduction

Interference and polarization are leading manifestations of coherence [1, 2]. In spite of being a very basic topic the complete relationship between coherence, interference and polarization for arbitrary fields is still an open question and is currently the subject of very active investigation [3–24].

For two electromagnetic fields $E_{1,2}$ we may say that interference reflects coherence between fields in the same vibration mode, while polarization displays coherence between fields in orthogonal vibration modes. Interference and polarization are easily interchangeable by means of simple energy-preserving transformations, such as phase plates, so that they are not independent field properties.

More specifically, coherence manifests in interference through the modulation of the intensity of the superposition $\langle |E_1 e^{i\phi_1} + E_2 e^{i\phi_2}|^2 \rangle$, where ϕ_j are the phases acquired by the waves within the interferometer. This modulation can be conveniently assessed by the fringe visibility

$$V(\mathbf{E}) = \frac{2|\langle E_1 E_2^* \rangle|}{\langle |E_1|^2 \rangle + \langle |E_2|^2 \rangle} \leq |\mu(\mathbf{E})| = \frac{|\langle E_1 E_2^* \rangle|}{\left(\langle |E_1|^2 \rangle \langle |E_2|^2 \rangle\right)^{1/2}}, \quad (1)$$

where μ is the complex second-order degree of coherence, and the angle brackets represent ensemble averages. In the polarization context, coherence manifests in the purity of the polarization state assessed through the degree of polarization

$$\mathcal{P} = \frac{|I_M - I_m|}{I_M + I_m} = \left[1 - 4 \frac{\det \Gamma}{(\text{tr} \Gamma)^2} \right]^{1/2}, \quad \Gamma(\mathbf{E}) = \begin{pmatrix} \langle |E_1|^2 \rangle & \langle E_1 E_2^* \rangle \\ \langle E_1^* E_2 \rangle & \langle |E_2|^2 \rangle \end{pmatrix}, \quad (2)$$

where $I_{M,m}$ are the eigenvalues of the correlation matrix Γ . A key point is that \mathcal{P} is invariant under deterministic unitary transformations $\mathbf{E} \rightarrow \mathbf{E}' = U\mathbf{E}$, $\Gamma(\mathbf{E}) \rightarrow \Gamma(\mathbf{E}') = U\Gamma(\mathbf{E})U^\dagger$, being U 2×2 unitary matrices, so we have $\mathcal{P}(\mathbf{E}') = \mathcal{P}(\mathbf{E})$ while $V(\mathbf{E}') \neq V(\mathbf{E})$ and $|\mu(\mathbf{E}')| \neq |\mu(\mathbf{E})|$. More specifically, it holds that V and \mathcal{P} satisfy the equality [21]

$$V^2 + R^2 = \mathcal{P}^2, \quad R(\mathbf{E}) = \frac{|\langle |E_1|^2 \rangle - \langle |E_2|^2 \rangle|}{\langle |E_1|^2 \rangle + \langle |E_2|^2 \rangle}. \quad (3)$$

This, along with Eq. (1), implies that \mathcal{P} is the maximum degree of coherence that can be obtained for a given field state after any deterministic unitary transformation

$$|\mu(U\mathbf{E})| \leq \mathcal{P}. \quad (4)$$

The standard definitions of the degrees of coherence and polarization in Eqs. (1) and (2) hold provided that three conditions are satisfied: i) we are restricted to two scalar electromagnetic fields, ii) we perform statistical evaluations of second order in the amplitudes, and iii) the fields obey Gaussian statistics. Beyond these three restrictions the standard approach may be undetermined or lack usefulness when applied to more general situations involving either more than two electric fields or nonGaussian statistics, as it is the case of the problems addressed in Refs. [3–33]. A particularly clear example of these limitations is provided by

quantum field states reaching maximum resolution in phase-shift detection, since they present vanishing degrees of polarization and coherence $\mu = \mathcal{P} = 0$ in spite of their interferometric usefulness [22, 32].

In this work we look for a more general approach to coherence that can be applied to any situation with multi-component fields and nonGaussian statistics, including naturally Eqs. (1) and (2) in the particular case of just two electric fields in a thermal-chaotic light state. To this end we focus on the practical application of interference and polarization to the detection of small phase shifts, as the cornerstone of the metrological applications of optics.

The key point is the idea of coherence as equivalent to good interference. This idea emerges after Eq. (1), where coherence is the maximum fringe visibility, as well as after Eq. (4), where visibility reaches its maximum provided that the unitary transformation U is properly chosen. This suggests that the degrees of coherence and polarization indicate the best interference that can be achieved with a suitable preparation of the interference for a given input field state.

Therefore, we will identify the generalized amount of coherence χ of a given field state with the maximum phase-shift resolution after a suitable optimization of the detection arrangement, so that the phase-shift measurement exploits all the interfering capabilities of the field state. Note that this approach can be equally well applied to quantum and classical fields.

The resolution can be properly assessed by means of the Fisher information entering in the Cramér-Rao lower bound for the uncertainty of phase-shift estimators [34, 35]. It is worth pointing out that the Fisher information has been extensively used to provide original analyses and interpretations of many different problems in science, from the formation of basic physical laws, to biology and financial economy, for example [36, 37].

2. Fisher information and detection of small phase shifts

2.1. Phase shifts and phase-shift detection

By phase shifts we will understand linear, energy-preserving, unitary transformations of the complex amplitudes \mathbf{E} of an arbitrary number k of complex electric fields, $\mathbf{E} \rightarrow U_\phi \mathbf{E}$, where U_ϕ is a $k \times k$ unitary matrix. In general terms we may express U_ϕ as $U_\phi = \exp(i\phi G)$, where ϕ is the phase shift, and G an Hermitian $k \times k$ matrix $G^\dagger = G$.

In quantum optics the amplitudes \mathbf{E} become complex-amplitude operators \mathbf{a} , with $[a_i, a_j^\dagger] = \delta_{i,j}$ and $[a_i, a_j] = 0$. The phase-shift transformation reads $\mathbf{a} \rightarrow U_\phi \mathbf{a} = T^\dagger \mathbf{a} T$, with $T = \exp(i\phi \hat{G})$ and $\hat{G} = \mathbf{a}^\dagger \mathbf{G} \mathbf{a}$ [38].

Both in classical and quantum optics the objective is to infer the value of the unknown phase shift ϕ from measurements performed on the transformed electric fields $U_\phi \mathbf{E}$. Let us call $P(\theta|\phi)$ the statistics of the measurement outcomes θ when the true value of the phase shift is ϕ . From a Bayesian perspective $P(\theta|\phi)$ defines a *posteriori* probability distribution $P(\tilde{\phi}|\theta)$ for the inferred value of the signal $\tilde{\phi}$ depending on the N outcomes θ obtained after N repetitions of the measurement. This is of the form

$$P(\tilde{\phi}|\theta) \propto \prod_{j=1}^N P(\theta_j|\tilde{\phi}) \Phi(\tilde{\phi}), \quad (5)$$

where $\Phi(\tilde{\phi})$ is the prior distribution for the signal.

2.2. Fisher information

A convenient assessment of the uncertainty $\Delta\phi$ on the value of ϕ after the measurement is the Fisher information F entering in the Cramér-Rao lower bound

$$(\Delta\phi)^2 \geq \frac{1}{NF}, \quad F = \int d\theta \frac{1}{P(\theta|\phi)} \left[\frac{dP(\theta|\phi)}{d\phi} \right]^2. \quad (6)$$

It is worth noting that the Cramér-Rao lower bound is not necessarily reachable in every situation, i. e., there is no guarantee that an efficient estimator can be found that reaches the minimum variance $1/(NF)$, specially for small N . However, for increasing N , the maximum likelihood estimator [the $\tilde{\phi}$ value that maximizes $P(\tilde{\phi}|\theta)$] is unbiased and asymptotically efficient, so that $\Delta\tilde{\phi} \rightarrow 1/\sqrt{NF}$ when $N \rightarrow \infty$ [34]. After a large enough number of repetitions, $1/(NF)$ becomes the width of $P(\tilde{\phi}|\phi)$ as a function of $\tilde{\phi}$ since $P(\tilde{\phi}|\phi) \propto \exp[-NF(\tilde{\phi} - \phi)^2/2]$, which is consistent with the above mentioned asymptotic efficiency.

Besides, the Fisher information is also the approximate form of several distances, or divergences [39, 40], between the statistics associated with two close enough phase shifts ϕ and $\phi + \delta\phi$, such as the Hellinger distance,

$$H = \int d\theta \left[\sqrt{P(\theta|\phi + \delta\phi)} - \sqrt{P(\theta|\phi)} \right]^2 \simeq \frac{(\delta\phi)^2}{4} F, \quad (7)$$

the Kullback-Liebler divergence

$$K = \int d\theta P(\theta|\phi + \delta\phi) \ln \frac{P(\theta|\phi + \delta\phi)}{P(\theta|\phi)} \simeq \frac{(\delta\phi)^2}{2} F, \quad (8)$$

or the Rényi-Chernoff distance

$$C_s = -\ln \int d\theta P^s(\theta|\phi + \delta\phi) P^{1-s}(\theta|\phi) \simeq \frac{s(1-s)}{2} (\delta\phi)^2 F, \quad (9)$$

that becomes the Bhattacharyya distance for $s = 1/2$. All them have been already applied for the assessment of coherence and polarization in different contexts [29–31, 41, 42].

2.3. Generalized measure of coherence

In general, F depends on the light state, on the generator G , on the measurement performed, and also on ϕ . Following the idea of coherence as optimum visibility, we will consider that the measurement performed extracts from the transformed fields $U_\phi \mathbf{E}$ the maximum information possible about ϕ . Throughout we assume $\phi \rightarrow 0$ since precision metrology is always interested in the detection of small phase shifts, which can be, without loss of generality, brought back around a zero phase-shift value. Nevertheless, there would be no fundamental difficulty to apply this approach beyond precision metrology to large phase shifts, maybe with the drawback of lacking simple expressions and obtaining different conclusions depending on the particular values of ϕ considered.

The larger F the lesser the uncertainty and the larger the resolution so we may say that larger F means larger coherence. Since F is not bounded from above, $\infty \geq F \geq 0$, it might be convenient to normalize it, so we may introduced a generalized assessment of coherence χ comprised between 0 and 1 in the form

$$\chi^2 = \frac{F}{F + 4\text{tr}(G^2)}. \quad (10)$$

The factor $4\text{tr}(G^2)$ is introduced so that χ becomes exactly the degree of coherence for two electric fields in a classical thermal-chaotic light state [see Eq. (21) below]. The dependence on G means essentially that the usefulness of interference fringes depends on whether the phase-shift transformation exploits all the interfering capabilities of the light state. A coherence assessment depending just on the field state can be obtained as the maximum of χ when G is varied. This might be regarded as the generalization of the degree of polarization as the ultimate measure of coherence in accordance with the standard result for two electric fields and Gaussian statistics recalled in the introduction.

Quantum and classical statistics differ in very fundamental issues, such as the uncertainty relation for example. Thus, at this point the analysis must be split into classical and quantum channels, although we will see that they largely run parallel.

3. Classical sector

In classical optics every measurement statistics $P(\theta|\phi)$ can be derived as a marginal distribution of the probability distribution for the transformed amplitudes $W(\mathbf{E}|\phi) = W(U_\phi^\dagger \mathbf{E})$, where $W(\mathbf{E})$ is the probability distribution for the complex amplitudes in the field state before the signal-dependent transformation U_ϕ . More specifically, taking into account that any measured observable is a function of the complex amplitudes $\theta(\mathbf{E})$ we have

$$P(\theta|\phi) = \int d^{2k} E W(U_\phi^\dagger \mathbf{E}) \delta[\theta - \theta(\mathbf{E})]. \quad (11)$$

Formally we can consider a suitable change of variables including θ within the new set of arguments of the transformed distribution \mathscr{W} , so that

$$P(\theta|\phi) = \int d\vartheta \mathscr{W}(\theta, \vartheta), \quad (12)$$

where ϑ represents the rest of variables required for a complete specification of the system.

After Eqs. (11) and (12) we have that any measurement statistics $P(\theta|\phi)$ is extracted as a marginal distribution from the complete distribution $W(U_\phi^\dagger \mathbf{E})$. Therefore, $W(U_\phi^\dagger \mathbf{E})$ contains no less information than any $P(\theta|\phi)$, so that (see Appendix A)

$$F \leq F_C = \int d^{2k} E \frac{1}{W(U_\phi^\dagger \mathbf{E})} \left[\frac{dW(U_\phi^\dagger \mathbf{E})}{d\phi} \right]^2. \quad (13)$$

This is to say that F_C is the maximum Fisher information reachable by any measurement. Therefore,

$$\chi^2 = \frac{F_C}{F_C + 4\text{tr}(G^2)}, \quad (14)$$

provides a suitable estimation of the degree of coherence, understood as metrological usefulness under phase shifts generated by G . Let us consider some meaningful examples.

3.1. Thermal-chaotic light

A fundamental example is provided by the coherence between two electric fields in a thermal-chaotic light state. This allows us to check the compatibility of this approach with the standard definition. Thermal-chaotic light is fully characterized by second-order moments of the field amplitudes as

$$W(\mathbf{E}) = \frac{\det M}{\pi^k} \exp(-\mathbf{E}^\dagger \mathbf{M} \mathbf{E}), \quad (15)$$

where $M = \Gamma^{-1}$ and $\Gamma_{i,j} = \langle E_i E_j^* \rangle$, with $\Gamma_{j,j} = \langle |E_j|^2 \rangle \neq 0$ since otherwise we shall accordingly reduce the number k of electric fields. A simple calculus leads to (see Appendix B)

$$F_C = 2 [\text{tr}(\Gamma G \Gamma^{-1} G) - \text{tr}(G^2)]. \quad (16)$$

The dependence on G reflects the fact that for a given field state the usefulness of the interference depends on the particular interferometric arrangement considered and whether it exploits the capabilities of the incoming light state. To exploit all the coherence conveyed by the field

state we have to look for the maximum of F_C by varying G . To this end we particularize the above expressions to the field basis where M is diagonal, leading to

$$F_C = 4 \sum_{i,j=1}^k |G_{i,j}|^2 \frac{\mathcal{P}_{i,j}^2}{1 - \mathcal{P}_{i,j}^2} \leq 4 \text{tr}(G^2) \frac{\mathcal{P}_{\max}^2}{1 - \mathcal{P}_{\max}^2}, \quad (17)$$

where only the terms $i \neq j$ contribute, $\mathcal{P}_{i,j}$ is the two-field degree of polarization of modes i, j , and \mathcal{P}_{\max} is the maximum of $\mathcal{P}_{i,j}$ for all pairs i, j :

$$\mathcal{P}_{i,j} = \frac{|I_i - I_j|}{I_i + I_j} \leq \mathcal{P}_{\max} = \frac{I_M - I_m}{I_M + I_m}, \quad (18)$$

where $I_{M,m}$ are the maximum and minimum intensities. Therefore, after Eq. (14)

$$\chi \leq \mathcal{P}_{\max}. \quad (19)$$

The equality in Eqs. (17) and (19) is reached when G just mixes the two electric fields with the maximum degree of polarization.

The upperbound in Eq. (19) agrees with previous results showing that the maximum visibility in the interference of two partially polarized transversal waves is reached by combining just the two components with the largest degree of polarization [14]. The bound in Eq. (17) reproduces also basic results of classical metrology: i) the resolution increases without limit as $\mathcal{P}_{\max} \rightarrow 1$, and ii) resolution is independent of the total field intensity. We will see below that these two properties no longer hold in the quantum sector.

3.1.1. Interference between two scalar fields

Let us particularize this result to the standard case with two electric fields $k = 2$

$$\Gamma = \begin{pmatrix} \langle |E_1|^2 \rangle & \langle E_1 E_2^* \rangle \\ \langle E_1^* E_2 \rangle & \langle |E_2|^2 \rangle \end{pmatrix}, \quad G \propto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (20)$$

where G generates phase-difference shifts. In this case, after Eqs. (10) and (16) we obtain

$$F_C = 4 \text{tr}(G^2) \frac{|\mu|^2}{1 - |\mu|^2}, \quad \chi = |\mu|, \quad (21)$$

where μ is the standard two-mode degree of coherence in Eq. (1).

3.1.2. Interference between two partially polarized fields

Let us consider the interference between two partially polarized fields in schemes of the Young type. This is $k = 4$ and

$$G \propto \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (22)$$

where I is the 2×2 identity matrix. We are considering the field basis in which $E_1 = E_{1,x}$, $E_2 = E_{1,y}$ denote the two components, say x and y , of one of the fields, while $E_3 = E_{2,x}$, $E_4 = E_{2,y}$ represent the components of the other one. Note that as reflected by the form of G in this scheme the two components x, y of each wave experience the same phase shift. The corresponding 4×4 correlation matrix can be expressed as

$$\Gamma = \begin{pmatrix} \Gamma_1 & \Upsilon \\ \Upsilon^\dagger & \Gamma_2 \end{pmatrix} = \Gamma_0^{1/2} U \begin{pmatrix} I & \Omega \\ \Omega & I \end{pmatrix} U^\dagger \Gamma_0^{1/2}, \quad (23)$$

with

$$\Gamma_0 = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}, \quad \Gamma_j = \begin{pmatrix} \langle |E_{j,x}|^2 \rangle & \langle E_{j,x} E_{j,y}^* \rangle \\ \langle E_{j,x}^* E_{j,y} \rangle & \langle |E_{j,y}|^2 \rangle \end{pmatrix} \quad (24)$$

and

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \mu_S & 0 \\ 0 & \mu_I \end{pmatrix}, \quad (25)$$

where $U_{1,2}$ are the unitary matrices performing the singular-value decomposition of $\Gamma_1^{-1/2} \Upsilon \Gamma_2^{-1/2}$, this is $\Omega = U_1^\dagger \Gamma_1^{-1/2} \Upsilon \Gamma_2^{-1/2} U_2$, and $\mu_{S,I}$ are the corresponding real and positive singular values, with $\mu_S \geq \mu_I$ for example. These singular values were introduced in Ref. [10] to represent the coherence between two partially polarized waves independently of their polarization states.

Applying Eq. (16) to this case and taking into account that $\Gamma_0^{-1/2} G \Gamma_0^{1/2} = G$ and $U^\dagger G U = G$ we get

$$F_C = 2\text{tr}(G^2) \left(\frac{1}{1-\mu_S^2} + \frac{1}{1-\mu_I^2} - 2 \right), \quad \chi^2 = \frac{\mu_S^2 + \mu_I^2 - 2\mu_S^2\mu_I^2}{2 - \mu_S^2 - \mu_I^2}. \quad (26)$$

For example, if $\mu_S = \mu_I$ we get $\chi = \mu_S$ which is exactly the case of two scalar fields in Eq. (21). On the other hand F_C diverges when $\mu_S \rightarrow 1$. All this is fully consistent with the properties of $\mu_{S,I}$ as measures of coherence between two partially polarized waves [43,44].

3.2. Fields with vanishing second-order degrees of coherence and polarization

The above equivalences in Eqs. (21) and (26) between resolution and the standard degree of coherence apply just to two fields in a thermal-chaotic light state, but fails to be true in general for other more sophisticated field statistics, as already shown in Ref. [22].

This is the case of field states whose probability distribution for the field amplitudes is an even function of any of the amplitudes, say $W(-E_1, E_2) = W(E_1, E_2)$, since Eq. (1) leads automatically to $\mu = 0$ irrespectively of any other statistical properties of the field state. Some examples are provided by most of the quantum field states reaching maximum phase-shift resolution that have $\langle E_i E_j^* \rangle = C \delta_{i,j}$, for all i, j , where C is a constant [32] leading to $\mu = \mathcal{P} = 0$. This includes the so-called N00N states [45] that are examined in more detail in the quantum sector.

4. Quantum sector

In the quantum sector the upper bound F_C in Eq. (13) cannot be defined because there is no proper probability distribution $W(\mathbf{E})$ providing observable statistics via marginals. Actually, this is the distinctive feature of nonclassical light [1,46].

Nevertheless, it is possible to find an upper bound to F in terms of the so-called quantum Fisher information F_Q [35],

$$F \leq F_Q = 2 \sum_{i,j} \frac{(p_i - p_j)^2}{p_i + p_j} |\langle \psi_i | \hat{G} | \psi_j \rangle|^2, \quad (27)$$

where p_i and $|\psi_i\rangle$ are the eigenvalues and eigenstates of the density matrix ρ representing the field state

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|. \quad (28)$$

In Eq. (27) the sum is extended to all pairs i, j with $p_i + p_j \neq 0$. When the field state is pure, i. e., $\rho = |\psi\rangle \langle \psi|$ we have that Eq. (27) greatly simplifies becoming the variance of the generator

\hat{G} in the field state $|\psi\rangle$

$$F_Q = 4 (\Delta\hat{G})^2. \quad (29)$$

Thus, in any case F_Q is the quantum analog of F_C in Eq. (13). Accordingly, a suitable estimation of the degree of coherence conveyed by the field state under phase shifts generated by \hat{G} in the quantum sector is

$$\chi^2 = \frac{F_Q}{F_Q + 4\text{tr}(G^2)}. \quad (30)$$

4.1. Thermal-chaotic light

In order to compare the classical and quantum sectors with a meaningful example let us consider quantum thermal-chaotic light. In the mode basis where the correlation matrix is diagonal $\langle a_i^\dagger a_j \rangle = \bar{n}_i \delta_{i,j}$ the field state factorizes $\rho = \otimes_{i=1}^k \rho_i$, with [1]

$$\rho_i = \frac{1}{1 + \bar{n}_i} \sum_{n=0}^{\infty} \left(\frac{\bar{n}_i}{1 + \bar{n}_i} \right)^n |n\rangle_i \langle n|, \quad (31)$$

where $|n\rangle_i$ are the corresponding photon-number states.

In order to check compatibility with the classical sector as simply and clearly as possible let us consider the two-field case $k = 2$. By symmetry, any generator of the form $\hat{G} \propto \exp(i\varphi) a_1^\dagger a_2 + \exp(-i\varphi) a_1 a_2^\dagger$ is optimum for any φ , leading to a maximum Fisher information (see Appendix C)

$$F_Q = 4\text{tr}(G^2) \frac{\mathcal{P}^2}{1 - \mathcal{P}^2 + 2/\bar{n}}, \quad \chi = \mathcal{P} \left(\frac{\bar{n}}{\bar{n} + 2} \right)^{1/2}, \quad (32)$$

where \mathcal{P} is the standard degree of polarization in Eq. (2), and \bar{n} is the total mean number of photons:

$$\mathcal{P} = \frac{|\bar{n}_1 - \bar{n}_2|}{\bar{n}_1 + \bar{n}_2}, \quad \bar{n} = \bar{n}_1 + \bar{n}_2. \quad (33)$$

It can be appreciated that for $\mathcal{P} \neq 1$ the classical result in Eq. (17) is obtained in the limit $\bar{n} \rightarrow \infty$, as it should be expected. On the other hand, for $\mathcal{P} \rightarrow 1$ we have that: i) the Fisher information does not diverge, contrary to the equivalent situation in the classical sector, but remains always finite, and ii) the Fisher information depends on the field intensity, so that for $\mathcal{P} = 1$ it holds that $F_Q = 2\text{tr}(G^2)\bar{n}$. Actually, the scaling $F_Q \propto \bar{n}$ is known as the standard quantum limit, which is the largest Fisher information that can be reached with classical light in quantum optics [47,48]. This may be compared with the scaling $F_Q \propto \bar{n}^2$ obtained for suitable nonclassical light, such as N00N states, as shown next in more detail.

4.2. N00N state

In the photon-number basis, the N00N states are

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|n\rangle_1 |0\rangle_2 + |0\rangle_1 |n\rangle_2). \quad (34)$$

This is an example of field states with vanishing degrees of polarization and coherence $\mu = \mathcal{P} = 0$, since for all $n > 1$ it holds that $\langle E_1 E_2^* \rangle = 0$, and $\langle |E_1|^2 \rangle = \langle |E_2|^2 \rangle$. Despite this, these states are extremely useful in interference since they provide much larger resolution than other states with larger values of μ and \mathcal{P} .

To show this we note that the maximum quantum Fisher information holds for $\hat{G} \propto a_1^\dagger a_1 - a_2^\dagger a_2$ leading to $F_Q = 2\text{tr}(G^2)\bar{n}^2$, where $\bar{n} = n$ is the total mean number of photons [32]. The

scaling $F_Q \propto \bar{n}^2$ is known as Heisenberg limit, that clearly surpasses the scaling $F_Q \propto \bar{n}$ that can be obtained with classical light [49].

The optimum Fisher information can be approached via ideal phase measurement whose statistics is given by projection on the relative-phase states [50]

$$P(\theta) = |\langle \theta | \psi \rangle|^2, \quad |\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^N e^{im\theta} |n-m\rangle_1 |m\rangle_2, \quad (35)$$

leading to the phase-difference distribution

$$P(\theta|\phi) = \frac{1}{\pi} \cos^2 [n(\theta - \phi)/2], \quad (36)$$

with Fisher information $F = n^2$.

The relative phase states emerge from a polar decomposition of the product of complex amplitudes $a_1 a_2^\dagger$ into a product of amplitude modulus and exponential of the phase difference. This problem has unitary solution for the exponential of the relative phase at difference with the equivalent single-mode polar decompositions [50].

It is worth noting that the total number n and the relative phase ϕ are commuting observables as the phase-number counterpart of commutation between total momentum and relative position for two particles $[p_1 + p_2, x_1 - x_2] = 0$ since, roughly speaking, we may say that $n = n_1 + n_2$ and $\phi = \phi_1 - \phi_2$, where $n_{1,2}$ and $\phi_{1,2}$ are the corresponding variables in the respective field mode. This does not contradict that single-mode number and phase do not commute, $[n_j, \phi_j] \neq 0$.

5. Conclusions

The classic degrees of coherence and polarization may admit different legitimate generalizations focusing on different field properties. For example, the metrological-based approach presented here is naturally different from global assessments of coherence and polarization in terms of distances of the light state to fully incoherent or unpolarized light [25–28, 51]. This is also different from the degree of polarization in Ref. [52] defined in terms of the minimum overlap between the original and transformed field state under any SU(2) transformation. In this work we are considering exclusively infinitesimal transformations in the spirit of precision metrology, while typically minimum overlap is obtained for large phase shifts.

We highlight the applicability of one and the same formalism both to the classical and quantum optical realms, obtaining parallel and consistent results in both of them [18, 53–56].

We have checked the consistency of this approach in the case of thermal light and N00N states. We think that there will be no consistency problems in any situation with multi-component fields and nonGaussian statistics because of the practical nature of this formalism and its focus on the observables consequences of coherence.

A. Upper bound to Fisher information in classical optics

In this Appendix we demonstrate the upper bound for Fisher information in classical optics in Eq. (13). To simplify notation let us consider discrete indices instead of continuous variables so that the measured statistics P_j of the j variable can be expressed in terms of the joint statistics of the j and k variables $W_{j,k}$ as

$$P_j = \sum_k W_{j,k}, \quad (37)$$

where for simplicity we omit expressing explicitly the ϕ dependence. Let us demonstrate that $F \leq F_C$ where

$$\begin{aligned} F &= \sum_j \frac{P_j^2}{P_j} = -\sum_j P_j (\ln P_j)'' = -\sum_{j,k} W_{j,k} (\ln P_j)'' , \\ F_C &= \sum_{j,k} \frac{W_{j,k}^2}{W_{j,k}} = -\sum_{j,k} W_{j,k} (\ln W_{j,k})'' , \end{aligned} \quad (38)$$

where primes indicate derivation with respect to ϕ . Then

$$F_C - F = -\sum_{j,k} W_{j,k} \left(\ln \frac{W_{j,k}}{P_j} \right)'' = -\sum_j P_j \sum_k \Lambda_{j,k} (\ln \Lambda_{j,k})'' , \quad (39)$$

where $\Lambda_{j,k}$ is the conditional probability of k given j

$$\Lambda_{j,k} = \frac{W_{j,k}}{P_j}, \quad \sum_k \Lambda_{j,k} = 1. \quad (40)$$

Finally

$$F_C - F = \sum_j P_j \sum_k \frac{\Lambda_{j,k}^2}{\Lambda_{j,k}} \geq 0. \quad (41)$$

As a further alternative demonstration let us show that the Hellinger distance in Eq. (7) H_P between the probability distributions $P_j = P_j(\phi)$ and $\tilde{P}_j = \tilde{P}_j(\phi + \delta\phi)$ is smaller than the Hellinger distance H_W between $W_{j,k} = W_{j,k}(\phi)$ and $\tilde{W}_{j,k} = \tilde{W}_{j,k}(\phi + \delta\phi)$

$$H_P = 2 \left(1 - \sum_j \sqrt{P_j \tilde{P}_j} \right), \quad H_W = 2 \left(1 - \sum_{j,k} \sqrt{W_{j,k} \tilde{W}_{j,k}} \right). \quad (42)$$

This is because $\sqrt{P_j \tilde{P}_j} \geq \sum_k \sqrt{W_{j,k} \tilde{W}_{j,k}}$. To show this let us consider the square of the left-hand side

$$P_j \tilde{P}_j = \sum_{k,\ell} W_{j,k} \tilde{W}_{j,\ell} = \frac{1}{2} \sum_{k,\ell} (W_{j,k} \tilde{W}_{j,\ell} + W_{j,\ell} \tilde{W}_{j,k}), \quad (43)$$

and the square of the right-hand side

$$\left(\sum_k \sqrt{W_{j,k} \tilde{W}_{j,k}} \right)^2 = \sum_{k,\ell} \sqrt{W_{j,k} \tilde{W}_{j,\ell} W_{j,\ell} \tilde{W}_{j,k}}, \quad (44)$$

so that $\sqrt{P_j \tilde{P}_j} \geq \sum_k \sqrt{W_{j,k} \tilde{W}_{j,k}}$ holds after comparing the right-hand side of Eqs. (43) and (44) because $a + b \geq 2\sqrt{ab}$ for all $a, b \geq 0$. Finally this implies $H_P \leq H_W$ and thus $F \leq F_C$ after Eq. (7).

B. Fisher information for classical thermal-chaotic light

Let us particularize F_C in Eq. (13) to the case of thermal-chaotic light. After Eq. (15) we have

$$W(U_\phi^\dagger \mathbf{E}) = \frac{\det M}{\pi^k} \exp \left(-\mathbf{E}^\dagger U_\phi M U_\phi^\dagger \mathbf{E} \right), \quad (45)$$

so that

$$\left. \frac{W(U_\phi^\dagger \mathbf{E})}{d\phi} \right|_{\phi=0} = -\mathbf{E}^\dagger M' \mathbf{E} W(\mathbf{E}), \quad (46)$$

where

$$M' = \left. \frac{d(U_\phi M U_\phi^\dagger)}{d\phi} \right|_{\phi=0} = i[G, M]. \quad (47)$$

Thus Eq. (13) leads to

$$F_C = \int d^{2k} E (\mathbf{E}^\dagger M' \mathbf{E})^2 W(\mathbf{E}) = \sum_{i,j,\ell,m=1}^k M'_{i,j} M'_{\ell,m} \int d^{2k} E E E_i^* E_j E_\ell^* E_m W(\mathbf{E}). \quad (48)$$

Using the Gaussian momentum theorem [1]

$$\langle E_j E_m E_i^* E_\ell^* \rangle = \langle E_j E_i^* \rangle \langle E_m E_\ell^* \rangle + \langle E_m E_i^* \rangle \langle E_j E_\ell^* \rangle, \quad (49)$$

we readily get

$$F_C = [\text{tr}(M' M^{-1})]^2 + \text{tr}[(M' M^{-1})^2] = \text{tr}[(M' M^{-1})^2], \quad (50)$$

leading to

$$F_C = 2 [\text{tr}(M^{-1} G M G) - \text{tr}(G^2)]. \quad (51)$$

In the field basis where M is diagonal we get

$$F_C = \sum_{i,j} \frac{(I_i - I_j)^2}{I_i I_j} |G_{i,j}|^2, \quad (52)$$

where $I_i = \langle |E_i|^2 \rangle$. Note the formal similarity with the quantum Fisher information in Eq. (27). We finally get

$$F_C = 4 \sum_{i,j=1}^k |G_{i,j}|^2 \frac{\mathcal{P}_{i,j}^2}{1 - \mathcal{P}_{i,j}^2} \leq 4 \text{tr}(G^2) \frac{\mathcal{P}_{\max}^2}{1 - \mathcal{P}_{\max}^2}, \quad (53)$$

where $\mathcal{P}_{i,j}$ is the two-mode degree of polarization of modes i, j , and \mathcal{P}_{\max} is the maximum of $\mathcal{P}_{i,j}$ for all pairs i, j :

$$\mathcal{P}_{i,j} = \frac{|I_i - I_j|}{I_i + I_j} \leq \mathcal{P}_{\max} = \frac{I_M - I_m}{I_M + I_m}, \quad (54)$$

where $I_{M,m}$ are the maximum and minimum intensities. The equality in Eq. (53) is reached when G mixes just the electric fields with the maximum and minimum intensities.

C. Quantum Fisher information for thermal-chaotic light

Since the thermal-chaotic states in Eq. (31) are mixed we have to resort to Eq. (27) to compute its quantum Fisher information. Since the product of number states $|n_1\rangle|n_2\rangle$ are the eigenstates of ρ , for $\hat{G} = \exp(i\varphi) a_1^\dagger a_2 + \exp(-i\varphi) a_1 a_2^\dagger$ we have to compute

$$|\langle \psi_i | \hat{G} | \psi_j \rangle|^2 = |\langle n'_1 | \langle n'_2 | [\exp(i\varphi) a_1^\dagger a_2 + \exp(-i\varphi) a_1 a_2^\dagger] | n_1 \rangle | n_2 \rangle|^2. \quad (55)$$

In the first place we have

$$\begin{aligned} \langle n'_1 | \langle n'_2 | [\exp(i\varphi) a_1^\dagger a_2 + \exp(-i\varphi) a_1 a_2^\dagger] | n_1 \rangle | n_2 \rangle = \\ \exp(i\varphi) \sqrt{(n_1+1)n_2} \delta_{n'_1=n_1+1} \delta_{n'_2=n_2-1} \\ + \exp(-i\varphi) \sqrt{n_1(n_2+1)} \delta_{n'_1=n_1-1} \delta_{n'_2=n_2+1}. \end{aligned} \quad (56)$$

The two Kronecker deltas are incompatible so that in the modulus square there are no crossed contributions and

$$F_Q = 2 \sum_{n_1, n_2=0}^{\infty} \frac{(p_{n_1+1} p_{n_2-1} - p_{n_1} p_{n_2})^2}{p_{n_1+1} p_{n_2-1} + p_{n_1} p_{n_2}} (n_1+1) n_2 + \text{exchange } 1 \leftrightarrow 2, \quad (57)$$

where

$$p_{n_i} = \frac{1}{1 + \bar{n}_i} \left(\frac{\bar{n}_i}{1 + \bar{n}_i} \right)^{n_i}. \quad (58)$$

After some simple algebra

$$F_Q = 4 \frac{(\bar{n}_1 - \bar{n}_2)^2}{2\bar{n}_1\bar{n}_2 + \bar{n}_1 + \bar{n}_2}. \quad (59)$$

This expression can be fruitfully simplified using the total mean number of photons $\bar{n} = \bar{n}_1 + \bar{n}_2$ and the degree of polarization in Eq. (2) $\mathcal{P} = |\bar{n}_1 - \bar{n}_2| / (\bar{n}_1 + \bar{n}_2)$ where we have taken into account that for the thermal state in Eq. (31) $\langle E_1 E_2^* \rangle = 0$, so that

$$F_Q = 4 \frac{\mathcal{P}^2 \bar{n}^2}{\bar{n}^2 (1 - \mathcal{P}^2) / 2 + \bar{n}} = 4 \text{tr}(G^2) \frac{\mathcal{P}^2}{1 - \mathcal{P}^2 + 2/\bar{n}}, \quad (60)$$

where we have used that $\text{tr}(G^2) = 2$.

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