



Multistage optimal homotopy asymptotic method for solving initial-value problems

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Abstract

In this paper, a new approximate analytical algorithm namely multistage optimal homotopy asymptotic method (MOHAM) is presented for the first time to obtain approximate analytical solutions for linear, nonlinear and system of initial value problems (IVPs). This algorithm depends on the standard optimal homotopy asymptotic method (OHAM), in which it is treated as an algorithm in a sequence of subinterval. The main advantage of this study is to obtain continuous approximate analytical solutions for a long time span. Numerical examples are tested to highlight the important features of the new algorithm. Comparison of the MOHAM results, standard OHAM, available exact solution and the fourth-order Runge Kutta (RK4) reveal that this algorithm is effective, simple and more impressive than the standard OHAM for solving IVPs.

Keywords: Optimal homotopy asymptotic method (OHAM), multistage optimal homotopy asymptotic method (MOHAM), initial value problems, series solution, Mathematica 9.

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1. Introduction

Many problems in the fields of science and engineering are modelled by initial value problems (IVPs).

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Different methods were developed to get accurate approximate solutions of IVPs. The Taylor method, the Euler method and the Runge-kutta method [6] provide an introduction to numerical method of solving IVPs. Taylor method requires the calculation of higher order derivatives. Jang, Chen and Liy [17] introduced the application of the concept of the differential transformation of fixed grid size to get approximate solution of initial value problems linear and nonlinear. Recently, much research has been conducted on numerical solution of the system of initial value problems by using Adomian decomposition method (ADM) [9, 13, 26, 27] and homotopy perturbation method (HPM) [11]. Nevertheless, one difficulty ingrained in ADM is the calculation of Adomian polynomials which might be cumbersome in general.

Another analytical procedure which has been shown to be much simpler than the ADM known as the optimal homotopy asymptotic method (OHAM), first proposed by Marinca et al. [22, 23] and Marinca and Herisanu [20]. They have applied it successfully to many nonlinear problems in fluid mechanics and heat transfer. The OHAM is an approximate analytical approach that is simple and has been built in convergence criteria similar to homotopy analysis method (HAM)[2, 3, 7] but with more degree of flexibility. The validity and the applicability of OHAM is independent whether there exist small parameters in the governing equation or not. In a series of papers, several authors [1, 4, 5, 8, 10, 12, 14, 15, 16, 18, 19, 24, 25] have proved effectiveness, reliability and generalization of this method and obtained solutions of currently important applications in science and engineering.

In this study, we consider the following initial value problem

$$y'_i(t) = f_i(t, y_1(t), y_2(t), \dots, y_k(t)) + g_i(t), \quad a \leq t \leq b, \quad (1.1)$$

with the initial condition

$$y_i(a) = \alpha_i, \quad (1.2)$$

where $y'_i(t), i = 1, 2, \dots, k$, denotes the derivative of $y_i(t)$ with respect to t .

Based on our observations, the OHAM solution obtained using the simplest form of the auxiliary function $H_i(p)$ for the above IVPs is valid for a short time span as will be shown in this paper. Thus, a new modification based on the standard OHAM is needed to overcome this limitation by dividing the time span interval into a sequence of subinterval and applying OHAM to each of them. Through this algorithm a continuous approximate analytical solution for long time span can be obtained.

This paper is organized into four sections. Section 2 discusses the basic principles of OHAM and MOHAM. In Section 3, we apply OHAM and MOHAM for solving several examples of IVPs and comparing the obtained results with the available exact solution and fourth-order Runge-Kutta method (RK4). Finally, the conclusions of this study are presented in the last section.

2. The fundamental concepts of OHAM and MOHAM

2.1. optimal homotopy asymptotic method (OHAM)

In this section, we review the fundamental principles of OHAM as explained in Marinca et al. [20] and other researcher [1, 16]. Consider the IVP

$$L_i(y_i(t)) + g_i(t) + N_i(y_i(t)) = 0, \quad i = 1, 2, \dots, N, \quad y_i(a) = \alpha_i, \quad (2.1)$$

where L_i is the chosen linear operator, N_i is non-linear operator, $y_i(t)$ is an unknown function, t denotes an independent variable, $g_i(t)$ is a known function. A homotopy map $h_i(v_i(t, p), p) : \mathfrak{R} \times [0, 1] \rightarrow \mathfrak{R}$ which satisfies

$$(1 - p)[L_i(v_i(t, p)) - y_{i,0}(x)] = H_i(p, t)[L_i(v_i(t, p)) + g_i(t) + N_i(v_i(t, p))], \quad (2.2)$$

can be constructed. Here $t \in \mathfrak{R}$ and $p \in [0, 1]$ is an embedding parameter, $H_i(p)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$ and $v_i(t, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds that $v_i(t, 0) = y_{i,0}(t)$ and $v_i(t, 1) = y_i(t)$ respectively. Thus, as p varies from 0 to 1, the solution $v_i(t, p)$

approaches from $y_{i,0}(t)$ to $y_i(t)$ where $y_{i,0}(t)$ is the initial guess that satisfies the linear operator which is obtained from Eq. (2.2) for $p = 0$ as,

$$L_i(u_{i,0}(x)) = 0. \tag{2.3}$$

Next, we choose the auxiliary function $H(p)$ in the form

$$H_i(p) = pC_1 + p^2C_2 + p^3C_3 + \dots, \tag{2.4}$$

where C_1, C_2, C_3, \dots are convergence control parameters which can be determined later. $H_i(p)$ can be expressed in many forms as reported by Marinca and Herisanu [21].

To get an approximate solution, we expand $v_i(t, p, C_k)$ in Taylor’s series about p in the following manner,

$$v_i(t, p, C_1, C_2, \dots, C_k) = y_0(t) + \sum_{k=1}^{\infty} y_{i,k}(t, C_1, C_2, \dots, C_k)p^k. \tag{2.5}$$

Define the vectors

$$\begin{aligned} \vec{C}_i &= \{C_1, C_2, \dots, C_i\}, \\ \vec{y}_{i,s} &= \{y_{i,0}(x), y_{i,1}(x, C_1), \dots, y_{i,s}(x, \vec{C}_s)\}, \end{aligned}$$

where $s = 1, 2, 3, \dots$. Substituting (2.5) into (2.2) and equating the coefficient of like powers of p , we obtain the following linear equations. The zeroth-order problem is given by (2.3), the first- and second-order problems are given as

$$L_i(y_{i,1}(t)) + g_i(t) = C_1 N_0(\vec{y}_{i,0}(t)), \quad y_{i,1}(a) = 0 \tag{2.6}$$

and

$$L_i(y_{i,2}(t)) - L_i(y_{i,1}(t)) = C_2 N_{i,0}(\vec{y}_{i,0}) + C_1 [L_i(y_{i,1}(t)) + N_{i,1}(\vec{y}_{i,1})], \quad y_{i,2}(a) = 0. \tag{2.7}$$

The general governing equations for $y_{i,k}(t)$ are

$$\begin{aligned} L_i(y_{i,k}(t)) - L_i(y_{i,k-1}(t)) &= C_k N_{i,0}(y_{i,0}(t)) + \sum_{m=1}^{k-1} C_{i,m} [L(y_{i,k-m}(t)) + N_{i,k-m}(\vec{y}_{i,k-1})], \\ y_{i,k}(a) &= 0, \end{aligned} \tag{2.8}$$

where $k = 2, 3, \dots$ and $N_{i,m}(y_{i,0}(t), y_{i,1}(t), \dots, y_{i,m}(t))$ is the coefficient of p^m in the expansion of $N_i(v_i(t, p))$ about the embedding parameter p

$$N_i(v_i(t, p, \vec{C}_k)) = N_{i,0}(y_{i,0}(t)) + \sum_{m=1}^{\infty} N_{i,m}(\vec{y}_{i,m})p^m. \tag{2.9}$$

It has been observed that the convergence of the series (2.9) depends upon the convergence control parameters C_1, C_2, C_3, \dots . If it is convergent at $p = 1$, one has

$$y_i(t, \vec{C}_k) = y_{i,0}(t) + \sum_{k=1}^{\infty} y_{i,k}(t, \vec{C}_k). \tag{2.10}$$

The result of the m th-order approximation is given

$$\tilde{y}_i(t, \vec{C}_m) = y_{i,0}(t) + \sum_{k=1}^m y_{i,k}(t, \vec{C}_k). \tag{2.11}$$

Substituting (2.11) into (2.1) yields the following residual

$$R_i(t, \vec{C}_m) = L_i(\tilde{y}(t, \vec{C}_m)) + g_i(t) + N_i(\tilde{y}(t, \vec{C}_m)). \tag{2.12}$$

If $R_i = 0$, then \tilde{y}_i will be the exact solution. Generally such a case will not arise for nonlinear problems.

In order to find the values of the convergence control parameters C_i 's there are several methods like the method of Least Squares, Collocation method, Ritz method and Galerkins method. By applying the method of Least Squares we obtain the following equation:

$$J_i(C_1, C_2, C_3, \dots, C_m) = \int_a^b R_i^2(t, C_1, C_2, C_3, \dots, C_m) dt, \tag{2.13}$$

where a and b are two values, depending on the given problem. The unknown convergence control parameters C_i ($i = 1, 2, 3, \dots, m$) can be identified from the conditions

$$\frac{\partial J_i}{\partial C_1} = \frac{\partial J_i}{\partial C_2} = \dots = \frac{\partial J_i}{\partial C_m} = 0. \tag{2.14}$$

With these known convergence control parameters, the approximate solution (of order m) is well determined.

2.2. Multistage optimal Homotopy Asymptotic Method (MOHAM)

Although the OHAM is used to provide approximate solutions for a wide class of nonlinear problems in terms of convergent series with easily computable components, it has some drawbacks in evaluating nonlinear problems with large domain. To overcome this shortcoming, we present in this section an multistage OHAM to handle the nonlinear problem with long time span. The new algorithm established based on the assumption that the approximate solutions (2.11) are in general, as shown in the numerical experiments presented in this paper which are not valid for large time span T . A simple way to confirm the validity of the approximations of large T is by dividing the interval $[0, T]$ by subinterval as $[t_0, t_1), \dots, [t_{j-1}, t_j]$ where $t_j = T$ and applying the MOAHM solution on each subintervals. The initial approximation in each interval is taken from the solution in previous interval. Firstly, by assuming the general initial condition as

$$y_i(t_j) = \alpha_i. \tag{2.15}$$

Thus, we can choose the initial approximation $y_{i,0}(x) = \alpha$ and the zero-order becomes

$$(1 - p)[L(v_i(t, p)) - y_{i,0}(x)] = H_i(p)[L(v_i(t, p)) + g_i(t) + N_i(v_i(t, p))]. \tag{2.16}$$

Next, we choose the auxiliary function $H_i(p)$ in the form

$$H_i(p) = C_{1,j}p + C_{2,j}p^2 + C_{3,j}p^3 + \dots \tag{2.17}$$

or

$$H_i(p, t) = (C_{1,j} + C_{2,j}t + C_{3,j}t^2 + \dots)p. \tag{2.18}$$

Then, the first, second and m th order–approximate solution can be generated subject to initial condition $y_{i,1}(t_j) = y_{i,2}(t_j) = \dots = y_{i,m}(t_j) = 0$, and the approximate solution becomes

$$\tilde{y}_i(t, C_{1,j}, C_{2,j}, C_{3,j}, \dots, C_{m,j}) = y_{i,0}(t) + \sum_{k=1}^m y_{i,k}(t, C_{1,j}, C_{2,j}, \dots, C_{k,j}). \tag{2.19}$$

Substituting (2.19) into (2.1) yields the following residual

$$R_i(t, C_{1,j}, C_{2,j}, C_{3,j}, \dots, C_{m,j}) = L(\tilde{y}_i(t, C_{1,j}, C_{2,j}, C_{3,j}, \dots, C_{m,j})) + g_i(t) + N_i(\tilde{y}_i(t, C_{1,j}, C_{2,j}, C_{3,j}, \dots, C_{m,j})). \tag{2.20}$$

If $R_i = 0$, then \tilde{y}_i will be the exact solution. Generally such a case will not arise for nonlinear problems, but we can minimize the function

$$J_i(C_{1,j}, C_{2,j}, C_{3,j}, \dots, C_{m,j}) = \int_{t_j}^{t_j+h} R_i^2(t, C_{1,j}, C_{2,j}, C_{3,j}, \dots, C_{m,j}) dt, \tag{2.21}$$

where h be the length of subinterval $[t_j, t_{j+1})$ and $N = \lceil T/h \rceil$ the number of subinterval. Now, we can solve (2.21) at $j = 0, 1, \dots, N$ with changing the initial approximation α_i in each subinterval from the previous one. For example in the subinterval $[t_j, t_{j+1})$ we define $\alpha_i = \tilde{y}(t_j)$. The unknown convergence control parameters $C_{i,j}$ ($i = 1, 2, 3, \dots, m, j = 1, 2, \dots, N$) can be identified from the solution of the system of equations

$$\frac{\partial J_i}{\partial C_{1,j}} = \frac{\partial J_i}{\partial C_{2,j}} = \dots = \frac{\partial J_i}{\partial C_{m,j}} = 0. \tag{2.22}$$

Therefore, the approximate analytic solution will be as

$$\tilde{y}(t) = \begin{cases} \tilde{y}_1(t), & t_0 \leq t < t_1, \\ \tilde{y}_2(t), & t_1 \leq t < t_2, \\ \vdots \\ \tilde{y}_N(t) & t_{N-1} \leq t \leq T. \end{cases} \tag{2.23}$$

By this way, we successfully obtain the solution of initial value problem for large value of T analytically. It worth mentioning that when $j = 0$ the MOHAM gives the standard OHAM. It is very important to point out that MOHAM provides a simple way to control and adjust the convergence region through the auxiliary function $H_i(p)$ involving several convergent control parameters $C_{i,j}$'s. On the other hand, this algorithm, mainly overcomes the difficulty arising in finding approximate solution of problems with large domain.

2.3. Convergent theorem

In this subsection we prove the convergent of MOHAM. For that it is enough to prove the convergent in any stage j . The technique in [25] is used to prove the convergent.

Theorem 2.1. *For the stage j , if the series (2.10) converges to $y_i(t)$, where $y_{i,k}(t) \in L(R^+)$ is produced by Eq. (2.6) and the k -order deformation (2.8), then $y_i(t)$ is the exact solution of equation (2.1).*

Proof. Since the series

$$\sum_{k=1}^{\infty} y_{i,k}(t, C_1, C_2, \dots, C_k)$$

is convergent, it can be written as

$$S_i(t) = \sum_{k=1}^{\infty} y_{i,k}(t, C_1, C_2, \dots, C_k), \tag{2.24}$$

it holds

$$\lim_{k \rightarrow \infty} y_{i,k}(t, C_1, C_2, \dots, C_k) = 0.$$

The left hand-side of Eq. (2.8) satisfies

$$\begin{aligned} y_{i,1}(t, C_1) + \sum_{k=2}^n y_{i,k}(t, \vec{C}_k) - \sum_{k=2}^n y_{i,k-1}(t, \vec{C}_{k-1}) &= y_{i,2}(t, \vec{C}_2) - y_{i,1}(t, C_1) + \dots + y_{i,n}(t, \vec{C}_n) - y_{i,n-1}(t, \vec{C}_{n-1}) \\ &= y_{i,n}(t, \vec{C}_n). \end{aligned}$$

According to Eq. (2.24) we have

$$y_{i,1}(t, C_1) + \sum_{k=2}^n y_{i,k}(t, \vec{C}) - \sum_{k=2}^n y_{i,k-1}(t, \vec{C}_{k-1}) = \lim_{n \rightarrow \infty} y_{i,n}(t, \vec{C}_n) = 0.$$

Using the linear operator L_i

$$\begin{aligned} L_i(y_{i,1}(t, C_1)) + \sum_{k=2}^{\infty} L_i(y_{i,k}(t, \vec{C}_k)) - \sum_{k=2}^{\infty} L_i(y_{i,k-1}(t, \vec{C}_{k-1})) \\ = L_i(y_{i,1}(t, C_1)) + L_i \sum_{k=2}^{\infty} y_{i,k}(t, \vec{C}_k) - L_i \sum_{k=1}^{\infty} y_{i,k-1}(t, \vec{C}_{k-1}) = 0, \end{aligned}$$

which satisfies

$$\begin{aligned} L_i(y_{i,1}(t, C_1)) + L_i \sum_{k=2}^{\infty} y_{i,k}(t, \vec{C}_k) - L_i \sum_{k=1}^{\infty} y_{i,k-1}(t, \vec{C}_{k-1}) \\ = \sum_{k=2}^{\infty} \left[C_k N_{i,0}(u_{i,0}(x)) + \sum_{m=1}^{k-1} C_m [L_i(y_{i,k-m}(t, \vec{C}_{k-m})) + N_{i,k-1-m}(\vec{y}_{i,k-1})] \right] + g_i(t) = 0. \end{aligned}$$

Also the right hand side can be written as

$$\sum_{k=1}^{\infty} \left[\sum_{m=1}^k C_{m-k} [L_i(y_{i,m-1}(t, \vec{C}_{m-1})) + N_{i,m-1}(\vec{y}_{i,k-1})] \right] + g_i(t) = 0. \tag{2.25}$$

Now, if the $C_m, m = 1, 2, \dots$ probably chosen, then Eq. 2.25 leads to

$$L_i(y_i(x)) + N_i(y_i(t)) + g_i(x) = 0, \tag{2.26}$$

which is the exact solution. □

3. Examples

In this section, several examples are introduced to demonstrate the efficiency of the new modification.

Example 3.1. Consider the following linear initial-value problems [17]

$$y'(t) = y(t) - t^2 + 1, \quad y(0) = 0.5. \quad 0 \leq t \leq 2. \tag{3.1}$$

The exact solution of the given problem is

$$y(t) = (t + 1)^2 - \frac{1}{2}e^t. \tag{3.2}$$

To solve this problem by MOHAM, we consider the initial condition as

$$y_0(t_j) = \alpha. \tag{3.3}$$

According to (2.1), we choose the linear and the nonlinear operators in the following forms:

$$\begin{aligned} L[v(t, p)] &= \frac{dv(t, p)}{dt}, \\ N[v(t, p)] &= \frac{dv(t, p)}{dt} - v(t, p) + t^2 - 1. \end{aligned} \tag{3.4}$$

Now, we will consider the auxiliary function $H_i(p, t)$ in the form $H_i(p, t) = (C_{1,j} + C_{2,j}t + C_{3,j}t^2 + C_{4,j}t^4)p$ where $C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}$ are unknown at this moment.

The first-order approximate solution is given by Eq. (2.19) for $m = 1$ as follows

$$\tilde{y}(t) = y_0(t) + y_1(t), \tag{3.5}$$

where

$$y_1'(t) = (-1 - \alpha + t^2) (C_{1,j} + t(C_{2,j} + t(C_{3,j} + tC_{4,j}))), \quad y_1(t_j) = 0. \tag{3.6}$$

By substituting the solution of Eq. (3.6) and using (3.3) into Eq. (3.5) we obtain,

$$\begin{aligned} \tilde{y}(t) = & \alpha - tC_{1,j} - \alpha tC_{1,j} + \frac{1}{3}t^3C_{1,j} + t_jC_{1,j} + \alpha t_jC_{1,j} - \frac{1}{3}t_j^3C_{1,j} \\ & - \frac{1}{2}t^2C_{2,j} - \frac{1}{2}\alpha t^2C_{2,j} + \frac{1}{4}t^4C_{2,j} + \frac{1}{2}t_j^2C_{2,j} + \frac{1}{2}\alpha t_j^2C_{2,j} \\ & - \frac{1}{4}t_j^4C_{2,j} - \frac{1}{3}t^3C_{3,j} - \frac{1}{3}\alpha t^3C_{3,j} + \frac{1}{5}t^5C_{3,j} + \frac{1}{3}t_j^3C_{3,j} \\ & + \frac{1}{3}\alpha t_j^3C_{3,j} - \frac{1}{5}t_j^5C_{3,j} - \frac{1}{4}t^4C_{4,j} - \frac{1}{4}\alpha t^4C_{4,j} + \frac{1}{6}t^6C_{4,j} \\ & + \frac{1}{4}\alpha t_j^4C_{4,j} + \frac{1}{4}t_j^4C_{4,j} - \frac{1}{6}t_j^6C_{4,j}. \end{aligned} \tag{3.7}$$

Substituting Eq. (3.7) into Eq. (2.20), yields the residual and the functional J respectively,

$$\begin{aligned} R(C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) &= \tilde{y}'(t) - \tilde{y}(t) + t^2 - 1, \\ J(C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) &= \int_{t_j}^{t_j+h} R^2(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j})dt. \end{aligned} \tag{3.8}$$

From the conditions (2.22), which become

$$\frac{\partial J}{\partial C_{1,j}} = \frac{\partial J}{\partial C_{2,j}} = \frac{\partial J}{\partial C_{3,j}} = \frac{\partial J}{\partial C_{4,j}} = 0, \quad j = 1, 2, \dots, N. \tag{3.9}$$

The values of the convergence control parameters $C_{i,j}$'s are obtained and presented in Table 1, by using $h = 0.2$ and starting with $t_0 = 0$ to $t_{10} = T = 2$.

Table 1: values of the convergent control parameters $C_{i,j}$'s for Example 3.1.

j	$C_{1,j}$	$C_{2,j}$	$C_{3,j}$	$C_{4,j}$
1	-0.815625	-0.877524	-0.0435768	-0.888789
2	-0.638505	-0.967322	0.564008	-1.0123
3	-0.403662	-1.40153	1.37327	-1.15637
4	-0.00219524	-2.34023	2.42728	-1.32627
5	0.742104	-4.01617	3.80987	-1.53577
6	2.1317	-6.82216	5.68324	-1.81072
7	4.74185	-11.5094	8.37232	-2.20055
8	9.82476	-19.7153	12.5895	-2.80986
9	20.6422	-35.6658	20.1529	-3.89901
10	48.0477	-73.1338	36.8612	-6.27809

By considering these values of the convergent control parameters, the first-order AOHM approximate solution (3.7) can be written in the following form,

$$\tilde{y}(t) = \begin{cases} 0.5 + 1.49998t + 0.75089t^2 - 0.0960768t^3 + 0.0463108t^4 - 0.094901t^5 - 0.131826t^6 & 0. \leq t < 0.2 \\ 0.500464 + 1.49202t + 0.802627t^2 - 0.245303t^3 + 0.187084t^4 - 0.00871536t^5 - 0.148132t^6 & 0.2 \leq t < 0.4 \\ 0.508911 + 1.41371t + 1.07087t^2 - 0.62909t^3 + 0.318501t^4 + 0.112802t^5 - 0.168717t^6 & 0.4 \leq t < 0.6 \\ 0.563891 + 1.06928t + 1.85628t^2 - 1.34712t^3 + 0.415405t^4 + 0.274654t^5 - 0.192728t^6 & 0.6 \leq t < 0.8 \\ 0.789489 + 0.006865t + 3.65921t^2 - 2.53095t^3 + 0.451831t^4 + 0.485455t^5 - 0.221045t^6 & 0.8 \leq t < 1. \\ 1.50812 - 2.7019t + 7.31116t^2 - 4.37637t^3 + 0.393834t^4 + 0.761974t^5 - 0.255961t^6 & 1. \leq t < 1.2 \\ 3.48188 - 8.9104t + 14.2581t^2 - 7.20797t^3 + 0.186641t^4 + 1.13665t^5 - 0.301787t^6 & 1.2 \leq t < 1.4 \\ 8.48191 - 22.4403t + 27.2336t^2 - 11.6264t^3 - 0.273883t^4 + 1.67446t^5 - 0.366759t^6 & 1.4 \leq t < 1.6 \\ 20.8417 - 51.909t + 52.0828t^2 - 18.8972t^3 - 1.21737t^4 + 2.51789t^5 - 0.46831t^6 & 1.6 \leq t < 1.8 \\ 52.624 - 120.038t + 103.701t^2 - 32.1835t^3 - 3.24808t^4 + 4.03058t^5 - 0.649835t^6 & 1.8 \leq t \leq 2. \end{cases}$$

The standard OHAM approximate solutions for the problem (3.1) is obtained when $t_j = 0$, $\alpha = 0.5$, and $h = 2$, by applying the technique as discussed in section (2.1), we obtain the third-order OHAM approximate solutions as follows

$$\tilde{y}(t) = 0.5 + 1.67153t + 0.116784t^2 + 0.297269t^3 - 0.012976t^4 - 0.0445813t^5. \tag{3.10}$$

A comparison between the first-order MOHAM approximate solution, three-order OHAM approximate solution and the exact one (3.2) are presented in Fig. 1 and Table 2. The absolute errors between the exact and MOHAM solutions for various times are tabulated in Tables 2. It is clear that the approximate analytical solutions obtained by MOHAM along with the use of the auxiliary convergent function $H_i(p, t)$ proved to be more accurate than the results obtained by the standard OHAM and the results obtained by using fourth-order differential transform method [17] since the absolute maximum error obtained is 1.2×10^{-4} .

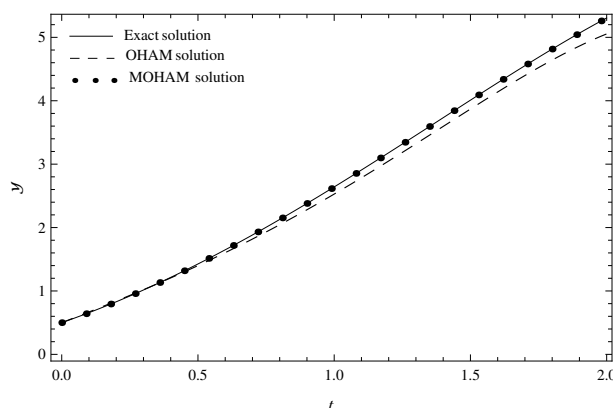


Figure 1: Comparison of MOHAM, OHAM and exact solutions (3.2) related to Example 3.1.

Table 2: Comparison between the first-order approximate solution obtained by MOHAM, three-order approximate solution obtained by OHAM and the exact solution (3.2) for Example 3.1.

t_j	Exact solution	MOHAM solution	OHAM solution	Absolute error
0.2	0.829298621	0.829298621	0.841319499	1.08646×10^{-11}
0.4	1.214087651	1.214087651	1.205532015	3.24201×10^{-11}
0.6	1.648940600	1.648940600	1.604019157	7.35692×10^{-11}
0.8	2.127229536	2.127229536	2.044240408	1.48915×10^{-11}
1.0	2.640859086	2.640859085	2.528021212	2.83469×10^{-10}
1.2	3.179941539	3.179941538	3.049841043	5.22911×10^{-10}
1.4	3.732400017	3.732400016	3.595121493	9.57685×10^{-10}
1.6	4.283483788	4.283483786	4.138514343	1.79265×10^{-9}
1.8	4.815176268	4.815176264	4.642189645	3.57537×10^{-9}
2.0	5.305471951	5.305471942	5.054123801	8.06698×10^{-9}

Example 3.2. Consider the nonlinear initial value problem [17]

$$y'(t) = -(y + 1)(y + 3), \quad y(0) = -2, \quad 0 \leq t \leq 3, \tag{3.11}$$

with the exact solution

$$y(t) = -3 + 2(1 + e^{-2t})^{-1}. \tag{3.12}$$

To solve this problem by MOHAM, we consider the initial condition as

$$y_0(t_j) = \alpha. \tag{3.13}$$

According to (2.1), we choose the linear and nonlinear operators in the following forms

$$\begin{aligned} L[v(t, p)] &= \frac{dv(t, p)}{dt}, \\ N[v(t, p)] &= \frac{dv(t, p)}{dt} + (v(t, p) + 1)(v(t, p) + 3). \end{aligned} \tag{3.14}$$

In this example we will consider the auxiliary function $H_i(p, t)$ in the form $H_i(p, t) = (C_{1,j} + C_{2,j}t)p$ where $C_{1,j}$ and $C_{2,j}$ yet to be determined. The first-order approximate solution is given by Eq. (2.19) for $m = 1$ as follows

$$\tilde{y}(t, C_{1,j}, C_{2,j}) = y_0(t) + y_1(t, C_{1,j}, C_{2,j}), \tag{3.15}$$

where

$$y'_1(t) = (1 + \alpha)(3 + \alpha)(C_{1,j} + tC_{2,j}), \quad y_1(t_j) = 0. \tag{3.16}$$

The solution of Eq. (3.16) is given by

$$\begin{aligned} y_1(t) &= 3tC_{1,j} + 4atC_{1,j} + \alpha^2tC_{1,j} - 3t_jC_{1,j} - 4at_jC_{1,j} \\ &\quad - \alpha^2t_jC_{1,j} + \frac{3}{2}t^2C_{2,j} + 2at^2C_{2,j} + \frac{1}{2}\alpha^2t^2C_{2,j} \\ &\quad - \frac{3}{2}t_j^2C_{2,j} - 2at_j^2C_{2,j} - \frac{1}{2}\alpha^2t_j^2C_{2,j}. \end{aligned} \tag{3.17}$$

The values of the convergence control parameters $C_{i,j}$'s are obtained and displayed in Table 3 by using (3.15) into (2.20) and applying the technique as discussed in Eqs. (2.21) and (2.22), with $h = 0.3$ and by starting with $t_0 = 0$ to $t_{10} = T = 3$. By using these values of the convergence control parameters $C_{i,j}$'s into Eq. (3.15), the first-order MOHAM approximate solution becomes

Table 3: values of the convergent control parameters $C_{i,j}$'s for Example 3.2.

j	$C_{1,j}$	$C_{2,j}$
1	-1.23169	0.748025
2	-1.63274	1.05743
3	-2.11221	1.2465
4	-2.61211	1.35618
5	-3.10826	1.41811
6	-3.59364	1.45261
7	-4.06826	1.47169
8	-4.53431	1.4822
9	-4.9942	1.48799
10	-5.44992	1.49117

$$\tilde{y}(t) = \begin{cases} -2. + 1.01394t - 0.143093t^2 & 0. \leq t < 0.3 \\ -2.01604 + 1.12717t - 0.342274t^2 & 0.3 \leq t < 0.6 \\ -2.02462 + 1.16184t - 0.376225t^2 & 0.6 \leq t < 0.9 \\ -1.96353 + 1.02849t - 0.303476t^2 & 0.9 \leq t < 1.2 \\ -1.82463 + 0.796774t - 0.206839t^2 & 1.2 \leq t < 1.5 \\ -1.64913 + 0.561726t - 0.128141t^2 & 1.5 \leq t < 1.8 \\ -1.47943 + 0.372201t - 0.075225t^2 & 1.8 \leq t < 2.1 \\ -1.3381 + 0.236915t - 0.0428518t^2 & 2.1 \leq t < 2.4 \\ -1.23056 + 0.146879t - 0.0240064t^2 & 2.4 \leq t < 2.7 \\ -1.15337 + 0.0894498t - 0.0133255t^2 & 2.7 \leq t \leq 3. \end{cases}$$

The standard OHAM approximate solutions for the problem (3.11) is obtained when $t_j = 0$, $\alpha = -2$ and $h = 3$, by applying the technique as discussed in section (2.1), we obtain the third-order OHAM approximate solutions as follows

$$\tilde{y}(t) = -2 + 0.964477t - 0.203569t^3. \tag{3.18}$$

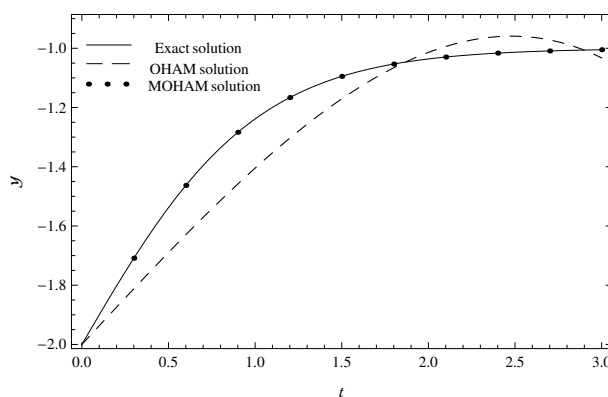


Figure 2: Comparison of MOHAM, OHAM and exact solutions (3.12) related to Example 3.2.

Comparison between the results obtained by using the first-order MOHAM approximate solution, three-order OHAM approximate solution and the exact one (3.12) are presented in Fig. 2 and Table 4. The

absolute error between the exact solution and the MOHAM solution for various times are are tabulated and recorded in the fourth column of Table 4. It can be seen that the approximate analytical solution obtained by MOHAM is more accurate and almost identical with that given by the exact solution.

Table 4: Comparison between the first-order approximate solution obtained by MOHAM, three-order approximate solution obtained by OHAM and the exact solution for Example 3.2.

t	Exact solution	MOHAM solution	OHAM solution	Absolute error
0.3	-1.708687388	-1.708695833	-1.812036159	8.44542×10^{-6}
0.6	-1.462950433	-1.462957634	-1.629601436	7.20134×10^{-6}
0.9	-1.283702130	-1.283707693	-1.458224949	5.56369×10^{-6}
1.2	-1.166345393	-1.166351921	-1.303435819	6.52886×10^{-6}
1.5	-1.094851746	-1.094859314	-1.170763162	7.56753×10^{-6}
1.8	-1.053193987	-1.053201320	-1.065736097	7.33246×10^{-6}
2.1	-1.029548063	-1.029554201	-0.993883743	6.13753×10^{-6}
2.4	-1.016325142	-1.016329789	-0.960735219	4.64715×10^{-6}
2.7	-1.008992546	-1.008995833	-0.971819641	3.28755×10^{-6}
3.0	-1.004945246	-1.004947465	-1.032666130	2.21874×10^{-6}

Example 3.3. Consider the stiff initial-value problem [17]

$$y'(t) = 5e^{5t} (y(t) - t)^2 + 1, \quad y(0) = -1, \quad 0 \leq t \leq 1, \tag{3.19}$$

with the exact solution

$$y(t) = t - e^{-5t}. \tag{3.20}$$

To solve this problem by MOHAM, we consider the initial conditions as

$$y_0(t) = (t - t_j) + \alpha. \tag{3.21}$$

According to (2.1), we start with

$$\begin{aligned} L[v(t, p)] &= \frac{dv(t, p)}{dt}, \\ N[v(t, p)] &= \frac{dv(t, p)}{dt} - 5 \sum_{i=1}^m \frac{(5t)^i}{i!} p^i (v(t, p) - t)^2 - 1. \end{aligned} \tag{3.22}$$

We choose the auxiliary function $H_i(p, t)$ in the form $H(p, t) = (C_{1,j} + C_{2,j}t + C_{3,j}t^2)p$ which provides us with the most convenient way to adjust the convergence region and rate of approximation series. According to (2.19), the first-order MOHAM approximate solution is

$$\tilde{y}(t) = y_0(t) + y_1(t), \tag{3.23}$$

where

$$y_1'(t) = -5e^{5t} (\alpha - t_j)^2 (C_{1,j} + t(C_{2,j} + C_{3,j})), \quad y_1(t_j) = 0. \tag{3.24}$$

By substituting the solutions of Eq. (3.24) and using Eq. (3.21) into Eq. (3.23), we obtain

$$\begin{aligned}
 \tilde{y}(t) = & \alpha + t - t_j - \alpha^2 e^{5t} C_{1,j} + \alpha^2 e^{5t_j} C_{1,j} + 2\alpha e^{5t} t_j C_{1,j} \\
 & - 2\alpha e^{5t_j} t_j C_{1,j} - e^{5t} t_j^2 C_{1,j} + e^{5t_j} t_j^2 C_{1,j} + \frac{1}{5} \alpha^2 e^{5t} C_{2,j} \\
 & - \frac{1}{5} \alpha^2 e^{5t_j} C_{2,j} - \alpha^2 e^{5t} t C_{2,j} - \frac{2}{5} \alpha e^{5t} t_j C_{2,j} + \frac{2}{5} \alpha e^{5t_j} t_j C_{2,j} \\
 & + \alpha^2 e^{5t_j} t_j C_{2,j} + 2\alpha e^{5t} t t_j C_{2,j} + \frac{1}{5} e^{5t} t_j^2 C_{2,j} - \frac{1}{5} e^{5t_j} t_j^2 C_{2,j} \\
 & - 2\alpha e^{5t_j} t_j^2 C_{2,j} - e^{5t} t t_j^2 C_{2,j} + e^{5t_j} t_j^3 C_{2,j} - \frac{2}{25} \alpha^2 e^{5t} C_{3,j} \\
 & + \frac{2}{25} \alpha^2 e^{5t_j} C_{3,j} + \frac{2}{5} \alpha^2 e^{5t} t C_{3,j} - \alpha^2 e^{5t} t^2 C_{3,j} + \frac{4}{25} \alpha e^{5t} t_j C_{3,j} \\
 & - \frac{4}{25} \alpha e^{5t_j} t_j C_{3,j} - \frac{2}{5} \alpha^2 e^{5t_j} t_j C_{3,j} - \frac{4}{5} \alpha e^{5t} t t_j C_{3,j} \\
 & + 2\alpha e^{5t} t^2 t_j C_{3,j} - \frac{2}{25} e^{5t} t_j^2 C_{3,j} + \frac{2}{25} e^{5t_j} t_j^2 C_{3,j} \\
 & + \frac{4}{5} \alpha e^{5t_j} t_j^2 C_{3,j} + \alpha^2 e^{5t_j} t^2 C_{3,j} + \frac{2}{5} e^{5t} t t_j^2 C_{3,j} \\
 & - e^{5t} t^2 t_j^2 C_{3,j} - \frac{2}{5} e^{5t_j} t_j^3 C_{3,j} - 2\alpha e^{5t_j} t_j^3 C_{3,j} \\
 & + e^{5t_j} t_j^4 C_{3,j}.
 \end{aligned} \tag{3.25}$$

The values of the convergence control parameters $C_{i,j}$'s are obtained and displayed in Table 5, by substituting (3.25) into (2.20) and applying the technique as discussed in Eqs. (2.21) and (2.22), with $h = 0.2$ and by starting with $t_0 = 0$ to $t_5 = T = 1$. By considering these values of the convergence control parameters $C_{i,j}$'s into Eq. (3.25), the first-order MOHAM approximate solutio becomes

Table 5: values of the convergent control parameters $C_{i,j}$'s for Example 3.3.

j	$C_{1,j}$	$C_{2,j}$	$C_{3,j}$
1	-3.21614	14.9284	-18.1546
2	-4.79542	13.7653	-9.93329
3	0.430942	-1.0654	-0.110817
4	-0.313527	0.0723172	0.113094
5	-0.0264537	-0.00383572	0.00168781

$$\tilde{y}(t) = \begin{cases} -4.94221 + 3.94221e^{5t} + t - 14.9278e^{5t}t + 18.1536e^{5t}t^2 & 0. \leq t < 0.2 \\ -1.81839 + 1.03602e^{5t} + t - 3.00352e^{5t}t + 2.45728e^{5t}t^2 & 0.2 \leq t < 0.4 \\ -0.519366 + 0.152837e^{5t} + t - 0.324952e^{5t}t - 0.181967e^{5t}t^2 & 0.4 \leq t < 0.6 \\ -0.0507495 - 0.00158033e^{5t} + t + 0.00254053e^{5t}t + 0.000275723e^{5t}t^2 & 0.6 \leq t < 0.8 \\ -0.0197467 + 0.0000861117e^{5t} + t - 7.31129 \times 10^{-6}e^{5t}t - 0.0000305343e^{5t}t^2 & 0.8 \leq t < 1. \end{cases}$$

The standard OHAM approximate solutions for the problem (3.19) is obtained when $t_j = 0$, $\alpha = -1$, and $h = 1$ and applying the technique as discussed in section (2.1), we obtain the third-order OHAM approximate solutions as follows

$$\tilde{y}(t) = -1 + 3.28703t + 0.10769t^2 - 1.98383t^3 + 0.0834894t^4 + 0.581527t^5 - 0.141212t^6 + 0.00389258t^7. \tag{3.26}$$

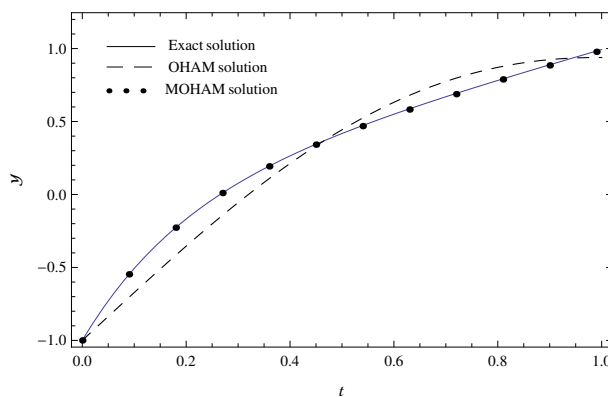


Figure 3: Comparison of MOHAM, OHAM and exact solutions (3.20) related to Example 3.3.

A comparison between the first-order MOHAM approximate solution, three-order OHAM approximate solution and the exact solution are presented in Fig. 3 and Table 6. The absolute errors between the exact solution and MOHAM solutions for various times are tabulated in Table 6. It can be concluded that the results obtained by means of MOHAM are nearly identical with the exact one, which proves the accuracy and the reliability of the method.

Table 6: Comparison between the first-order approximate solution obtained by MOHAM, three-order approximate solution obtained by OHAM and the exact solution for Example 3.3.

t	Exact solution	MOHAM solution	OHAM solution	Absolute error
0.2	-0.1678794412	-0.167903440	-0.353846958	2.39983×10^{-5}
0.4	0.26466471676	0.2646526384	0.212596160	1.20784×10^{-5}
0.6	0.55021293163	0.5501192179	0.632037427	9.37138×10^{-5}
0.8	0.78168436111	0.7835685861	0.871372076	1.88423×10^{-3}
1	0.99326205300	0.9874166761	0.938582516	5.84538×10^{-6}

Example 3.4. In this example, we consider the following liner system of ODEs [11]

$$\begin{aligned} y_1'(t) &= y_1(t) + y_2(t), \\ y_2'(t) &= -y_1(t) + y_2(t), \end{aligned} \tag{3.27}$$

subject to the initial conditions

$$y_1(0) = 0, \quad y_2(0) = 1. \tag{3.28}$$

The exact solution is

$$y_1(t) = e^t \sin t, \quad y_2(t) = e^t \cos t. \tag{3.29}$$

According to (2.1) for this example, we defined the linear and nonlinear operator as follows:

$$\begin{aligned} L_i[v_i(t, p)] &= \frac{dv_i(t, p)}{dt}, \quad i = 1, 2, \\ N_1[v_1(t, p)] &= \frac{dv_1(t, p)}{dt} - v_1(t, p) - v_2(t, p), \\ N_2[v_2(t, p)] &= \frac{dv_2(t, p)}{dt} + v_1(t, p) - v_2(t, p). \end{aligned} \tag{3.30}$$

It is straight forward to choose the initial conditions in the following form:

$$\begin{aligned} y_{1,0}(t_j) &= \alpha, \\ y_{2,0}(t_j) &= \beta. \end{aligned} \tag{3.31}$$

Now, based on OHAM, the first, second, third and the fourth-order problems are subject to the initial conditions as given bellow, respectively

$$\begin{aligned} y'_{1,1}(t) &= -\alpha C_{1,j} - \beta C_{1,j}, \quad y_{1,1}(t_j) = 0, \\ y'_{2,1}(t) &= \alpha C_{1,j} - \beta C_{1,j}, \quad y_{2,1}(t_j) = 0. \end{aligned} \tag{3.32}$$

$$\begin{aligned} y'_{1,2}(t) &= y'_{1,1}(t) - \alpha C_{2,j} - \beta C_{2,j} - C_{1,j}y_{1,1}(t) - C_{1,j}y_{2,1}(t) + C_{1,j}y'_{1,1}(t), \quad y_{1,2}(t_j) = 0, \\ y'_{2,2}(t) &= \alpha C_{2,j} - \beta C_{2,j} + C_{1,j}y_{1,1}(t) - C_{1,j}y_{2,1}(t) + y'_{2,1}(t) + C_{1,j}y'_{2,1}(t), \quad y_{2,2}(t_j) = 0, \end{aligned} \tag{3.33}$$

$$\begin{aligned} y'_{1,3}(t) &= y'_{1,2}(t) - \alpha C_{3,j} - \beta C_{3,j} - C_{2,j}y_{1,1}(t) - C_{1,j}y_{1,2}(t) - C_{2,j}y_{2,1}(t) \\ &\quad - C_{1,j}y_{2,2}(t) + C_{2,j}y'_{1,1}(t) + C_{1,j}y'_{1,2}(t), \quad y_{1,3}(t_j) = 0, \\ y'_{2,3}(t) &= \alpha C_{3,j} - \beta C_{3,j} + C_{2,j}y_{1,1}(t) + C_{1,j}y_{1,2}(t) - C_{2,j}y_{2,1}(t) - C_{1,j}y_{2,2}(t) \\ &\quad + C_{2,j}y'_{2,1}(t) + y'_{2,2}(t) + C_{1,j}y'_{2,2}(t), \quad y_{2,3}(t_j) = 0, \end{aligned} \tag{3.34}$$

$$\begin{aligned} y'_{1,4}(t) &= -\alpha C_{4,j} - \beta C_{4,j} - C_{3,j}y_{1,1}(t) - C_{2,j}y_{1,2} - C_{1,j}y_{1,3}(t) - C_{3,j}y_{2,1}(t) - C_{2,j}y_{2,2}(t) \\ &\quad - C_{1,j}y_{2,3}(t) + C_{3,j}y'_{1,1}(t) + C_{2,j}y'_{1,2}(t) + y'_{1,3}(t) + C_{1,j}y'_{1,3}(t), \quad y_{1,4}(t_j) = 0, \\ y'_{2,4}(t) &= \alpha C_{4,j} - \beta C_{4,j} + C_{3,j}y_{1,1}(t) + C_{2,j}y_{1,2} + C_{1,j}y_{1,3} - C_{3,j}y_{2,1} - C_{2,j}y_{2,2}(t) - C_{1,j}y_{2,3} \\ &\quad + C_{3,j}y'_{2,1}(t) + C_{2,j}y'_{2,2}(t) + C_{1,j}y'_{2,3} + y'_{2,3}(t) + y'_{1,3}(t) + C_{1,j}y'_{1,3}(t), \quad y_{2,4}(t_j) = 0. \end{aligned} \tag{3.35}$$

By substituting the solutions of Eqs. (3.32)–(3.35) into Eq. (2.19) for $m = 4$, the fourth-order MOHAM approximate solution will be

$$\begin{aligned} \tilde{y}_1(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) &= y_{1,0}(t) + y_{1,1}(t, C_{1,j}) + y_{1,2}(t, C_{1,j}, C_{2,j}) \\ &\quad + y_{1,3}(t, C_{1,j}, C_{2,j}, C_{3,j}) + y_{1,4}(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}), \\ \tilde{y}_2(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) &= y_{2,0}(t) + y_{2,1}(t, C_{1,j}) + y_{2,2}(t, C_{1,j}, C_{2,j}) \\ &\quad + y_{2,3}(t, C_{1,j}, C_{2,j}, C_{3,j}) + y_{2,4}(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}). \end{aligned} \tag{3.36}$$

The standard OHAM approximate solution for the problem (3.27) and (3.28) is obtained by substituting $t_0 = 0, \alpha = 0, \beta = 1$ and $h = 1$, and by applying the technique as discussed in Eqs. (2.11)–(2.14), we obtain the fourth-order OHAM approximate solutions as follow

$$\begin{aligned} \tilde{y}_1(t) &= 0.828043t + 1.31281t^2 + 0.117835t^3, \\ \tilde{y}_2(t) &= 1 + 0.828043t - 0.117835t^3 - 0.00836726t^4. \end{aligned} \tag{3.37}$$

The fourth-order MOHAM approximate solutions for $y_1(t)$ and $y_2(t)$ are obtained and plotted respectively in Figs. (4-a) and (4-b), by substituting Eqs. (3.36) into Eq. (2.20) and by applying the technique as discussed in Eqs. (2.21) and (2.22), with the time step $h = 0.001, \alpha = 0, \beta = 1$ and starting with $t_0 = 0$, to $t_{5000} = T = 5$. From this Figure, we observe that the MOHAM solutions are very close to the exact solution and this reflects that the MOHAM is producing an accurate approximate solutions.

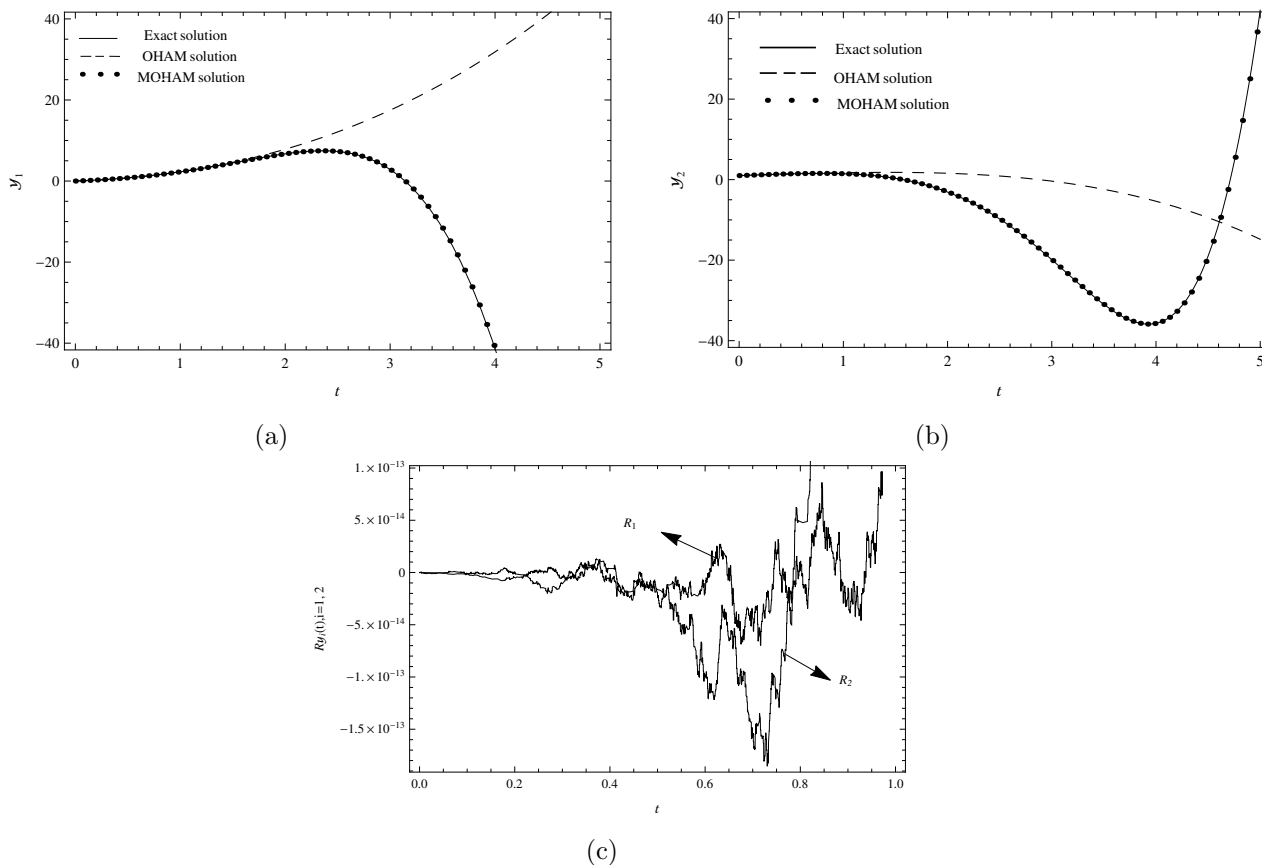


Figure 4: (4a) and (4b) Comparison of MOHAM, OHAM and exact solutions (3.29) for $y_1(t)$ and $y_2(t)$ respectively, related to Example 3.4 and (4c) the residual error $R_i(t), i = 1, 2$ by using the fourth-order MOHAM approximate solution.

Example 3.5. Consider the following nonlinear system of ODEs [11]

$$\begin{aligned} y_1'(t) &= y_1 (1 - y_2(t)), \\ y_2'(t) &= -y_2 (0.1 - y_1(t)), \end{aligned} \tag{3.38}$$

subject to the initial conditions

$$y_1(0) = 14 \quad y_2(0) = 18. \tag{3.39}$$

According to (2.1) for this example, we defined the linear and nonlinear operator as follows:

$$\begin{aligned} L_i[v_i(t,p)] &= \frac{dv_i(t,p)}{dt}, \quad i = 1, 2, \\ N_1[v_1(t,p)] &= \frac{dv_1(t,p)}{dt} - v_1 (1 - v_2(t,p), p), \\ N_2[v_2(t,p)] &= \frac{dv_2(t,p)}{dt} + v_2 (0.1 - v_1(t,p), p). \end{aligned} \tag{3.40}$$

It is straight forward to choose the initial conditions in the following form:

$$y_{1,0}(t_j) = \alpha, \quad y_{2,0}(t_j) = \beta. \tag{3.41}$$

Following the same manipulations as in the previous example, we have

$$\begin{aligned} y'_{1,1}(t) &= -\alpha C_{1,j} + \alpha\beta C_{1,j}, \quad y_{1,1}(t_j) = 0, \\ y'_{2,1}(t) &= 0.1\beta C_{1,j} - 1.0\alpha\beta C_{1,j}, \quad y_{2,1}(t_j) = 0, \end{aligned} \tag{3.42}$$

$$\begin{aligned}
 y'_{1,2}(t) &= -\alpha C_{2,j} + \alpha\beta C_{2,j} - C_{1,j}y_{1,1}(t) + \beta C_{1,j}y_{1,1}(t) + \alpha C_{1,j}y_{2,1}(t) + y'_{1,1}(t) \\
 &\quad + C_{1,j}y'_{1,1}(t), \quad y_{1,2}(t_j) = 0, \\
 y'_{2,2}(t) &= -\beta C_{1,j}y_{1,1}(t) + 0.1C_{1,j}y_{2,1} - \alpha C_{1,j}y_{2,1} + y'_{2,1}(t) + C_{1,j}y'_{2,1} \\
 &\quad + 0.1\beta C_{2,j} - \alpha\beta C_{2,j}, \quad y_{2,2}(t_j) = 0,
 \end{aligned}
 \tag{3.43}$$

$$\begin{aligned}
 y'_{1,3}(t) &= -\alpha C_{3,j} + \alpha\beta C_{3,j} - C_{2,j}y_{1,1}(t) + \beta C_{2,j}y_{1,1}(t) - C_{1,j}y_{1,2}(t) + \beta C_{1,j}y_{1,2} \\
 &\quad + \alpha C_{2,j}y_{2,1}(t) + C_{1,j}y_{1,1}(t)y_{2,1}(t) + \alpha C_{1,j}y_{2,2}(t) + C_{2,j}y'_{1,1}(t) + y'_{1,2}(t) \\
 &\quad + C_{1,j}y'_{1,2}(t) \quad y_{1,3}(t_j) = 0, \\
 y'_{2,3}(t) &= 0.1\beta C_{3,j} - \alpha\beta C_{3,j} - bC_{1,j}y_{1,2}(t) - C_{1,j}y_{1,1}(t)y_{2,1}(t) + 0.1C_{1,j}y_{2,2}(t) \\
 &\quad - \alpha C_{1,j}y_{2,2}(t) + C_{1,j}y'_{2,2}(t) - \beta C_{2,j}y_{1,1}(t) + 0.1C_{2,j}y_{2,1} - \alpha C_{2,j}y_{2,1} \\
 &\quad + C_{2,j}y'_{2,1}(t) + y'_{2,2}(t), \quad y_{2,3}(t_j) = 0.
 \end{aligned}
 \tag{3.44}$$

By substituting the solutions of the problems (3.42)-(3.44) into Eq. (2.19), the third-order MOHAM approximate solution is given as

$$\begin{aligned}
 \tilde{y}_1(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) &= y_{1,0}(t) + y_{1,1}(t, C_{1,j}) + y_{1,2}(t, C_{1,j}, C_{2,j}) + y_{1,3}(t, C_{1,j}, C_{2,j}, C_{1,3}), \\
 \tilde{y}_2(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) &= y_{2,0}(t) + y_{2,1}(t, C_{1,j}) + y_{2,2}(t, C_{1,j}, C_{2,j}) + y_{2,3}(t, C_{1,j}, C_{2,j}, C_{1,3}).
 \end{aligned}
 \tag{3.45}$$

The standard OHAM approximate solution for the problem (3.38) and (3.39) is obtained by substituting $t_j = 0$, $\alpha = 14$, $\beta = 18$ and $h = 1$ into Eqs. (3.42)–(3.44). Then by applying the technique as discussed in Eqs. (2.11)–(2.14), we obtain the third-order OHAM approximate solutions as follows

$$\begin{aligned}
 \tilde{y}_1(t) &= 14 + 0.00460661t + 1.88858 \times 10^{-14}t^2 - 2.84559 \times 10^{-27}t^3, \\
 \tilde{y}_2(t) &= 18 - 0.00484275t - 2.80304 \times 10^{-14}t^2 + 2.83093 \times 10^{-27}t^3.
 \end{aligned}
 \tag{3.46}$$

Figs. (5-a) and (5-b) show the solutions for $y_1(t)$ and $y_2(t)$ respectively by using three-order MOHAM approximate analytic solution with time step $h = 0.01$ and by starting with $t_0 = 0$, to $t_{100} = T = 1$ and the solutions obtained by using the fourth-order Runge-kutta method (RK4). It is observed that the the three-order MOHAM solutions agree very well with the solutions obtained by using the RK4.

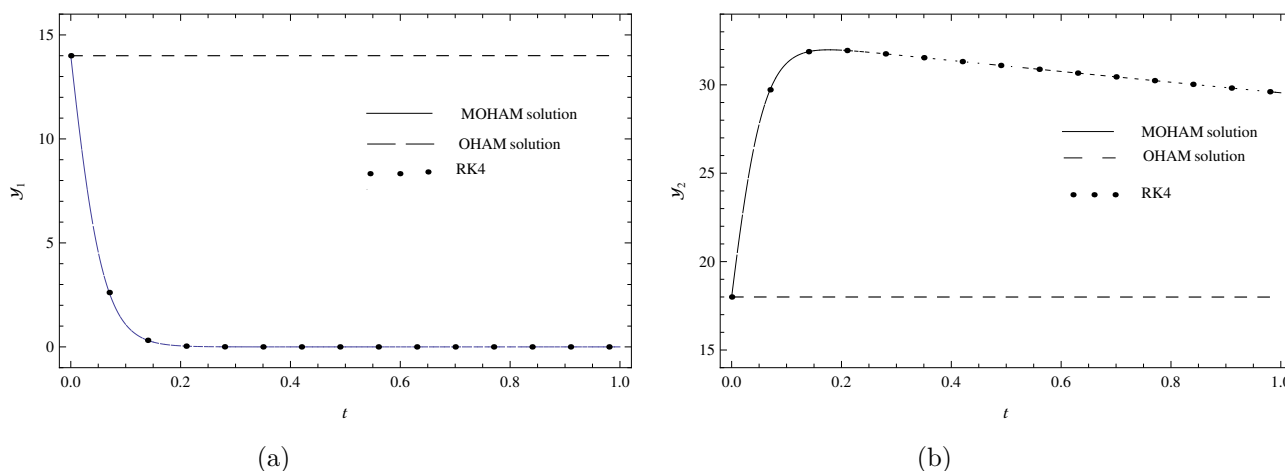


Figure 5: (5a) and (5b) Comparison of MOHAM, OHAM and the fourth-order Runge-kutta method for $y_1(t)$ and $y_2(t)$ respectively, related to Example 3.5.

4. Conclusion

In this paper, it is noticed that the solutions obtained by the standard OHAM were not valid for large time span. Therefore, a new modification based on standard OHAM has been successfully applied to get accurate approximate solutions of the first-order linear, nonlinear and system of IVPs by dividing the interval $[0, T]$ in a sequence of intervals $[t_0, t_1), \dots, [t_{j-1}, t_j]$ for large time span T . In MOHAM the calculations are straightforward and simple. The reliability of the algorithm and the reduction in the size of computational domain made this algorithm are more applicable, also this algorithm provides a convenient way of controlling the convergence region of the series solution. Achieved results indicate that MOHAM is a reliable and efficient procedure for finding approximate analytical solution of the IVPs.

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