TWO CARDINALS MODELS
WITH GAP ONE REVISITED
SH824

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Abstract. We succeed to say something on the identities of \((\mu^+, \mu)\) when \(\mu > \theta > \text{cf}(\mu), \mu\) strong limit \(\theta\)-compact or even \(\mu\) limit of compact cardinals.

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§0 Introduction

[We give the basic definitions.]

§1 2-simplicity for gap one

[We prove that if $\mu = 2^{<\mu}$ then the family of identities of $(\mu^+, \mu)$ is 2-simple. So this applies to $\mu$ singular strong limit but also, e.g., to triples $(\mu^+, \mu, \kappa), \mu = 2^{<\mu} > \kappa$.]

§2 Successor of strong limit above supercompact: 2-identities

[Consider a pair $(\mu^+, \mu)$ with $\mu$ strong limit singular $> \theta > \text{cf}(\mu), \theta$ a compact cardinal. We point out quite simply 2-identities which belong to $\text{ID}_2(\mu^+, \mu)$ but not to $\text{ID}_2(\aleph_1, \aleph_0)$.]
§0 Introduction

There has been much work on $\kappa$-compactness of pairs $(\lambda, \mu)$ of cardinals, i.e., when: if $T$ is a set of first order sentences of cardinality $\leq \kappa$ and every finite subset has a $(\lambda, \mu)$-model (i.e., a model $M$ of cardinality $\lambda$, $|P^M| = \mu$ for a fixed unary $P$). Then $T$ has a $(\lambda, \mu)$-model.

A particularly important case is $\lambda = \mu^+$ in which case this can be represented as a problem on the $\kappa$-compactness of the logic $\mathbb{L}(\mathbb{Q}_{\lambda}^{\text{card}})$, i.e., $(\mathbb{Q}_{\geq \lambda}^{\text{card}} x) \varphi$ says that there are at least $\lambda$ element $x$ satisfying $\varphi$. We deal here only with this case. See Furfken [Fu65], Morley and Vaught [MoVa62], Keisler [Ke70], Mitchel [Mi72]; for more history see [Sh 604].

Now two cardinal theorems can be translated to partition problems so-called identities (0.2): see [Sh 8], [Sh:E17], lately Shelah and Vaananan [ShVa 790] or [ShVa:E47].

Restricting ourselves to pairs $(\mu^+, \mu)$, the identities of $(\aleph_1, \aleph_0)$ were sorted out in [Sh 74], but we do not know of the identities of any really different pair $(\mu^+, \mu)$, i.e., one for which $(\aleph_1, \aleph_0) \not\rightarrow (\mu^+, \mu)$. We know that (consistently) some pairs $(\mu^+, \mu)$ have a different set of identities than $(\aleph_1, \aleph_0)$ but we do not have a characterization in any of those cases. By Mitchel [Mi72] this applies to $(\aleph_2, \aleph_1)$ in the universe gotten by forcing: suitably collapsing of a Mahlo strongly inaccessible to $\aleph_2$. The other such case is when there is a compact cardinal in the interval (cf $(\mu)$) by Litman and Shelah. So it would be nice to know (taking the extreme case):

0.1 Question: Assume $\mu$ is a singular cardinal the limit of compact and even super-compact cardinals.

1) What are the identities of $(\mu^+, \mu)$?
2) Is $(\mu^+, \mu)$ $\aleph_0$-compact (equivalently $\mu$-compact)?

Note that though we already know that there are some identities of $(\mu^+, \mu)$ which are not identities of $(\aleph_1, \aleph_0)$ we have no explicit example. We give here a partial solution to 0.1(1) by finding families of such identities.

Another problem is consistency of failure of compactness. In [Sh 604] we have dealt with the simplest case for pairs $(\lambda, \mu)$ by a reasonable criterion: including no use of large cardinals. From another perspective the simplest case is the consistency of non-compactness of $\mathbb{L}(\mathbb{Q})$, $\mathbb{Q}$ one cardinality quantifier, and the simplest one is $\mathbb{Q} = \exists \geq \mu^+$. So we are again drawn to pairs $(\mu^+, \mu)$, that is gap one instead of gap 2 as in [Sh 604], so necessarily we need to use large cardinals as if, e.g., $\neg 0^#$ then every such pair is compact.

0.2 Definition. 1) A partial identity $^1 s$ is a pair $(a, e) = (\text{Dom}_s, e_s)$ where $a$ is a

\[^1\text{identification in the terminology of [Sh 8]}\]
finite set and $e$ is an equivalence relation on a subfamily of the family of the finite subsets of $a$, having the property

$$b e c \Rightarrow |b| = |c|.$$  

The equivalence class of $b$ with respect to $e$ will be denoted $b/e$.

1A) We say $s$ is a full identity or identity if $\text{Dom}(e) = \mathcal{P}(a)$.

1B) We say that partial identities $s_1 = (a_1, e_1), s_2 = (a_2, e_2)$ are isomorphic if there is an isomorphism $h$ from $s_1$ onto $s_2$ which mean that $h$ is a one-to-one function from $a_1$ onto $a_2$ such that for every $b_1, c_1 \subseteq a_1$ we have $(b_1 e_1 c_1) \equiv h(b_1) e_2 h(c_1)$ (so $h$ maps $\text{Dom}(e_1)$ onto $\text{Dom}(e_2)$). We define similarly “$h$ is an embedding of $s_1$ into $s_2$” when $b_1 e_1 c_1 \Rightarrow h(b_1) e_2 h(c_1)$.

2) We say that $\lambda \rightarrow (a, e)_\mu$, if $(a, e)$ is an identity or a partial identity and for every function $f : [\lambda]^{<\aleph_0} \rightarrow \mu$, there is a one-to-one function $h : a \rightarrow \lambda$ such that

$$b e c \Rightarrow f(h''(b)) = f(h''(c)).$$  

(Instead $\text{Rang}(f) \subseteq \mu$ we may just require $|\text{Rang}(f)| \leq \mu$, this is equivalent).

3) We define

$$\text{ID}(\lambda, \mu) =: \{(n, e) : n < \omega \& (n, e) \text{ is an identity and } \lambda \rightarrow (n, e)_\mu\}$$  

and for $f : [\lambda]^{<\aleph_0} \rightarrow X$ we let

$$\text{ID}(f) =: \{(n, e) : (n, e) \text{ is an identity such that for some one-to-one function } h \text{ from } n = \{0, \ldots, n - 1\} \text{ to } \lambda \text{ we have } (\forall b, c \subseteq n)(b e c \Rightarrow f(h''(b)) = f(h''(c)))\}.$$  

Clearly two-place functions are easier to understand; this motivates:

**0.3 Definition.** 1) A two-identity or 2-identity\(^2\) is a pair $(a, e)$ where $a$ is a finite set and $e$ is an equivalence relation on $[a]^2$. Let $\lambda \rightarrow (a, e)_\mu$ mean $\lambda \rightarrow (a, e^+)_\mu$ where $b e^+ c \rightarrow [(b e c) \lor (b = c \subseteq a)]$ for any $b, c \subseteq a$.

2) We define

$$\text{ID}_2(\lambda, \mu) =: \{(n, e) : (n, e) \text{ is a 2-identity and } \lambda \rightarrow (n, e)_\mu\}$$

\(^2\)it is not an identity as $e$ is an equivalence relation on too small set but it is a partial identity
we define $\text{ID}_2(f)$ when $f : [\lambda]^2 \to X$ as

$$\{(n, e) : (n, e) \text{ is a two-identity such that for some } h, \text{ a one-to-one function from } \{0, \ldots, n-1\} \text{ into } \lambda \text{ we have } \{\ell_1, \ell_2\} e \{k_1, k_2\} \text{ implies that } \ell_1 \neq \ell_2 \in \{0, \ldots, n-1\}, \ k_1 \neq k_2 \in \{0, \ldots, n-1\} \text{ and } f(\{h(\ell_1), h(\ell_2)\}) = f(\{h(k_1), h(k_2)\})\}.$$  

3) Let us define $\text{ID}^\circ_2 =: \{(^{n}2, e) : (^{n}2, e) \text{ is a two-identity and if } \{\eta_1, \eta_2\} \neq \{\nu_1, \nu_2\} \text{ are } \subseteq ^n2, \text{ then } \{\eta_1, \eta_2\} e \{\nu_1, \nu_2\} \Rightarrow \eta_1 \cap \eta_2 = \nu_1 \cap \nu_2\}.$

4) In parts (1) and (2) we may replace $2$ by $k < \omega$ (only $k < |a_s|$ is interesting) and by $(\leq k)$.

0.4 Discussion: By [Sh 49], under the assumption $\aleph_\omega < 2^{\aleph_0}$, the families $\text{ID}_2(\aleph_\omega, \aleph_0)$ and $\text{ID}^\circ_2$ coincide (up to an isomorphism of identities). In Gilchrist and Shelah [GcSh 491] and [GcSh 583] we considered the question of the equality between these $\text{ID}_2(2^{\aleph_0}, \aleph_0)$ and $\text{ID}^\circ_2$ under the assumption $2^{\aleph_0} = \aleph_2$. We showed that consistently the answer may be “yes” and may be “no”.

Note that $(\aleph_n, \aleph_0) \not\rightarrow (\aleph_\omega, \aleph_0)$ so $\text{ID}(\aleph_2, \aleph_0) \neq \text{ID}(\aleph_\omega, \aleph_0)$, but for identities for pairs (i.e. $\text{ID}_2$) the question is meaningful.

We can look more at ordered identities

0.5 Definition. 1) An ord-identity or order identity is an identity $s$ such that $a_s \subseteq \text{Ord}$ or just: $a$ is an ordered set.

2) $\lambda \rightarrow^{\text{ord}} (s)_\mu$ if $s$ is an ord-identity and for every $c : [\lambda]^{< \aleph_0} \to \mu$ we have $s \in \text{OID}(c)$, see below (equivalently Dom$(c) = [\lambda]^{< \aleph_0}, |\text{Rang}(c)| \leq \mu$).

3) For $c : [\lambda]^{< \aleph_0} \to \mu$ let $\text{OID}(c) = \{(a, e) : a \text{ is a set of ordinals and there is an order preserving function } f : a \to \lambda \text{ such that } b_1 e b_2 \Rightarrow c(f''(b_1)) = c(f''(b_2))\}$.

4) $\text{OID}(\lambda, \mu) = \{(n, e) : (n, e) \in \text{OID}(c) \text{ for every } c : [\lambda]^{< \aleph_0} \to \mu\}$.

5) Similarly $\text{OID}_2$, $\text{OID}_k$, $\text{OID}_{\leq k}$.

Of course,
0.6 Claim. 1) $\text{ID}(\lambda, \mu)$ can be computed from $\text{OID}(\lambda, \mu)$.
2) Let $a$ be a finite set of ordinals and $e$ an equivalence relation. If $(a, e)$ is an identity, $a$ a set of ordinals and $\lambda > \mu$, then $(a, e) \in \text{ID}(\lambda, \mu)$ iff for some permutation $\pi$ of $a$ we have $(a, e^\pi) \in \text{OID}(\lambda, \mu)$ where $e^\pi = \{(b, c) : (\pi''(b), \pi''(c)) \in e\}$.
3) Let $A$ be a set of ordinals, $(a, e)$ an ord-identity and $c$ a function with domain $[A]^{<\aleph_0}$. Then $(a, e) \in \text{ID}(c)$ iff for some permutation $\pi$ of $a$, $(a, e^\pi) \in \text{OID}(c)$.
4) Similarly for $2$-identities and $k$-identities and $(\leq k)$-identities and partial identities.

0.7 Claim. If $n \in [1, \omega)$ and $s$ an ordered partial identity then there is a first order sentence $\psi_s$ such that: $\psi_s$ has a $(\mu^+ + n, \mu)$-model iff $s \not\in \text{OID}(\mu^+ + n, \mu)$.

Proof. Easy as for some first order $\psi$ sentence if $M$ is a $(\mu^+ + n, \mu)$-model of $\psi$ then $<_M$ is a linear order of $M$ (of cardinality $\mu^+ + n$) which is $\mu^+ + n$-like (i.e. every initial segment has cardinality). $\square_{0.7}$

We define simplicity:

0.8 Definition. 1) For $k < \aleph_0$, we say $(\lambda, \mu)$ has $k$-simple identities when $(a, e) \in \text{ID}(\lambda, \mu) \Rightarrow (a, e') \in \text{ID}(\lambda, \mu)$ whenever:

\[ (*)_k \quad a \subseteq \omega, (a, e) \text{ is an identity of } (\lambda, \mu) \text{ and } e' \text{ is defined by } \]

\[ b'c \iff |b| = |c| \text{ and } (\forall b'c)[b' \subseteq b \text{ and } |b'| \leq k \text{ and } c' = \text{OP}_{c,b'}(b') \Rightarrow b'ec'] \text{;} \]

recall $\text{OP}_{A,B}(\alpha) = \beta$ iff $\alpha \in A$ and $\beta \in B$ and otp($\alpha \cap A$) = otp($\beta \cap B$).

2) We define “$(\lambda, \mu)$ has $k$-simple ordered identities”, similarly.

We can ask

0.9 Question: 1) Define reasonably a pair $(\lambda, \mu)$ such that consistently

- $\oplus$ $\text{ID}(\lambda, \mu)$ is not recursive
- $\otimes'$ $\text{ID}(\lambda, \mu)$ is not, in a reasonable way, finitely generated.

2) Similarly for $\text{ID}_2(\lambda, \mu)$.
3) Restrict yourself to $(\mu^+, \mu)$. 
1.1 Claim. 1) If $\mu$ is strong limit singular then $\text{ID}_2(\mu^+, \mu)$ is 2-simple.

2) If $\mu = 2^{<\mu}$ and $c_0 : [\mu^+]^{<\aleph_0} \to \mu$ then we can find $c^* : [\mu^+]^2 \to \mu$ such that:

   (a) if $n \in [2, \omega)$ and $\alpha_0, \ldots, \alpha_{n-1} < \mu^+$ are with no repetitions and $\beta_0, \ldots, \beta_{n-1} < \mu^+$ are with no repetitions and $\ell < k < n \Rightarrow c^*\{\alpha_\ell, \alpha_k\} = c^*\{\beta_\ell, \beta_k\}$
   
   $c_0\{\alpha_0, \ldots, \alpha_{n-1}\} = c_0\{\beta_0, \ldots, \beta_{n-1}\}$

   (b) if in addition $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \mu^+$ are with no repetitions and $\beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-2} < \beta_{n-1}$

1.2 Remark. 1) We may wonder what is the gain in 1.1(2) as compared to 1.1(1), as if $\mu = 2^{<\mu}$ is regular then we know all relevant theory on $(\mu^+, \mu)$? The answer is that it clarifies identities of triples $(\mu^+, \mu, \kappa)$, e.g.

   (a) $(\mu^+, \mu, \kappa), \mu$ strong limit singular $> \kappa \geq \text{cf}(\mu)$
   
   (b) $(\mu^+, \mu, \kappa), \mu = \mu^{\beth_\omega(\kappa)}$.

2) Replacing $\mu + 2$ by $\mu + k, k + 1 \geq 2$ is similar and easier.

Proof. 1) By part (2).

2) By subclaims 1.3 - 1.7 below the claim is easy (see details in the end).

1.3 Subclaim. There is $c_1 : [\mu^+]^2 \to \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0, \beta_1, \beta_2 < \mu^+$ are with no repetitions and $c_1\{\beta_\ell, \beta_k\} = c_1\{\alpha_\ell, \alpha_k\}$ for $\ell < k < 3$ then at least two of the following holds $\beta_0 < \beta_1, \beta_0 < \beta_2, \beta_1 < \beta_2$.

Notice, that we have only three possibilities (not four):

(i) $\beta_0 < \beta_1 < \beta_2$

(ii) $\beta_1 < \beta_0 < \beta_2$

(iii) $\beta_0 < \beta_2 < \beta_1$.

Proof. Let $\eta_\alpha \in 2^{\mu^+}$ for $\alpha < \mu^+$ be pairwise distinct and for $\alpha \neq \beta < \mu^+$ let $\varepsilon\{\alpha, \beta\} = \text{Min}\{\varepsilon : \eta_\alpha \upharpoonright \varepsilon \neq \eta_\beta \upharpoonright \varepsilon\}$ and define the function $c_1'$ with domain $[\mu^+]^2$ by $c_1'\{\alpha, \beta\} = \{\eta_\alpha \upharpoonright \varepsilon\{\alpha, \beta\}, \eta_\beta \upharpoonright \varepsilon\{\alpha, \beta\}\}$, now $|\text{Rang}(c_1')| \leq \mu$ holds because $\mu = 2^{<\mu}$. For $\alpha \neq \beta$, let $c_1'\{\alpha, \beta\}$ be 1 if $(\eta_\alpha <_{\text{lex}} \eta_\beta) \equiv (\alpha < \beta)$ and 0 otherwise (the Sierpinski colouring). Lastly, define $c_1$ by: $c_1\{\alpha, \beta\} = (c_1'\{\alpha, \beta\}, c_1''\{\alpha, \beta\})$, it is a function with domain $[\mu^+]^2$ and range of cardinality $\leq \mu$ and easily it is as required. \hfill $\square_{1.3}$
1.4 Subclaim. For every $c : [\mu^+]^{<\aleph_0} \to \mu$ there is $c_2 : [\mu^+]^2 \to \mu$ such that: if $n \geq 2, \alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \mu^+, \beta_0 < \beta_1 < \ldots < \beta_{n-1} < \mu^+$ and $\ell < k < n \Rightarrow c_2(\alpha_\ell, \alpha_k) = c_2(\beta_\ell, \beta_k)$ then $c(\alpha_0, \ldots, \alpha_{n-1}) = c(\beta_0, \ldots, \beta_{n-1})$.

Proof. We are given $c : [\mu^+]^{<\aleph_0} \to \mu$ and for each $\alpha < \mu^+$ let $f_\alpha$ be a one-to-one function from $\alpha$ onto the ordinal $|\alpha| \leq \mu$ and we shall use those $f_\alpha$'s also later. We define an equivalence relation $E$ on $[\mu^+]^2$ such that:

(\*) for $\alpha_1 < \beta_1 < \mu^+$ and $\alpha_2 < \beta_2 < \mu^+$ we have $\{\alpha_1, \beta_1\}E\{\alpha_2, \beta_2\}$ iff 
\begin{enumerate}
  \item\ $(a)$ $f_{\beta_1}(\alpha_1) = f_{\beta_2}(\alpha_2)$ and
  \item\ $(b)$ for any $n < \omega$ and $\gamma_0 < \ldots < \gamma_{n-1} < f_{\beta_1}(\alpha_1)$ we have
\end{enumerate}

$$c(\alpha_1, \beta_1, f_{\beta_1}^{-1}(\gamma_0), \ldots, f_{\beta_1}^{-1}(\gamma_{n-1})) = c(\alpha_2, \beta_2, f_{\beta_2}^{-1}(\gamma_0), \ldots, f_{\beta_2}^{-1}(\gamma_{n-1}))$$

and similarly if we omit $\alpha_1, \alpha_2$ and/or $\beta_1, \beta_2$.

So $[\mu^+]^2/E$ has cardinality $\leq \mu^>2 = \mu$ and let $c_2 : [\mu^+]^2 \to \mu$ be such that $c_2(\alpha_1, \beta_1) = c_2(\alpha_2, \beta_2)$ iff $\{\alpha_1, \beta_1\}/E = \{\alpha_2, \beta_2\}/E$. We now check that it is as required in 1.4. Let $n, (\alpha_\ell : \ell < n), (\beta_\ell : \ell < n)$ be as in 1.4; so $\ell < k < n \Rightarrow c_2(\alpha_\ell, \alpha_k) = c_2(\beta_\ell, \beta_k)$, hence by (\*)(a) above (for $k = n - 1$) we have $\ell < n - 1 \Rightarrow f_{\alpha_{n-1}}(\alpha_\ell) = f_{\beta_{n-1}}(\beta_\ell)$, call it $\gamma_\ell$; as $f_{\alpha_{n-1}}$ is one to one, clearly $(\gamma_\ell : \ell < n - 2)$ is with no repetitions. Let $\ell(\ast) < n$ be such that $\gamma_{\ell(\ast)}$ is maximal and for $\ell < n - 2$ let $\gamma'_\ell$ be $\gamma_\ell$ if $\ell < \ell(\ast)$ and by $\gamma'_{\ell+1}$ if $\ell \in [\ell(\ast), n - 1)$. Now apply (\*)(b) with $\alpha_{\ell(\ast)}, \alpha_{n-1}, \beta_{\ell(\ast)}, \beta_{n-2}, (\gamma'_\ell : \ell < n - 2)$ here standing for $\alpha_1, \beta_1, \alpha_2, \beta_2, (\gamma_\ell : \ell < n - 2)$ there and we get the desired result. \(1.4\)

1.5 Subclaim. In 1.4, using $f_\alpha : \alpha \to \mu$ as in its proof, we have $c(\alpha_0, \ldots, \alpha_{n-1}) = c(\beta_0, \ldots, \beta_{n-2})$ also when

(\*) $n \geq 2, \alpha_0 < \alpha_1 < \ldots < \alpha_{n-3} < \alpha_{n-2} < \alpha_{n-1} < \mu^+, \beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-2} < \beta_{n-1}$ and $\ell < n - 2 \Rightarrow f_{\alpha_{n-1}}(\alpha_\ell) = f_{\alpha_{n-2}}(\alpha_\ell)$ and $\ell < k < n \Rightarrow c_2(\alpha_\ell, \alpha_k) = c_2(\beta_\ell, \beta_k)$.

Proof. Just the same proof. \(1.5\)
1.6 Subclaim. There is $c_4 : [\mu^+]^2 \to \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0, \beta_1, \beta_2 < \mu^+$ with no repetitions and $c_4(\beta_\ell, \beta_k) = c_4(\alpha_\ell, \alpha_k)$ for $\ell < k < 3$ then $\beta_0 < \beta_1 \& \beta_0 < \beta_2$.

Proof. For $\alpha < \beta < \mu^+$ we let $c'(\alpha, \beta) = \{f_\beta(\gamma) : \gamma < \alpha \& f_\beta(\gamma) < f_\beta(\alpha)\}$ and let $c_4(\alpha, \beta) = (c'(\alpha, \beta), c_1(\alpha, \beta), f_\beta(\alpha))$ where $c_1$ is from 1.3 and $\langle f_\gamma : \gamma < \mu^+ \rangle$ is from the proof of 1.4. Clearly $\|\text{Rang}(c')\| \leq \sum_{\zeta < \mu} 2^{[\zeta]} = \mu$ hence $\|\text{Rang}(c_4)\| \leq \mu^3 = \mu$.

If $\alpha_\ell, \beta_\ell (\ell < 3)$ form a counterexample, then $c_4(\alpha_\ell, \alpha_k) = c_4(\beta_\ell, \beta_k)$ for $\ell < k < 3$ hence by 1.3 we have three cases according to which one of the inequalities $\beta_\ell < \beta_k, \ell < k < 3$ fail.

Case (ii): $\beta_0 < \beta_1 < \beta_2$.

Trivial: the desired conclusion holds.

Case (ii): $\beta_1 < \beta_0$ so $\beta_1 < \beta_0 < \beta_2$.

Let $\zeta_\ell = f_{\alpha_2}(\alpha_\ell)$ for $\ell = 0, 1$ hence $\zeta_0 \neq \zeta_1$ as $f_{\alpha_2}$ is one to one and $\zeta_\ell = f_{\beta_2}(\beta_\ell)$.

Now on the one hand if $\zeta_0 < \zeta_1$ then $c'(\alpha_1, \alpha_2) \neq c'(\beta_1, \beta_2)$ (as $\zeta_0 \in c'(\alpha_1, \alpha_2)$, $\zeta_0 \notin c'(\beta_1, \beta_2)$), contradiction. On the other hand if $\zeta_1 < \zeta_0$ then $c'(\alpha_0, \alpha_2) \neq c'(\beta_0, \beta_2)$ (as $\zeta_1 \in c'(\beta_0, \beta_2)$, $\zeta_1 \notin c'(\alpha_0, \alpha_2)$), a contradiction, too.

Case (iii): $\beta_2 < \beta_1$.

By Subclaim 1.3 we have $\beta_0 < \beta_2 < \beta_1$.

This is O.K. for 1.6. \(\square_{1.6}\)

1.7 Subclaim. For every $c : [\mu^+]^2 \to \mu$ there is $c_5 : [\mu^+]^2 \to \mu$ such that

(a) $c_5(\alpha_1, \beta_1) = c_5(\alpha_2, \beta_2) \Rightarrow c_2(\alpha_1, \beta_1) = c_2(\alpha_2, \beta_2)$ where $c_2$ is from 1.5 (so also Subclaim 1.5)

(b) there are no $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0 < \beta_1 < \beta_2 < \mu^+$ such that $f_{\alpha_2}(\alpha_0) \neq f_{\alpha_1}(\alpha_0), c_5(\alpha_0, \alpha_1) = c_5(\beta_0, \beta_2), c_5(\alpha_0, \alpha_2) = c_5(\beta_0, \beta_1)$ and $c_5(\alpha_1, \alpha_2) = c_5(\beta_1, \beta_2)$

(c) $c_5(\alpha_1, \beta_1) = c_5(\alpha_2, \beta_2) \Rightarrow c_4(\alpha_1, \beta_1) = c_4(\alpha_2, \beta_2)$ where $c_4$ is from Subclaim 1.6.

Proof. Let $\kappa = \text{cf}(\mu) \leq \mu$ and $\mu = \sum_{i < \kappa} \lambda_i$ be such that if $\mu$ is a limit cardinal then $\lambda_i$ is (strictly) increasing continuous and if $\mu$ is a successor cardinal then $\mu = \lambda^+, \kappa = \mu$ and $\lambda_i = \lambda$ for $i < \kappa$. We can find $d : [\mu^+]^2 \to \kappa$ and $\bar{g}$ such that
\(\mathcal{U}_0 (i)\) for \(\beta < \mu^+, i < \kappa\) the set \(A_{\beta,i} = \{\alpha < \beta : d\{\alpha, \beta\} \leq i\}\) has cardinality \(\leq \lambda_i\).

(ii) if \(\alpha < \beta < \gamma < \mu^+\) then \(d\{\alpha, \gamma\} \leq \max\{d\{\alpha, \beta\}, d\{\beta, \gamma\}\}\)

(iii) \(\vec{g}\) is a sequence \((g_\alpha : \alpha < \mu^+)\)

(iv) \(g_\alpha : \alpha \to \mu\) is one to one and 
\[\lambda_i^+ < \mu \land \ i < \kappa \land \ \alpha < \beta \Rightarrow ((g_\beta(\alpha) < \lambda_i^+) \equiv (d\{\alpha, \beta\} \leq i))\]

(v) if \(\alpha < \beta, d\{\alpha, \beta\} = i\) and \(\lambda_i^+ = \mu\) then \(g_\beta(\alpha) < d\{\alpha, \beta\}\).

[Why we can find them? By induction on \(\beta < \mu^+\) by induction on \(i < \mu\) for \(\alpha = f_{\beta}^{-1}(i)\) we choose \(d\{\alpha, \beta\}\) and \(g_\beta(\alpha)\) as required.]

Define the functions \(c'_6\) and \(c'_7\) with domain \([\mu^+]^2\) as follows: if \(\alpha < \beta\) then 
\[c'_6(\alpha, \beta) = \{(t, \zeta_1, \zeta_2) : \zeta_1, \zeta_2 \leq g_\beta(\alpha), t < 2 \land t = 0 \Rightarrow g_\beta^{-1}(\zeta_1) < g_\beta^{-1}(\zeta_2) \land t = 1 \Rightarrow g_\beta^{-1}(\zeta_1) > g_\beta^{-1}(\zeta_2)\}\] and 
\[c'_7(\alpha, \beta) = \{(t, \zeta, \xi) : \zeta \in \lambda_i^+ \cap \text{Rang}(g_\beta) \land \xi \in \lambda_i^+ \cap \text{Rang}(g_\beta)\} \land g_\beta^{-1}(\zeta) = g_\beta^{-1}(\xi) \land t = 0 \lor g_\beta^{-1}(\zeta) > g_\beta^{-1}(\xi) \land t = 1\].

Now for \(\alpha < \beta < \mu^+\) we define \(c'_5(\alpha, \beta) \in \Pi\{\lambda_j^+ : j \leq d\{\alpha, \beta\}\}\), we do this by induction on \(\beta\) and for a fixed \(\beta\) by induction on \(i = d\{\alpha, \beta\}\) and for a fixed \(\beta\) and \(i\) by induction on \(\alpha\).

Arriving to \(\alpha < \beta\), for each \(j \leq d\{\alpha, \beta\}\), let \((c'_5(\alpha, \beta))(j)\) be the first ordinal \(\xi < \lambda_j^+\) such that:

\(\mathcal{U}_1 (i)\) if \(\gamma < \beta \land d\{\gamma, \beta\} \leq j \land (d\{\gamma, \beta\} = d\{\alpha, \beta\} \Rightarrow \gamma < \alpha)\) then 
\[(c'_5(\alpha, \gamma))(j) < \xi.\]

Clearly possible. The colouring we use is \(c_5\) where for \(\alpha < \beta < \mu^+\) we let \(c_5(\alpha, \beta) = (d\{\alpha, \beta\}, g_\beta(\alpha), f_\beta(\alpha), c_2(\alpha, \beta), c'_5(\alpha, \beta), c'_6(\alpha, \beta), c'_7(\alpha, \beta), c_4(\alpha, \beta))\), recalling \(c_4\) is from Subclaim 1.6 and \(c_2\) is from Subclaim 1.4. Obviously, \(|\text{Rang}(c_5)| \leq \mu\) and clauses \((a) + (c)\) of Subclaim 1.7 holds. So assume \(\alpha_0 < \alpha_1 < \alpha_2, \beta_0 < \beta_1 < \beta_2\) form a counterexample to clause \((b)\) of Subclaim 1.7 and we shall eventually derive a contradiction.

Clearly

\(\mathcal{U}_2 (i)\) if \(d\{\alpha_0, \alpha_2\} = d\{\beta_0, \beta_1\}, d\{\alpha_0, \alpha_1\} = d\{\beta_0, \beta_2\}, d\{\alpha_1, \alpha_2\} = d\{\beta_1, \beta_2\}\)

(ii) similarly for \(c_4, c'_5, c'_6, c'_7\).
By clause $\otimes_0(ii)$ above we have $d(\alpha_0, \alpha_2) \leq \max\{d(\alpha_0, \alpha_1), d(\alpha_1, \alpha_2)\}$, and applying clause $\otimes_0(ii)$ to $\beta_0 < \beta_1 < \beta_2$ and using $\otimes_2$ we have $d(\alpha_0, \alpha_1) = d(\beta_0, \beta_2) \leq \max\{d(\beta_0, \beta_1), d(\beta_1, \beta_2)\} = \max\{d(\alpha_0, \alpha_2), d(\alpha_1, \alpha_2)\}$. Hence $d(\alpha_0, \alpha_1) = d(\alpha_0, \alpha_2) > d(\omega_{\alpha_0, \alpha_2})$ or $\bigwedge_{\ell=1}^2 [d(\omega_{\alpha_0, \alpha_2}) < d(\alpha_1, \alpha_2)]$; we deal with those two cases separately.

Case 1: $\delta = d(\alpha_0, \alpha_1) = d(\alpha_0, \alpha_2) > d(\alpha_1, \alpha_2)$.

So (see the definition of $c_5$, with $\alpha_0, \alpha_2, \alpha_1, \delta$ here standing for $\alpha, \beta, \gamma, \xi$ there recalling that $\alpha_0 < \alpha_1 < \alpha_2$ we have $\lambda_+^\beta > (c_5(\alpha_0, \alpha_2))(\delta) > (c_5(\alpha_0, \alpha_1))(\delta)$. Similarly, $\lambda_+^\beta > (c_5(\beta_0, \beta_2))(\delta) > (c_5(\beta_0, \beta_1))(\delta)$. This contradicts $c_5(\alpha_0, \alpha_2) = c_5(\beta_0, \beta_{3-\ell})$ for $\ell = 1, 2$.

Case 2: $d(\alpha_0, \alpha_2) \leq d(\alpha_1, \alpha_2)$ for $\ell = 1, 2$.

Let $\delta = d(\alpha_1, \alpha_2)$. Let $\gamma_\ell = g_{\alpha_\ell}(\alpha_0)$ for $\ell = 1, 2$ so $\gamma_\ell = g_{\beta_{3-\ell}}(\beta_0)$ for $\ell = 1, 2$. By the assumption toward contradiction, i.e., by a demand in clause (b) of 1.7 we have $\gamma_1 \neq \gamma_2$. Clearly $\gamma_\ell < \lambda_+^\beta(d(\alpha_0, \alpha_2)) \leq \lambda_+^\beta(d(\alpha_1, \alpha_2)) = \lambda_+^\beta(\mu) \Rightarrow \gamma_\ell < d(\alpha_0, \alpha_\ell) \leq d(\alpha_1, \alpha_2) = \epsilon$.

As $c_5(\alpha_0, \alpha_2) = c_5(\beta_1, \beta_2)$ and $g_{\alpha_1}^{-1}(\gamma_1) = \alpha_0 = g_{\alpha_2}^{-1}(\gamma_2)$ clearly $g_{\beta_1}^{-1}(\gamma_1) = g_{\beta_2}^{-1}(\gamma_2)$ and they are well defined.

For $\ell = 1, 2$ as $c_5(\alpha_0, \alpha_\ell) = c_5(\beta_0, \beta_{3-\ell})$ by the choice of $\gamma_\ell$ (that is $\gamma_\ell = g_{\alpha_\ell}(\alpha_0)$) we have $g_{\beta_1}(\beta_0) = \gamma_3 - \ell$ so $g_{\beta_1}^{-1}(\gamma_3 - \ell) = \beta_0$ for $\ell = 1, 2$ hence $g_{\beta_1}^{-1}(\gamma_2) = g_{\beta_2}^{-1}(\gamma_1)$. As $c_5(\alpha_0, \alpha_2) = c_5(\beta_0, \beta_2)$ we have $c_5(\alpha_0, \alpha_2) = c_5(\beta_1, \beta_2)$ but $\gamma_1, \gamma_2 \leq g_{\alpha_2}(\alpha_1)$ hence $g_{\beta_1}^{-1}(\gamma_1) = g_{\beta_2}^{-1}(\gamma_2)$ for $\ell = 1, 2$.

As $\gamma_1 \neq \gamma_2$ we have $g_{\alpha_2}^{-1}(\gamma_1) \neq g_{\alpha_2}^{-1}(\gamma_2)$.

By symmetry without loss of generality $\gamma_1 > \gamma_2$. We can form an equivalence chain, starting with $g_{\beta_1}^{-1}(\gamma_1) < g_{\beta_1}^{-1}(\gamma_2)$ and arriving to $g_{\beta_1}^{-1}(\gamma_2) < g_{\beta_1}^{-1}(\gamma_1)$, a clear contradiction. Well, $g_{\beta_1}^{-1}(\gamma_1) < g_{\beta_1}^{-1}(\gamma_2)$ iff $g_{\beta_1}^{-1}(\gamma_2) < g_{\beta_1}^{-1}(\gamma_1)$ (by the equalities above) iff $g_{\beta_1}^{-1}(\gamma_2) < g_{\beta_2}^{-1}(\gamma_1)$ (by $\oplus 3$) iff $g_{\beta_1}^{-1}(\gamma_2) < g_{\beta_1}^{-1}(\gamma_1)$ (by $\oplus 5$) iff $g_{\beta_1}^{-1}(\gamma_2) < g_{\beta_1}^{-1}(\gamma_1) = c_5(\beta_0, \beta_1)$ and use the parameter $\ell$ in the triple $(t, \gamma_1, \gamma_2)$).

So we have proved Subclaim 1.7. $\square_{1.7}$

We can now sum up, i.e.:

Proof of 1.1(2) from Subclaims 1.3-1.7. We are given $c_0 : [\mu^+]^{<K_0} \rightarrow \mu$. First we apply Subclaim 1.4 for $c = c_0$ and get $c_2 : [\mu^+]^2 \rightarrow \mu$ as there and let $c_4$ be as in 1.6.

Second, we apply Subclaim 1.7 for $c = c_2$ and get $c_5$ as there. Let us check that $c_5$ is as required on $c^*$ in 1.1(2). So assume (*)$a_0 + (*)_1$ below and (as the case $n = 2$ is trivial) assume $n \geq 3$ where
\((**)_0\) \(\{\alpha_0, \ldots, \alpha_{n-1}\} \in [\mu^+]^n\) and \(\{\beta_0, \ldots, \beta_{n-1}\} \in [\mu^+]^n\) and
\((**)_1\) \(\ell < k < n \Rightarrow c_5\{\alpha_\ell, \alpha_k\} = c_5\{\beta_\ell, \beta_k\}\).

Without loss of generality (by renaming)
\((**)_2\) \(\alpha_0 < \ldots < \alpha_{n-1}\).

and it is enough to prove that \(c_0\{\alpha_0, \ldots, \alpha_{n-1}\} = c_0\{\beta_0, \ldots, \beta_{n-1}\}\). By clause (a) of Subclaim 1.7 we have
\((**)_3\) \(\ell < k < n \Rightarrow c_2\{\alpha_\ell, \alpha_k\} = c_2\{\beta_\ell, \beta_k\}\).

By clause (c) of Subclaim 1.7 we have
\((**)_4\) \(\ell < k < n \Rightarrow c_4\{\alpha_\ell, \alpha_k\} = c_4\{\beta_\ell, \beta_k\}\).

Hence by Subclaim 1.6 we have
\((**)_5\) if \(\ell < k < n\) and \(\ell < n - 2\) then \(\beta_\ell < \beta_k\).

[Why? Apply Subclaim 1.6 to \(\alpha_\ell, \alpha_{\ell+1}, \alpha_k; \beta_\ell, \beta_{\ell+1}, \beta_k\) if \(\ell + 1 < k\), and apply 1.6 to \(\alpha_\ell, \alpha_{\ell+1}, \alpha_{\ell+2}; \beta_\ell, \beta_{\ell+1}, \beta_{\ell+2}\) if \(\ell + 1 = k\).]

So
\((**)_6(i)\) \(\beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-2} < \beta_{n-1}\) or
\((**)_6(ii)\) \(\beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-1} < \beta_{n-2}\).

So clause (\(\beta\)) of 1.1 holds.

If (i) of \((**)_6\) holds, then the choice of \(c_2\), i.e., by Subclaim 1.4 and \((**)_3\) above we get \(c_0\{\alpha_0, \ldots, \alpha_{n-1}\} = c_0\{\beta_0, \ldots, \beta_{n-1}\}\) so we are done. Otherwise we have (ii) of \((**)_6\) so by clause (b) of Subclaim 1.7 we have
\((**)_7\) if \(\ell < n - 2\) then \(f_{\alpha_{n-1}}(\alpha_\ell) = f_{\alpha_{n-2}}(\alpha_\ell)\).

[Why? Apply clause (b) of Subclaim 1.7 to \(\alpha_\ell, \alpha_{n-2}, \alpha_{n-1}; \beta_\ell, \beta_{n-1}, \beta_{n-2}\).]

So by Subclaim 1.5 we get \(c_0\{\alpha_0, \ldots, \alpha_{n-1}\} = c_0\{\beta_0, \ldots, \beta_{n-1}\}\) finishing. \(\square_{1.1}\)

1.8 Claim. Defining \(\text{ID}(\lambda, \mu)\), we can restrict ourselves to \(c : [\lambda]^{<\aleph_0} \to \mu\) such that \(c \upharpoonright [\lambda]^1\) is constant if \(\text{cf}(\lambda) > \mu\).

1.9 Claim. Assume \(\mu = 2^{<\mu}\) and \(n \in [1, \omega)\). The identities of \(\text{ID}(\mu^+^n, \mu)\) are \((n + 1)\)-simple (and also \(\text{OID}(\mu^+, \mu)\)).

Proof. As in 1.1, only easier in the additional cases. \(\square_{2.1}\)
§2 Successor of strong limit above supercompact: 2-identities

So we know that if $\mu$ is strong limit singular and there is a compact cardinal in $(\text{cf}(\mu), \mu)$ then $\text{ID}_2(\mu^+, \mu) \neq \text{ID}_2(\aleph_1, \aleph_0)$. It seems desirable to find explicitly such 2-identities.

The proof of the following does much more.

2.1 Claim. Assume

(a) $s_k = (k + {k \choose 2}, e_{s_k})$ where the non-singleton $e_{s_k}$-equivalence classes are the sets:

\[
\{\{\ell_0, \ell_2\} : \ell_0 < k \text{ and for some } \ell_1 \in \{\ell_0 + 1, \ldots, k - 1\} \text{ we have } \\
\ell_2 = k + {\ell_1 \choose 2} + \ell_0\} \text{ and } \{\{\ell_1, \ell_2\} : \ell_1 < k \text{ and for some } \ell_0 < \ell_1 \text{ we have } \\
\ell_2 = k + {\ell_1 \choose 2} + \ell_0\}.
\]

We stipulate $\left({k \choose 2}\right) = 0$ here.

(b) $\mu$ is strong limit, $\theta$ a compact cardinal and $\text{cf}(\mu) < \theta < \mu$.

Then

1) $s_k \in \text{ID}_2(\mu^+, \mu)$, moreover $s_k \in \text{OID}_2(\mu^+, \mu)$.

2) $s_k \notin \text{ID}_2(\aleph_1, \aleph_0)$ for $k \geq 3$ so for $k = 3$ we have $s_k = (6, e_{s_k})$ and the non-singleton equivalence classes, after permuting $\{3, 5\}$ are $\{\{1, 3\}, \{0, 4\}, \{0, 5\}\}$ and $\{\{1, 5\}, \{2, 3\}, \{2, 4\}\}$.

Proof. Part (1) follows from subclaim 2.2(3) below and part (2) follows from 2.3 below. \(\Box_{2.1}\)

2.2 Claim. Assume

(a) $\mu$ is strong limit,

(b) $\theta$ is compact and $\text{cf}(\mu) < \theta < \mu$

(c) $\kappa = \text{cf}(\mu), \langle \lambda_i : i < \kappa \rangle$ is increasing with limit $\mu$

(d) $c : [\mu^+]^2 \to \mu$

(e) $d\{\alpha, \beta\} = \text{Min}\{i : c\{\alpha, \beta\} < \lambda_i\}$.

1) We can find $i(*)$, $A, f$ such that

\[
(*) \quad i(*) < \kappa, A \in [\mu^+]^{\mu^+} \text{ and } i(*) < \kappa
\]

(ii) for every set $B \subseteq A$ of cardinality $< \theta$ there are $\mu^+$ ordinals $\gamma \in A$ satisfying $\forall \alpha \in B \exists d\{\alpha, \gamma\} = i(*)$. 

Proof. Part (1) follows from subclaim 2.2(3) below and part (2) follows from 2.3 below. \(\Box_{2.1}\)
2) In part (1) we also have: if $A_1 \subseteq A, |A_1| \geq \beth_n(\lambda)^+$ and $\lambda_{i(\ast)} \leq \lambda < \mu$, then there are $\langle \gamma_\ell : \ell < n \rangle \in \beth_n(\lambda_{i(\ast)})$ and $B \in |A_1|^\lambda$ such that for every $\alpha_0 < \ldots < \alpha_{n-1}$ from $B$ for arbitrarily large $\beta < \lambda$ we have $\ell < n \Rightarrow c(\alpha_\ell, \beta) = \gamma_\ell$.

3) $s_\ell \in \text{ID}_2(c)$ where $s_\ell$ is from clause (a) of 2.1.

Proof. 1) Let $D$ be a uniform $\theta$-complete ultrafilter on $\mu^+$.

Define $f : \mu^+ \rightarrow \kappa$ by $f(\alpha) = i \iff \{ \gamma < \mu^+ : d(\alpha, \gamma) = i \} \in D$, note that the function $f$ is well defined as $D$ is a $\theta$-complete ultrafilter on $\mu^+$ and $\theta > \kappa \supseteq \text{Rang}(d)$. So for some $i(*)$, the set $A = \{ \alpha < \mu^+ : f(\alpha) = i(*) \}$ belongs to $D$ and check that $(*)$ holds, that is (i) + (ii) hold.

2) Define $c^* : [A]^n \rightarrow \beth_n(\lambda_{i(\ast)})$ such that

$$\forall \alpha_0 < \ldots < \alpha_{n-1} \text{ are from } A \text{ then for } \mu^+ \text{ ordinals } \beta < \mu^+ \text{ we have } c(\alpha_\ell, \beta) : \ell < n = c^* \{ \alpha_0, \ldots, \alpha_{n-1} \}.$$

So $\text{Rang}(c^*)$ has cardinality $\leq (\lambda_{i(\ast)})^n = \lambda_{i(\ast)}$ hence by the Erdős-Rado theorem there is $B \subseteq A_1$ infinite (even of any pregiven cardinality $< \lambda$) such that $c^* \upharpoonright [B]^n$ is constant.

3) Straight: in part (2) use $n = 2, A_1 = A$ and get $B$ and $\langle \gamma_0, \gamma_1 \rangle \in 2(\lambda_{i(\ast)})$ as there and choose $\alpha_0 < \ldots < \alpha_{k-1}$ from $B$. Next choose $\alpha_\ell$ for $\ell = 0, 1, \ldots, (k_2) - 1$, choosing $\beta_\ell$ by induction on $\ell$. If $\ell = (\ell_1, \ell_0)$ and $\ell_0 < \ell_1 < k$ choose $\beta_\ell \in A$ satisfying $\beta_\ell > \alpha_{\ell-1}$ and $\beta_\ell > \beta_m$ for $m < \ell$ such that $c(\alpha_\gamma, \beta_\ell) = \gamma_0, c(\alpha_\ell, \beta_\ell) = \gamma_1$.

Now let $\alpha_{k+\ell} = \beta_\ell$ for $\ell < (k_2)$, and clearly $\langle \alpha_\ell : \ell < k + (k_2) \rangle$ realize the identity $s_k$.

$\square_{2.2}$

2.3 Subclaim. 1) If $s \in \text{ID}_2(\aleph_1, \aleph_0)$, then we can find a function $h : |\text{Dom}_s|^2/\epsilon_s \rightarrow \omega$ respecting $e_s$ (i.e. $\ell_1, \ell_2 \in \epsilon_s \{ \ell_3, \ell_4 \} \Rightarrow h(\ell_1, \ell_2) = h(\ell_3, \ell_4)$) and there is a linear order $< \text{ of Dom}_s$ satisfying

$$\forall \text{ for any equivalence class } a \text{ of } e_s \text{ there are } a_0, a_1 \text{ such that }$$

$$\begin{align*}
(i) & \quad a_0, a_1 \text{ are disjoint finite subsets of Dom}_s \\
(ii) & \quad \text{if } \{ \ell_0, \ell_1 \} \in a \text{ and } \ell_0 < \ell_1 \text{ then } \ell_0 \in a_0 \& \ell_1 \in a_1 \\
(iii) & \quad \text{if } \ell_0 \neq \ell_1 \text{ are from } a_0 \cup a_1 \text{ and } \{ \ell_0, \ell_1 \} \notin a \text{ and } \{ \ell_0, \ell_1 \} \in a \text{ then } \\
& \quad h(\{ \ell_0, \ell_1 \}) > h(\{ \ell_0, \ell_1 \}).
\end{align*}$$

2) We can add in $\oplus$

$$\begin{align*}
(iv) & \quad \text{if } a_0, a_1 \text{ are distinct } e_s\text{-equivalence classes then for some } m \in \{ 0, 1 \} \text{ we have } |\cup a_m|^2/\epsilon_m \text{ is disjoint to } a_{1-m}.
\end{align*}$$
(v) in $\oplus$ above $a_0, a_1$ can be defined as $\{\ell_0 : \{\ell_0, \ell_1\} \in a, \ell_0 < \ell_1\}$, $\{\ell_1 : \{\ell_0, \ell_1\} \in a, \ell_0 < \ell_1\}$ respectively.

3) If $k \geq 3, s_k$ from 2.1 clause (a) then $s_k$ does not belong to ID$_2(N_1, N_0)$.

Proof. 1) Remember that by 0.6 we can deal with OID($N_1, N_0$). By [Sh 74] we know what is OID($N_1, N_0$), i.e., the family of identities in OID($N_1, N_0$) is generated by two operations; one is called duplication and the other of restriction (see below) from the trivial identity (i.e. $[\text{dom}_s] = 1$) and we prove $\oplus$ by induction on $n$, the number of times we need to apply the operations.

Recall that $(a, e)$ is gotten by duplication if we can find sets $a_0, a_1, a_2$ and a function $g$ such that

$$\oplus_1(a) \quad a_0 < a_1 < a_2 \quad \text{(i.e. } \ell_0 \in a_0, \ell_1 \in a_1, \ell_2 \in a_2 \Rightarrow \ell_0 < \ell_1 < \ell_2)$$

(a) $a = a_0 \cup a_1 \cup a_2$

(c) $g$ a one-to-one order preserving function from $a_0 \cup a_1$ onto $a_0 \cup a_1$ (so $g \restriction a_0 = \text{id}_{a_0}$; let $g_1 = g, g_2 = g^{-1}$

(d) for $\ell_0 \neq \ell_1 \in (a_0 \cup a_1)$ we have $\{\ell_0, \ell_1\}e\{g(\ell_0), g(\ell_1)\}$

(e) if $\ell_1 \in a_1, \ell_2 \in a_2$ then $\{\ell_1, \ell_2\}/e$ is a singleton

(f) $s_\ell = (a_0 \cup a_2, e \restriction [a_0 \cup a_2]^2)$ is from a lower level (up to isomorphism), for $\ell \in \{1, 2\}$.

Recall that $(a, e)$ is gotten by restriction from $(a', e')$ if $a \subseteq a', e = e' \restriction [a]^2$.

Now we prove the existence of $h$ as required by induction on the level. If $|\text{Dom}_s| = 1$ this is trivial. If $s$ is gotten by restriction it is trivial too, (as if $s = (a, e), s' = (a', e'), a' \subseteq a, e' = e \restriction a'$ and $h : [a]^2 \to \omega$ is as guaranteed then we let $h' \langle \{\ell_0, \ell_1\} = h(\{\ell_0, \ell_1\})$ for $\ell_0 < \ell_1$. Easily $h'$ is as required). So assume $s = (a, e)$ is gotten by duplication, so let $a_0, a_1, a_2, g_1, g_2$ be as in $\oplus_1$ and let $h_1$ be as required for $s_1 = (a_0 \cup a_1, e \restriction [a_0 \cup a_1]^2)$ and similarly define $h_2$ by $h_2\{\alpha, \beta\} = h_1\{g_2(\alpha), g_2(\beta)\}$.

Let $n^* = \sup \text{Rang}(h_1)$ and define $h : [a_0 \cup a_1 \cup a_2]^2 \to \omega$ by $h \geq h_1, h \geq h_2$ and if $k \in a_1, \ell \in a_2$ then we let $h\{k, \ell\} = n^* + 1$. Now check.

2) By symmetry, without loss of generality $h(a_0) < h(a_1)$ and now $m = 1$ satisfies the requirement by applying $\oplus_1$ to the equivalence class $a = a_1$.

3) It is enough to deal with $s_3$. By direct checking the criterion in part (2) fails.

$\Box_{2.3}$
2.4 Claim. Assume

(a) $s'_n \in \text{OID}_2$ is $(2n + n^2, e_{s'_n})$ where the non-singleton $e_{s'_n}$-equivalence classes are
$$\{{}\ell_0, 2n + n\ell_0 + \ell_1\} : \ell_0, \ell_1 < n\} \text{ and } \{{}\ell_0, 2n + n\ell_0 + \ell_1\} : \ell_0, \ell_1 < n\}$$

(b) $\mu$ is a limit cardinal, $\mu > \theta > \text{cf}(\mu)$ and $\theta$ is a compact cardinal

(c) $s''_n \in \text{OID}_2$ is $(2^n + 2^{2^n}, e_{s''_n})$ where the non-singleton $e_{s''_n}$-equivalence classes are: for $m < n, \eta \in \mathbb{m}2, i = 0, 1$ let $a^i_\eta = \{{}\ell_i, 2^n\left(2^n\right)^\eta + \ell_1\} : \ell_0, \ell_1 < 2^n$ and for some $\nu_0, \nu_1 \in ^n2$ we have $\eta^\nu(0) \leq \nu_0, \eta^\nu(1) \leq \nu_1$ and $\ell_0 = \Sigma(\nu_0(j)2^j : j < n}$ and $\ell_1 = \Sigma(\nu_1(j)2^j : j < n}$.

Then

1) $s'_n \in \text{ID}_2(\mu^+, \mu)$, moreover $s'_n \in \text{OID}_2(\mu^+, \mu)$ similarly for $s''_n$.
2) $s'_n \notin \text{ID}_2(\mathbb{N}_1, \mathbb{N}_0)$ for $n \geq 2$, similarly for $s''_n$.

Proof. 1) Like the proof of 2.2 using [Sh 49] (or just [Sh 604, 5.13]) instead of the Erdős-Rado theorem.

2) Otherwise there is $(a, e) \in \text{ID}_2(\mathbb{N}_1, \mathbb{N}_0)$ and an embedding $h$ of $s'_n$ into $(a, e)$ and by 0.6 without loss of generality $(a, e) \in \text{OID}_2(\mathbb{N}_1, \mathbb{N}_0)$. Now

$(*)_1$ if $\ell_0 < n, \ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$ then $h(\ell_0) < h(\ell)$.
[Why? Choose $\ell'_1 < n, \ell'_1 \neq \ell_1$, $\ell' = 2n + n\ell_0 + \ell'_1$, so $\ell \neq \ell'$ and 
$\{{}\ell_0, \ell\} \not\in \ell_1, \ell'_1\} \not\in \ell_1, \ell'_1\} \text{ hence the pairs } \{{}\ell(\ell_0), h(\ell)\}, \{{}\ell(\ell_0), h(\ell')\} \text{ are e-equivalent and } h(\ell) \neq h(\ell').$ But on $(a, e)$ we know that if $\{m_0, m_1, m_2\}$ has three members and $\{m_0, m_1\} e \{m_0, m_2\}$ then $m_2 < m_1$ and $m_2 < m_1$ are impossible (see 2.5(2) below) so we are done.]

$(*)_2$ if $\ell_0 < n, \ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$ then $h(\ell_1) < h(\ell)$.
[Why? Like $(*)_1$.]

Now we apply 2.3(1) $(2)$ above so $s'_n \notin \text{ID}_2(\mathbb{N}_1, \mathbb{N}_0)$. The conclusion about $s''_n$ follows. $\square_{2.4}$

2.5 Observation. 1) If $k \geq 2, \mathbf{s} = (n, e) \in \text{OID}_2(\mu^+, \mu)$ then we can find $s' = (n', e')$ in fact $n' = 2n - 1$ such that:

(i) $e' \upharpoonright [n]^2 = e$

(ii) $s' \in \text{ID}(\mu^+, \mu)$

(iii) for every $c : [\mu^+]^{<\mathbb{N}_0} \rightarrow \mu$ there is $c' : [\mu^+]^{<\mathbb{N}_0} \rightarrow \mu$ refining $c$ (i.e. $c'(u_1) = c'(u_2) \Rightarrow c(u_1) = c(u_2)$) such that: if $h : \{0, \ldots, 2n - 2\} \rightarrow \mu^+$ is one to
one and satisfies $u_1 e' u_2 \Rightarrow c'(h''(u_1)) = c'(h''(u_2))$ then $h \upharpoonright \{0, \ldots, n-1\}$ is increasing.

2) There is $c : [\mu^+]^2 \rightarrow \mu$ such that:
   - if $\alpha, \beta, \gamma$ are distinct and $c\{\alpha, \beta\} = c\{\alpha, \gamma\}$ then $\alpha < \beta$ & $\alpha < \gamma$.
3) We can replace in (1), $(\mu^+, \mu)$ by $(\lambda, \mu)$ if there is $s = (n, e) \in \text{ID}(\lambda, \mu)$ such that for some $c : [\lambda]^{< R_0} \rightarrow \mu$ such that
   - $\oplus$ if $h : n \rightarrow \lambda$ induces $e_s$ then $h(0) < h(1)$.

Proof. 1) Define $e' : u_1 e' u_2$ iff $u_1 e u_2 \vee u_1 = u_2 \vee \bigvee_{\ell < n-1} (u_1 = \{\ell, n + \ell + 1\} \& u_2 = \{\ell, \ell + 1\}) \vee \bigvee_{\ell < n-1} (u_2 = \{\ell, n + \ell + 1\} \& u_1 = \{\ell, \ell + 1\})$. Now use (2).

2) Let $f_\alpha : \alpha \rightarrow \mu$ be one to one for $\alpha < \mu^+$ and let $<^*$ a dense linear order on $\mu^+$ with $\{\alpha : \alpha < \mu\}$ a dense subset. Now choose $c_1 : [\mu^+]^2 \rightarrow \mu$ such that $\alpha <^* \beta \Rightarrow \alpha <^* c_1\{\alpha, \beta\} <^* \beta$ and define $c_0 : [\mu^+]^2 \rightarrow \{0, 1\} \& c_0\{\alpha, \beta\} = 1 \Leftrightarrow (\alpha < \beta \equiv \alpha <^* y)$.

Lastly, let $c : [\mu^+]^2 \rightarrow \mu$ be $\alpha < \beta \Rightarrow c\{\alpha, \beta\} = \text{pr}(2f_\beta(\alpha) + c_0\{\alpha, \beta\}, c_1\{\alpha, \beta\})$ for some pairing function pr.

3) Similar to part (1) only $|\text{Dom}_s'|$ is larger. $\square_{2.5}$
REFERENCES.


