Maximally Robust Capon Beamformer

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Abstract—The standard Capon beamformer (SCB) achieves the maximum output signal-to-interference-plus-noise ratio in the error-free case. However, estimation errors of the signal steering vector and the array covariance matrix can result in severe performance deteriorations of the SCB, especially if the training data contains the desired signal component. A popular technique to improve the robustness against model errors is to compute the Capon beamformer with the maximum output power, considering an uncertainty set for the signal steering vector. We show that the MRCB can be implemented efficiently using Lagrange duality. Simulation results demonstrate that the MRCB outperforms state-of-the-art robust adaptive beamformers in many scenarios.

Index Terms—Robust adaptive beamforming.

I. INTRODUCTION

BEAMFORMERS are spatial filters, which are used in a number of application areas to receive the signal with a specific spatial signature while interferers and noise are suppressed [1]–[4]. The performance of narrowband beamformers is commonly measured in terms of the signal-to-interference-plus-noise ratio (SINR) at the output of the beamformer. The output SINR can be maximized by minimizing the total beamformer output power subject to a distortionless constraint for the desired signal. This leads to the standard Capon beamformer (SCB) of [5]. The SCB is known to provide an excellent performance and a fast convergence rate if the beamformer training data does not contain the desired signal component [6]. The latter condition is satisfied in some active radar and sonar systems. However, in other application areas such as wireless communications and passive radar and sonar systems, the training data is “contaminated” by the desired signal component. Then, even small estimation errors of the signal steering vector or the array covariance matrix can lead to a severe performance deterioration of the SCB. Such estimation errors can be caused by environmental nonstationarities, look direction errors, array calibration errors, signal wavefront distortions, array geometry distortions, near-far mismatches, local scattering, etc. If the presumed signal steering vector deviates from the actual signal steering vector, then the SCB tends to erroneously suppress the desired signal component. This effect is commonly referred to as signal self-nulling [4]. A similar effect occurs in the case of estimation errors of the array covariance matrix.

A number of techniques have been proposed to improve the robustness of the SCB against model errors [4], [7]. For example, the array response towards the desired signal can be stabilized by enforcing multiple distortionless or derivative constraints. However, this approach is only applicable if the array manifold is known and each distortionless and derivative constraint reduces the beamformer degrees of freedom by one. Moreover, these constraints do not provide robustness against arbitrary-type signal steering vector estimation errors, which can occur, e.g., due to signal wavefront distortions.

An alternative approach to improve the robustness of the SCB is to project the estimate of the signal steering vector onto the estimate of the signal-plus-interference subspace [8], [9]. The corresponding eigenspace beamformer provides a good performance for moderate values of the input signal-to-noise ratio (SNR) especially if the number of sensors is significantly larger than the number of sources. However, the eigenspace beamformer is highly sensitive to estimation errors in the number of sources, and for low SNRs, signal subspace swaps can result in severe performance losses [10], [11].

The beamformer sensitivity to model errors can be limited by enforcing a distortionless constraint for the presumed signal steering vector and an upper bound on the norm of the weight vector [12], [13]. The corresponding norm-bounded Capon beamformer (NBCB) is equivalent to a regularization of the sample covariance matrix, where the regularization parameter depends on the upper bound on the norm of the beamformer vector. The choice of this upper bound is critical for the beamformer performance.

More recently, several set-based worst-case beamformers have been developed [14]–[22]. These beamformers are based on an uncertainty set for the signal steering vector. For all vectors within the presumed uncertainty set, the magnitude of the array response is constrained to be larger than or equal to one. In the case of spherical or ellipsoidal uncertainty sets, the infinite number of constraints can be simplified analytically using the worst-case principle. This allows to compute the...
beamformer weight vector by solving a convex second-order
cone programming (SOCP) problem [23]. Similar to the
NBCB, the set-based worst-case beamformer with a spherical
uncertainty set is equivalent to a regularization of the sample
covariance matrix [14].

It has been shown in [24] that the constraints of the set-based
worst-case beamformer of [14] limit the sensitivity to model
errors. Additionally, if the signal steering vector lies within the
interior of the presumed uncertainty set, then the constraints of this
beamformer result in an incentive for a low sensitivity, which
becomes stronger if the power of the desired signal increases.
Such an incentive is desirable since signal self-nulling effects
are most severe in the high SNR regime. A similar incentive for
a low sensitivity, which becomes stronger with the power of the
desired signal, does not exist for the NBCB [25].

In [26], it has been shown that the set-based worst-case beam-
formers of [14] and [17] are equivalent to the Capon beam-
former with the maximum output power, considering the un-
certainty set for the signal steering vector. In other words, the
set-based worst-case beamformers of [14] and [17] can be ob-
tained by replacing the estimate of the signal steering vector in
the SCB formula by the vector from the presumed uncertainty
set, which results in the maximum output power. Therefore,
we refer to this beamformer as maximum output power Capon
beamformer (MOPCB).

It can be expected that maximizing the Capon beamformer
output power results in an insufficient suppression of interferers
and noise. As an alternative approach, we propose to compute
the Capon beamformer with the minimum sensitivity to model
errors, considering the uncertainty set for the signal steering
vector. This beamformer is referred to as maximally robust
beamformer (MRCB). By definition, the sensitivity of the
MRCB is less than or equal to that of the MOPCB with the
same uncertainty set.

The proposed approach leads to a quadratically constrained
quadratic programming (QCQP) problem [23]. We show that
this problem can be solved efficiently using Lagrange duality.
Our simulation results suggest that the asymptotic growth rate
of the computational complexity of the proposed algorithm is
dominated by the eigendecomposition of the sample covariance
matrix.

We address the relation of the MRCB to the maximally robust
beamformer (MRB), that is the delay-and-sum beamformer. We
prove that the MRCB is equivalent to the MRB if the un-
certainty set for the signal steering vector is sufficiently large. Con-
sidering the exact array covariance matrix, we also show that
the MRCB is equivalent to the MRB if the true signal steering
vector lies within the interior of the presumed uncertainty set,
and if the power of the desired signal is sufficiently high.

The SINR performance of the MRB is compared numerically
with that of the SCB, MRB, NBCB, MOPCB, and the
eigenspace beamformer. Our simulation results demonstrate
that the MRCB frequently outperforms the other beamformers
tested. It is also shown that the MOPCB often achieves its
best SINR performance if the uncertainty set for the signal
steering vector is significantly larger as compared to the actual
error in the presumed signal steering vector, even if the exact
array covariance matrix is used. The MRCB typically achieves
its best SINR performance for smaller signal steering vector
uncertainty sets than the MOPCB. This property of the MRCB
is desirable, since small signal steering vector uncertainty sets
reach a high spatial resolution capability.

II. BACKGROUND

The narrowband beamformer output at the \( k \)th time instant is

\[
y(k) = w^H x(k)
\]  

(1)

where \( x(k) \in \mathbb{C}^{N \times 1} \) and \( w \in \mathbb{C}^{N \times 1} \) are the array snapshot and
beamformer weight vectors, respectively, \( N \) is the number of
sensors, \( \mathbb{C} \) denotes the set of complex numbers, and \( (\cdot)^H \) stands
for the Hermitian transpose. The snapshot vectors are modeled as

\[
x(k) = \sum_{l=1}^{L} a_l s_l(k) + n(k);
\]  

(2)

where \( L \) is the number of sources, \( a_l \in \mathbb{C}^{N \times 1} \) is the steering
vector (spatial signature) of the \( l \)th source, \( s_l(k) \) is the base-
band waveform of the \( l \)th source at the \( k \)th time instant, \( n(k) \in \mathbb{C}^{N \times 1} \) is the noise vector, and \( (\cdot)^T \) stands for the transpose.
Without loss of generality, we assume that the source with index
\( l = 1 \) is the source-of-interest, while the sources with indices
\( l \in \{2, \ldots, L\} \) are interferers. Hence, the received snapshot
vector can be expressed as

\[
x(k) = x_s(k) + x_i(k) + n(k);
\]  

(3)

where \( x_s(k) = a_1 s_1(k) \) and \( x_i(k) = a_2 s_2(k) + \cdots + a_L s_L(k) \) are the desired signal and interferer components, respectively.
In the sequel, we assume that the desired signal, interferer, and
noise waveforms are quasi-stationary zero-mean random pro-
cesses and that the desired signal waveform is uncorrelated with
the interferer and noise waveforms. Then, the array covariance
matrix can be expressed as

\[
R_n = E\{x(k) x^H(k)\} = P I + R_{i+n}
\]  

(4)

where \( P = E\{|s_1(k)|^2\} \) is the power of the desired signal,
\( E\{\cdot\} \) denotes the statistical expectation, and

\[
R_{i+n} = E\{(x_i(k) + n(k))(x_i(k) + n(k))^H\}
\]  

(5)

is the interference-plus-noise covariance matrix. The beam-
former performance is commonly measured in terms of the
output SINR, defined as [4]

\[
\text{SINR} = \frac{P_I |w^H a_1|^2}{w^H R_{i+n} w}.
\]  

(6)

The output SINR can be maximized by minimizing the output
interference-plus-noise power subject to a distortionless con-
straint for the desired signal. This can be formulated as

\[
\min_w w^H R_{i+n} w \quad \text{s.t.} \quad w^H a_1 = 1.
\]  

(7)
The optimum weight vector of (7) is [4]

\[ \mathbf{w}_{\text{opt}} = \frac{\mathbf{R}_{i+n}^{-1} \mathbf{a}_1}{\mathbf{a}_1^H \mathbf{R}_{i+n}^{-1} \mathbf{a}_1} \]  

(8)

where we assumed that \( \mathbf{R}_{i+n} \) is non-singular. In practice, this assumption is usually satisfied due to the presence of a white noise component. Substituting (8) in (6) yields the optimum SINR

\[ \text{SINR}_{\text{opt}} = P_1 \mathbf{a}_1^H \mathbf{R}_{i+n}^{-1} \mathbf{a}_1. \]  

(9)

Due to (4) and the distortionless constraint in (7), replacing \( \mathbf{R}_{i+n} \) by \( \mathbf{R}_s \) in the objective function of (7) leads to an additional constant term. Thus, the optimum weight vector of (7) does not change if \( \mathbf{R}_{i+n} \) is replaced by \( \mathbf{R}_s \). The array covariance matrix can be estimated as

\[ \hat{\mathbf{R}}_s = \frac{1}{K} \sum_{k=1}^{K} \mathbf{x}(k) \mathbf{x}^H(k); \]  

(10)

where \( K \) is the number of training snapshot vectors. Replacing \( \mathbf{R}_{i+n} \) in (8) by the sample covariance matrix \( \hat{\mathbf{R}}_s \) and \( \mathbf{a}_1 \) by the presumed (estimated) signal steering vector \( \hat{\mathbf{a}}_1 \) leads to the SCB [4]

\[ \mathbf{w}_{\text{SCB}} = \frac{\hat{\mathbf{R}}_s^{-1} \hat{\mathbf{a}}_1}{\hat{\mathbf{a}}_1^H \hat{\mathbf{R}}_s^{-1} \hat{\mathbf{a}}_1} \]  

(11)

where \( \hat{\mathbf{R}}_s \) is assumed to be nonsingular. It is well-known that estimation errors in \( \hat{\mathbf{R}}_s \) and \( \hat{\mathbf{a}}_1 \) can lead to a severe performance degradation of the SCB, especially if the SNR is high.

**A. Beamformer Sensitivity**

Let \( \epsilon \) denote the presumed upper bound on the norm of the signal steering vector estimation error. That is, we consider the uncertainty set for the signal steering vector

\[ \mathcal{S}_{\epsilon} = \{ \hat{\mathbf{a}}_1 | \| \hat{\mathbf{a}}_1 - \mathbf{a}_1 \|_2 < \epsilon \} \]  

(12)

where \( \| \cdot \|_2 \) is the Euclidean vector norm. The array response for the presumed (estimated) signal steering vector is \( \mathbf{w}^H \hat{\mathbf{a}}_1 \). The maximum deviation of the array response from \( \mathbf{w}^H \hat{\mathbf{a}}_1 \) for all steering vectors in \( \mathcal{S}_{\epsilon} \) is

\[ \max_{\hat{\mathbf{a}}_1 \in \mathcal{S}_{\epsilon}} \| \mathbf{w}^H \hat{\mathbf{a}}_1 - \mathbf{w}^H \mathbf{a}_1 \|_2 = \epsilon \| \mathbf{w} \|_2. \]  

(13)

Clearly, the larger the uncertainty in the signal steering vector, the larger can be the deviation of the array response from \( \mathbf{w}^H \hat{\mathbf{a}}_1 \). To obtain a measure for the beamformer sensitivity, we normalize (13) by \( \epsilon \). Moreover, we divide (13) by \( \mathbf{w}^H \hat{\mathbf{a}}_1 \), since we are interested in the relative size of the deviation in the array response. This leads to the definition of the beamformer sensitivity [4, 27, 28]

\[ T_{\text{sc}} = \frac{\| \mathbf{w} \|_2^2}{\| \mathbf{w}^H \mathbf{a}_1 \|}. \]  

(14)

where the square has been taken for convenience. If \( T_{\text{sc}} \) is small, then the beamformer is robust against signal steering vector estimation errors. Note that the beamformer sensitivity is defined as the squared norm of the scaled weight vector \( \mathbf{w} / (\hat{\mathbf{a}}_1^H \mathbf{w}) \), which satisfies \( \hat{\mathbf{a}}_1^H \hat{\mathbf{a}}_1 = 1 \). The normalization with respect to \( \| \mathbf{w} \|_2 \) in (14) is important, since otherwise an arbitrarily low sensitivity could be achieved by scaling the beamformer weight vector without changing the SINR performance.

The definition of the beamformer sensitivity in (14) can also be motivated by considering the effect of estimation errors in the sample covariance matrix on the beamformer output power. Let \( \kappa \) denote the presumed upper bound on the Frobenius norm of the estimation error in the sample covariance matrix. Then, the corresponding maximum variation of the beamformer output power is

\[ \max_{\| \hat{\mathbf{R}}_s - \mathbf{R}_s \|_F \leq \kappa} \| \mathbf{w}^H (\hat{\mathbf{R}}_s - \mathbf{R}_s) \mathbf{w} - \kappa \| \| \mathbf{w} \|_2^2 \]  

(15)

where \( \| \cdot \|_F \) denotes the Frobenius norm. Similar to (13), (15) increases with the norm of the beamformer weight vector. Hence, if \( T_{\text{sc}} \) is small, then the beamformer is robust against estimation errors in \( \hat{\mathbf{a}}_1 \) and \( \hat{\mathbf{R}}_s \).

The sensitivity (14) is minimized by the MRB

\[ \mathbf{w}_{\text{MRB}} = \hat{\mathbf{a}}_1 \]  

(16)

and all non-zero scalings of this weight vector. Since the MRB is non-adaptive, it does not suffer from signal self-nulling. At the same time, it does not adaptively suppress interferers. For this reason, the performance of the MRB is usually significantly worse than that of the SCB in the low SNR regime.

Over the last decades, a number of techniques have been proposed to improve the robustness of the SCB against model errors. Two popular robust adaptive beamformers, the NBCB and the set-based worst-case beamformer, are reviewed next.

**B. Norm-Bounded Capon Beamformer**

The beamformer sensitivity can be limited by enforcing a distortionless constraint and an upper bound on the norm of the beamformer weight vector. This leads to the NBCB [13, 28]

\[ \min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}}_s \mathbf{w} \quad \text{s.t.} \quad \mathbf{w}^H \hat{\mathbf{a}}_1 = 1, \| \mathbf{w} \|_2 \leq \alpha_w \]  

(17)

where \( \alpha_w \) is a user-defined parameter. Applying the Cauchy-Schwarz inequality to the equality constraint of (17) yields \( \| \mathbf{w} \|_2 \| \hat{\mathbf{a}}_1 \|_2 \geq 1 \). Hence, (17) is feasible if and only if \( \alpha_w \geq \| \hat{\mathbf{a}}_1 \|_2^2 \). Numerical results show that choosing \( \alpha_w \) moderately (1–3 dB) above this lower bound often leads to a good performance [24, 28].

The problem (17) belongs to the class of SOCP problems [23]. An efficient algorithm to solve this problem has been presented in [26]. Simulation results suggest that the asymptotic growth rate of the complexity of this algorithm is dominated by the eigendecomposition of \( \hat{\mathbf{R}}_s \), which is \( \mathcal{O}(N^3) \) [29].
C. Set-Based Worst-Case Beamformer

The set-based worst-case beamformer of [14] minimizes the expected output power subject to the constraint that the magnitude of the array response is larger than or equal to one for all steering vectors in \( \mathcal{S}_{a_1} \). This can be formulated as

\[
\begin{align*}
\min_{w, \hat{\mathbf{a}}_1} & \quad w^H \hat{\mathbf{R}}_x w \\
\text{s.t.} & \quad |w^H \hat{\mathbf{a}}_1| \geq 1 \quad \forall \hat{\mathbf{a}}_1 \in \mathcal{S}_{a_1}.
\end{align*}
\]

The constraints in (18) are satisfied if and only if they are satisfied for the worst-case steering vector, which minimizes the magnitude of the array response. In the sequel, we assume that \( \epsilon < ||\hat{\mathbf{a}}_1||_2 \). Then, [14]

\[
\min_{\hat{\mathbf{a}}_1 \in \mathcal{S}_{a_1}} |w^H \hat{\mathbf{a}}_1| = w^H \hat{\mathbf{a}}_1 - \epsilon ||w||_2.
\]

Thus, (18) can be written as

\[
\begin{align*}
\min_{w} & \quad w^H \hat{\mathbf{R}}_x w \\
\text{s.t.} & \quad |w^H \hat{\mathbf{a}}_1| - \epsilon ||w||_2 \geq 1.
\end{align*}
\]

In the literature, the beamformer (20) is often referred to as “worst-case performance optimization based beamformer.” However, it has been explained in [24] that, even if \( \hat{\mathbf{R}}_x = \hat{\mathbf{R}}_x \) and \( \hat{\mathbf{a}}_1 = \mathbf{a}_1 \), the optimum weight vector of (20) does not maximize the worst-case output SINR, defined as

\[
\min_{\hat{\mathbf{a}}_1 \in \mathcal{S}_{a_1}} \frac{P_1 |w^H \hat{\mathbf{a}}_1|^2}{w^H \hat{\mathbf{R}}_x w}.
\]

Since the beamformer (20) does not maximize the worst-case SINR, \( \epsilon \) should not necessarily be chosen according to the expected size of the signal steering vector estimation errors. In fact, numerical results show that, even if \( \hat{\mathbf{R}}_x = \hat{\mathbf{R}}_x \), the best SINR performance of the beamformer (20) is often attained if \( \epsilon \) is chosen significantly larger than the actual norm of the signal steering vector estimation error.

The robustness of the set-based worst-case beamformer (20) can be explained as follows [24]. Any weight vector, which satisfies the constraint of (20) with strict inequality, can be scaled such that the constraint is still satisfied, but a lower value of the objective function is attained. Hence, the optimum weight vector has to satisfy the constraint of (20) with equality. Thus, the sensitivity of the beamformer (20) is

\[
T_{\text{se}} = \frac{||w||_2^2}{(1 + \epsilon ||w||_2)^2} \leq \frac{1}{\epsilon^2}.
\]

That means, the constraint in (20) limits the beamformer sensitivity. Moreover, if \( \hat{\mathbf{a}}_1 = \mathbf{a}_1 - \mathbf{a}_1 \) denotes the signal steering vector estimation error, then the constraint in (20) yields

\[
|w^H \mathbf{a}_1| \geq 1 + (\epsilon - ||\hat{\mathbf{a}}_1||_2) ||w||_2.
\]

The minimization of the total beamformer output power leads to an incentive for a small value of \( ||w^H \mathbf{a}_1||_2 \). The strength of this incentive increases with the power of the desired signal. The inequality (23) results in an incentive for a small value of \( ||w||_2 || \) if the signal steering vector lies within the interior of \( \mathcal{S}_{a_1} \), i.e., if \( \epsilon > ||\hat{\mathbf{a}}_1||_2 \). The sensitivity (22) is strictly increasing in \( ||w||_2 \). Thus, if \( \epsilon > ||\hat{\mathbf{a}}_1||_2 \), then the constraint in (20) leads to an incentive for a low sensitivity, which becomes stronger with the power of the desired signal. A similar signal-dependent incentive for a low sensitivity does not exist for the NBCB [25].

Due to (23), the strength of the incentive for a low sensitivity of the set-based worst-case beamformer increases also with \( \epsilon \), assuming that \( \epsilon > ||\hat{\mathbf{a}}_1||_2 \). That means, the larger the uncertainty set for the signal steering vector, the stronger is the incentive for a low sensitivity.

The beamformer (18) is based on the spherical uncertainty set \( \mathcal{S}_{a_1} \). The generalization to ellipsoidal uncertainty sets can be solved in a similar way [17]. For the sake of simplicity, we do not further consider ellipsoidal signal steering vector uncertainty sets, even though the generalization to these sets is straightforward.

It has been shown in [26] that the set-based worst-case beamformer (18) can be obtained by solving

\[
\begin{align*}
\max_{\hat{\mathbf{a}}_1, w} & \quad w^H \hat{\mathbf{R}}_x w \\
\text{s.t.} & \quad w = \frac{\hat{\mathbf{R}}_x^{-1} \hat{\mathbf{a}}_1}{\hat{\mathbf{a}}_1^H \hat{\mathbf{R}}_x \hat{\mathbf{a}}_1}, \\
& \quad \hat{\mathbf{a}}_1 \in \mathcal{S}_{a_1}.
\end{align*}
\]

Hence, the beamformer (18) is equivalent to the MOPCB, i.e., the Capon beamformer with the maximum output power, considering the uncertainty set for the signal steering vector. An efficient technique to solve (24) based on the eigendecomposition of the sample covariance matrix has been presented in [26]. Simulation results suggest that the complexity of this technique is \( \mathcal{O}(N^3) \).

III. Maximally Robust Capon Beamformer

In general, the beamformer output power consists of a desired signal, an interference, and a noise component. The maximization of the Capon beamformer output power in (24) diminishes the erroneous suppression of the desired signal component. At the same time, it may lead to a poor suppression of interferers and noise. As an alternative approach, we propose to compute the Capon beamformer with the minimum sensitivity to model errors, considering the uncertainty set for the signal steering vector. The proposed beamforming problem can be formulated as

\[
\begin{align*}
\min_{\hat{\mathbf{a}}_1, w} & \quad w^H \mathbf{w} \\
\text{s.t.} & \quad w = \frac{\hat{\mathbf{R}}_x^{-1} \hat{\mathbf{a}}_1}{\hat{\mathbf{a}}_1^H \hat{\mathbf{R}}_x \hat{\mathbf{a}}_1}, \\
& \quad \hat{\mathbf{a}}_1 \in \mathcal{S}_{a_1}.
\end{align*}
\]

Obviously, the sensitivity of the proposed beamformer is less than or equal to that of the MOPCB with the same uncertainty set. In other words, the proposed beamformer is at least as robust against model errors as the MOPCB with the same uncertainty set.
Substituting the equality constraint of (25) in the objective function yields

\[
\min_{\tilde{a}_1} \frac{\tilde{a}_1^H \hat{R}_x^{-2} \tilde{a}_1}{\tilde{a}_1^H \hat{R}_x^{-1} \tilde{a}_1}^2 \quad \text{s.t.} \quad \tilde{a}_1 \in \mathcal{C}_{a_1}. \quad (26)
\]

Since any scaling of \( \tilde{a}_1 \) has no impact on the objective function of (26), the constraint of this optimization problem can be replaced by

\[
\min_{\zeta} \| \tilde{a}_1 \zeta - \tilde{a}_1 \|_2 \leq \epsilon. \quad (27)
\]

The minimum in (27) is attained for \( \zeta = \frac{\tilde{a}_1^H \tilde{a}_1}{\| \tilde{a}_1 \|_2^2} \). Using this result, (27) can be written as

\[
| \Pi_{a_1} \tilde{a}_1 - \tilde{a}_1 |_2 \leq \epsilon \quad (28)
\]

where \( \Pi_{a_1} = \tilde{a}_1 \tilde{a}_1^H / \| \tilde{a}_1 \|_2^2 \) is the orthogonal projection matrix onto the range space of \( \tilde{a}_1 \). Straightforward transformations allow to write (28) as

\[
| \tilde{a}_1 |^2 \sqrt{\| \tilde{a}_1 \|_2^2 - \epsilon^2} \leq \tilde{a}_1^H \tilde{a}_1. \quad (29)
\]

Thus, (26) can be formulated as

\[
\min_{\tilde{a}_1} \frac{\tilde{a}_1^H \hat{R}_x^{-2} \tilde{a}_1}{\tilde{a}_1^H \hat{R}_x^{-1} \tilde{a}_1} \quad \text{s.t.} \quad \nu \| \tilde{a}_1 \|_2 \leq | \tilde{a}_1^H \tilde{a}_1 | \quad (30)
\]

where

\[
\nu = \sqrt{\| \tilde{a}_1 \|_2^2 - \epsilon^2}. \quad (31)
\]

The objective function and the constraint in the latter optimization problem are invariant with respect to the scaling of \( \tilde{a}_1 \). We choose to scale this vector such that \( \tilde{a}_1^H \hat{R}_x^{-1} \tilde{a}_1 = 1 \). This leads to the optimization problem

\[
\min_{\tilde{a}_1} \tilde{a}_1^H \hat{R}_x^{-2} \tilde{a}_1 \quad \text{s.t.} \quad \tilde{a}_1^H \hat{R}_x^{-1} \tilde{a}_1 = 1 \quad (32)
\]

Proposition 1: The optimization problem (32) and its relaxation

\[
\min_{\tilde{a}_1} \tilde{a}_1^H \hat{R}_x^{-2} \tilde{a}_1 \quad \text{s.t.} \quad \tilde{a}_1^H \hat{R}_x^{-1} \tilde{a}_1 = 1 \quad \nu \| \tilde{a}_1 \|_2 \leq | \tilde{a}_1^H \tilde{a}_1 | \quad (33)
\]

where \( \Re \{ \cdot \} \) denotes the real part operator, have the same optimum point. Moreover, strong duality holds for (33).

Proof: See Appendix A.
Moreover, (40) can be reformulated as
\[
\max_{\lambda \geq 0} \frac{1}{\hat{A}_1^H \tilde{R}_z^{-1} \hat{A}_1} \quad \text{s.t.} \quad C_\lambda \succeq 0 \quad \text{and} \quad \tilde{R}_z^{-1} \hat{a}_1 \in \mathcal{R}\{C_\lambda\}.
\]  
(43)

Since the objective function of (43) is positive, this optimization problem can be written equivalently as
\[
\max_{\lambda \geq 0} \frac{1}{\hat{A}_1^H \tilde{R}_z^{-1} \hat{A}_1} \quad \text{s.t.} \quad C_\lambda \succeq 0 \quad \text{and} \quad \tilde{R}_z^{-1} \hat{a}_1 \in \mathcal{R}\{C_\lambda\}.
\]  
(44)

The semidefinite constraint \(C_\lambda \succeq 0\) is convex in \(\lambda\) and it is satisfied for \(\lambda = 0\). For \(\lambda \geq 0\),
\[
G_\lambda \triangleq \tilde{R}_z^{-2} + \lambda \nu^2 I_N
\]  
(45)
is positive definite, so we can write
\[
C_\lambda = G^{1/2}_\lambda \left( I_N - \lambda G^{-1/2}_\lambda \hat{a}_1 \hat{a}_1^H G^{1/2}_\lambda \right) G^{1/2}_\lambda.
\]  
(46)

Thus, \(C_\lambda \succeq 0\) for \(\lambda \geq 0\) if and only if
\[
f(\lambda) \triangleq \lambda \left| G^{1/2}_\lambda \hat{a}_1 \right|^2 \leq 1.
\]  
(47)
The function \(f(\lambda)\) can be rewritten using the eigendecomposition of \(\tilde{R}_z\), that is
\[
\tilde{R}_z = U \Gamma U^H
\]  
(48)
where the unitary matrix \(U = [u_1, \ldots, u_N]\) contains the orthonormal eigenvectors, and the diagonal matrix \(\Gamma = \text{diag}\{\gamma_1, \ldots, \gamma_N\}\) contains the eigenvalues. In the sequel, we assume without loss of generality that the eigenvalues are sorted in decreasing order, i.e., \(\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_N\).

Using (48), we obtain
\[
f(\lambda) = \lambda \sum_{n=1}^{N} \frac{|z_n|^2}{\gamma_n^2 + \lambda \nu^2}.
\]  
(49)

where \(z_n\) is the \(n\)th component of \(z = U^H \hat{a}_1\). It can easily shown that \(f(0) = 0\) and that \(f(\lambda)\) is strictly monotonically increasing for positive values of \(\lambda\). Moreover,
\[
\lim_{\lambda \to \infty} f(\lambda) - \frac{||\hat{a}_1||^2_2}{\nu^2} > 1.
\]  
(50)

Consequently, there is a unique positive value \(\lambda_{\text{max}}\) such that
\[
f(\lambda_{\text{max}}) = \lambda_{\text{max}} \sum_{n=1}^{N} \frac{|z_n|^2}{\gamma_n^2 + \lambda_{\text{max}} \nu^2} = 1.
\]  
(51)

Then, \(C_\lambda \succeq 0\) for \(0 < \lambda < \lambda_{\text{max}}\) and \(C_\lambda \not\succeq 0\) for \(\lambda > \lambda_{\text{max}}\). The optimization problem (44) can therefore be written as
\[
\begin{align*}
\min_{\lambda} & \quad \hat{a}_1^H \tilde{R}_z^{-1} C_\lambda \tilde{R}_z^{-1} \hat{a}_1 \\
\text{s.t.} & \quad 0 \leq \lambda \leq \lambda_{\text{max}} \quad \text{and} \quad \tilde{R}_z^{-1} \hat{a}_1 \in \mathcal{R}\{C_\lambda\}
\end{align*}
\]  
(52)

where \(\lambda_{\text{max}}\) can be determined, e.g., by means of the secant method. This method involves the evaluation of \(f(\lambda)\) at a large number of points. Once \(\Gamma\) and \(\hat{a}\) have been computed, (49) allows to efficiently determine \(f(\lambda)\) using only \(\mathcal{O}(N)\) operations. From (51), we obtain
\[
\lambda_{\text{max}} \sum_{n=1}^{N} \frac{|z_n|^2}{\gamma_n^2 + \lambda_{\text{max}} \nu^2} \geq 1 \implies \lambda_{\text{max}} \geq \frac{1}{\hat{a}_1^H \tilde{R}_z^{-1} \hat{a}_1}.
\]  
(53)
\[
\lambda_{\text{max}} \sum_{n=1}^{N} \frac{|z_n|^2}{\gamma_n^2 + \lambda_{\text{max}} \nu^2} \geq 1 \implies \lambda_{\text{max}} \geq \frac{1}{\gamma_1^2 \nu^2}.
\]  
(54)
\[
\lambda_{\text{max}} \sum_{n=1}^{N} \frac{|z_n|^2}{\gamma_n^2 + \lambda_{\text{max}} \nu^2} \leq 1 \implies \lambda_{\text{max}} \leq \frac{1}{\gamma_1^2 \nu^2}.
\]  
(55)

These upper and lower bounds for \(\lambda_{\text{max}}\) may be used as initial values for the secant method.

Proposition 2: The constraint
\[
\tilde{R}_z^{-1} \hat{a}_1 \in \mathcal{R}\{C_\lambda\}
\]  
(56)
is satisfied for \(0 \leq \lambda < \lambda_{\text{max}}\), but not for \(\lambda = \lambda_{\text{max}}\).

Proof: See Appendix B.

The latter proposition allows to formulate (52) as
\[
\begin{align*}
\min_{\lambda} & \quad \hat{a}_1^H \tilde{R}_z^{-1} C_\lambda \tilde{R}_z^{-1} \hat{a}_1 \\
\text{s.t.} & \quad 0 \leq \lambda < \lambda_{\text{max}}.
\end{align*}
\]  
(57)

The dual function is always concave ([23], Section 5.1.2). The objective function of (43) has been obtained by maximizing the dual function \(g(\lambda, \mu)\) over \(\mu\). Hence, the objective function of (43) is concave for \(0 < \lambda < \lambda_{\text{max}}\) ([23], Section 3.2.5). Due to the positiveness of this function for \(0 \leq \lambda < \lambda_{\text{max}}\), its reciprocal
\[
h(\lambda) \triangleq \hat{a}_1^H \tilde{R}_z^{-1} C_\lambda \tilde{R}_z^{-1} \hat{a}_1
\]  
(58)
is convex for \(0 \leq \lambda < \lambda_{\text{max}}\) ([23], Section 3.2.4). Thus, (57) is a convex optimization problem.

The convexity of (57) implies that the optimum point is \(\lambda^* = 0\) if and only if \(h'(0) \geq 0\). For matrix-valued functions, we have [4]
\[
\frac{\partial M^{-1}(\kappa)}{\partial \kappa} = -M^{-1}(\kappa) \frac{\partial M(\kappa)}{\partial \kappa} M^{-1}(\kappa).
\]  
(59)

Hence,
\[
h'(\lambda) = -\hat{a}_1^H \tilde{R}_z^{-1} C_\lambda \left( \nu^2 I_N - \hat{a}_1 \hat{a}_1^H \right) C_\lambda \tilde{R}_z^{-1} \hat{a}_1.
\]  
(60)

Consequently, the optimum point of (57) is \(\lambda^* = 0\) if and only if
\[
\left( \hat{a}_1^H \tilde{R}_z \hat{a}_1 \right)^2 - \nu^2 \left| \tilde{R}_z \hat{a}_1 \right|^2 \geq 0. \tag{61}
\]

Based on Proposition 2, it can be shown that \(h(\lambda)\) is unbounded above for \(\lambda \nrightarrow \lambda_{\text{max}}\), where \(\nrightarrow\) means convergence from below. Therefore, if (61) is not satisfied, then the optimum point of (57) can be determined by searching for the root of \(h'(\lambda)\) using, e.g.,
the secant method \cite{31}. This method requires the evaluation of $h(\lambda)$ at a large number of points. We show in Appendix C that $h(\lambda)$ and $h'(\lambda)$ can be computed efficiently based on the eigen-decomposition of $\hat{R}_x$. In particular, if $\Gamma$ and $z$ are available, then the evaluation of $h(\lambda)$ and $h'(\lambda)$ requires only $O(N)$ operations.

In summary, the MRCB can be obtained as follows:

1: Compute the eigendecomposition of $\hat{R}_x$ and $z = U^H \hat{a}_1$.
2: Search for $\lambda_{\text{max}}$, which satisfies (51) and (53)–(55).
3: Search for the optimum point $\lambda^*$ of (57).
4: Compute the beamformer weight vector

$$ w_{\text{MRCB}} = \frac{\hat{R}_x^{-1} a_{1,\lambda^*}}{a_{1,\lambda^*}^H \hat{R}_x^{-1} a_{1,\lambda^*}} $$

(62)

where $a_{1,\lambda^*}$ is given by (42).

**Remark 1:** The steps 2 and 3 of the proposed algorithm have comparable computational complexities if they are implemented using the secant method, because $f(\lambda)$ and $h'(\lambda)$ can both be evaluated using $O(N)$ operations, assuming that $\Gamma$ and $z$ are known. Moreover, step 4 can be implemented efficiently based on the eigendecomposition of $\hat{R}_x$. Our simulation results suggest that the computational complexity of the proposed algorithm is asymptotically (with $N \to \infty$) dominated by the eigendecomposition in step 1. Hence, it can be expected that the computational complexity of the proposed algorithm is $O(N^3)$.

**Remark 2:** Substituting (42) in (62) yields

$$ w_{\text{MRCB}} = \zeta_{\lambda^*} \left( \hat{R}_x^2 + \frac{1}{\lambda^*} \frac{1}{\nu^2} \hat{R}_x \hat{a}_1 \hat{a}_1^H \hat{R}_x \right)^{-1} \hat{a}_1 $$

(63)

where the scalar $\zeta_{\lambda^*}$ has no effect on the SINR performance. Consequently, the proposed beamformer is not equivalent to a regularization of $\hat{R}_x$. This is in contrast to the NBCB and MOPCB, see \cite{13} and \cite{14}, respectively.

**Remark 3:** If (61) holds, then the optimum steering vector of (33) follows from (42) as

$$ a_{1,0} = \frac{\hat{R}_x \hat{a}_1}{\|\hat{a}_1\|^2} $$

(64)

and the corresponding beamformer weight vector is

$$ w_{\text{MRCB},0} = \frac{\|\hat{a}_1\|^2}{\hat{a}_1^H \hat{R}_x \hat{a}_1} \hat{a}_1. $$

(65)

Hence, if (61) holds, then the MRCB provides the same SINR performance as the MRB. Using (31), it can be shown that (61) is satisfied if

$$ \epsilon \geq \sqrt{\frac{\|a_{1,0}\|^2}{\|\hat{R}_x \hat{a}_1\|^2}}. $$

(66)

Consequently, the MRCB is equivalent to the MRB if the uncertainty set for the signal steering vector is sufficiently large, so that (66) holds.

Replacing $\hat{R}_x$ in (61) by $R_x$ gives

$$ \left( \hat{a}_1^H R_x \hat{a}_1 \right)^2 - \nu^2 \| R_x \hat{a}_1 \|^2 \geq 0. $$

(67)

Using (4), the latter inequality can be written as

$$ \beta_1 P_1^2 + \beta_2 P_1 + \beta_3 > 0 $$

(68)

where

$$ \beta_1 = \left| \hat{a}_1^H a_1 \right|^2 \left( \left| \hat{a}_1^H a_1 \right|^2 - \nu^2 \| a_1 \|^2 \right) $$

(69)

$$ \beta_2 = 2 \Im \left\{ \left( \hat{a}_1^H a_1 \right)^2 \hat{a}_1^H - \nu^2 \hat{a}_1^H a_1 a_1^H R_{i+n} \hat{a}_1 \right\} $$

(70)

$$ \beta_3 = \left( \hat{a}_1^H R_{i+n} \hat{a}_1 \right)^2 - \nu^2 \hat{a}_1^H R_{i+n} \hat{a}_1. $$

(71)

If the actual signal steering vector lies in the interior of $\mathcal{S}_a$, then (see (29))

$$ \left| \hat{a}_1^H a_1 \right|^2 - \nu^2 \| a_1 \|^2 > 0 $$

(72)

so $\beta_1 > 0$. The quadratic term in (68) dominates over the linear and constant terms if $P_1$ is large. Consequently, if the true signal steering vector lies in the interior of $\mathcal{S}_a$, and if the power of the desired signal is sufficiently high, then (68) is satisfied. In this case, the MRCB using the exact array covariance matrix provides the same SINR performance as the MRB. It can be expected that the MRCB shows a similar behavior if the sample covariance matrix is used. This will be evaluated numerically in the subsequent section.

**IV. SIMULATION RESULTS**

Let us assume three far-field sources that impinge on a uniform linear array (ULA) of $N = 10$ identical isotropic sensors. The displacement between adjacent sensors is half a wavelength. The directions-of-arrival (DOAs) of the two interferers are $\theta_2 = 15^\circ$ and $\theta_3 = 30^\circ$ relative to the broadside direction of the ULA. The presumed (estimated) DOA of the desired signal is $\hat{\theta}_1 = -10^\circ$. The actual steering vector of the desired signal is generated randomly as

$$ a_1 = \hat{a}_1 - \delta_1 $$

(73)

where $\hat{a}_1$ corresponds to $\hat{\theta}_1$, and $\delta_1$ is drawn in each Monte-Carlo run independently from a zero-mean circularly symmetric complex Gaussian distribution with covariance $\xi I_N$. The parameter $\xi$ is chosen such that the average norm of $\delta_1$ is one. The desired signal, interferer, and noise waveforms are generated as uncorrelated white standard complex Gaussian random processes. The noise waveforms are assumed to have the same power in all sensors. The input interference-to-noise ratios (INRs) of the two interferers are set to 20 dB. The array covariance matrix is estimated using $K = 30$ independent snapshot vectors.

We compare the performance of the MRCB with that of the SCB, the MRB, the NBCB, the MOPCB, and the eigenspace beamformer of \cite{8}, \cite{9}. For the NBCB, we set $\alpha_n = \nu \| \hat{a}_1 \|^2$, where $\| \hat{a}_1 \|^2 = \sqrt{N}$. For the MOPCB and MRCB, we use $\epsilon = 1.269$ such that $\| \hat{\delta}_1 \|_2 \leq \epsilon$ in 95% of the Monte-Carlo runs. Note that the MOPCB is equivalent to the set-based worst-case
beamformer of [14]. As a reference, we also plot the optimum output SINR (9).

First, we study the beamformer output SINR versus the input SNR (averaged over the sensors). Fig. 1 shows that the performance of the SCB deteriorates severely for high input SNRs due to the signal self-nulling effect. The other beamformers tested provide a significantly improved robustness against signal self-nulling as compared to the SCB. The MRB outperforms the conventional adaptive beamformers in the high SNR regime as it does not suffer from signal self-nulling. Fig. 1 demonstrates that the performance of the MRCB is similar to that of the MRB for high input SNRs. However, the MRCB outperforms the MRB for low input SNRs. Overall, it can be observed that the MRCB is an attractive alternative to the other beamformers tested for the whole range of input SNRs.

In the next simulation example, we study the beamformer performance versus the number of training snapshot vectors that are used to estimate the array covariance matrix. The input SNR is set to $-10$ dB. All other parameters are chosen as before. The performance of the MRB is not depicted as it is below the SINR range plotted. Fig. 2 shows that the MRCB outperforms the other beamformers tested for $K \leq 200$ snapshot vectors. For $K \geq 300$ snapshot vectors, the eigenspace beamformer provides a better performance as compared to the MRCB.

Next, we analyze the effect of the angular distance between the desired signal and the interferers. Thereto, we vary the presumed (estimated) DOA of the desired signal. The number of training snapshot vectors is $K = 30$. All other parameters are chosen as before. Fig. 3 shows that the performance of all beamformers deteriorates if the distance between $\hat{\theta}_1$ and $\theta_2 = 15^\circ$ (the DOA of one of the interferers) decreases. The MRCB outperforms the other beamformers tested for the entire range of presumed signal DOAs considered.

In Fig. 1, the MRCB provides a similar performance as the MRB for high input SNRs, but for low input SNRs, the MRCB significantly outperforms the MRB. In Section III, we have shown that the MRCB provides the same SINR performance as the MRB if $\epsilon$ is sufficiently large so that (66) holds. Next, we analyze the effect of $\epsilon$ on the performance of the MRCB and MOPCB. Thereto, we change the presumed upper bound on the norm of the signal steering vector estimation errors. At the same time, we do not change the statistics of the actual signal steering vector estimation errors. Hence, we still draw the signal steering vector estimation errors from a Gaussian distribution such that the average norm of $\tilde{\delta}_1$ is one. The presumed DOA of the desired signal is set to $\hat{\theta}_1 = -10^\circ$. Figs. 4, 5, and 6 depict the beamformer performance for input SNRs of $-10$, $10$, and $30$ dB, respectively. All other parameters are chosen as before.

In Fig. 4, the MRCB outperforms the other beamformers tested if $0.15 \leq \epsilon \leq 2.75$. Clearly, if $\epsilon \to 0$, then the optimum steering vectors of (24) and (25) converge to $\hat{\alpha}_1$. Therefore, the performance of the MRB and MOPCB converges to the performance of the SCB if $\epsilon \to 0$. The performance of the MRB is not depicted in Fig. 4 as it is below the SINR range plotted. Hence, the performance of the MRCB degrades to that of the MRB in the narrow interval $3 \leq \epsilon < \|\hat{\alpha}_1\|_2$, where $\|\hat{\alpha}_1\|_2 = \sqrt{N}$. In fact, (66) was satisfied in none of the Monte-Carlo runs of Fig. 4.
Fig. 4. Beamformer performance versus the presumed upper bound on the norm of the signal steering vector estimation errors (for $\text{SNR} = -10$ dB, $K = 30$ snapshot vectors, and $\theta_1 = -10^\circ$).

Fig. 5. Beamformer performance versus the presumed upper bound on the norm of the signal steering vector estimation errors (for $\text{SNR} = 10$ dB, $K = 30$ snapshot vectors, and $\theta_1 = -10^\circ$).

Fig. 6. Beamformer performance versus the presumed upper bound on the norm of the signal steering vector estimation errors (for $\text{SNR} = 30$ dB, $K = 30$ snapshot vectors, and $\theta_1 = -10^\circ$).

Fig. 7. Beamformer performance versus the presumed upper bound on the norm of the signal steering vector estimation errors (for $\text{SNR} = 30$ dB, $\theta_1 = -10^\circ$, and the exact array covariance matrix).

Fig. 5 shows that, for an input SNR of 10 dB, the MRCB achieves its maximum output SINR at $\epsilon \approx 1.4$. If $\epsilon$ increases beyond this value, then the performance of the MRCB degrades, and it eventually converges to the performance of the MRB for large values of $\epsilon$.

For an input SNR of 30 dB, Fig. 6 shows that the performance of the MRCB is close to that of the MRB if $\epsilon \geq 1.4$. Furthermore, the MOPCB provides a similar performance as the MRB for large values of $\epsilon$. If the true signal steering vector lies in the interior of $S_{t_1}$, then the MOPCB has an incentive for a low sensitivity (see Section II-C). This incentive becomes stronger if the power of the desired signal or $\epsilon$ increases. Hence, it comes at no surprise that the MOPCB and MRB provide a similar performance if $\lambda_I$ and $\epsilon$ are large.

In Figs. 4–6, the MOPCB achieves an excellent performance for large values of $\epsilon$. Since this may be partially due to the estimation errors in the sample covariance matrix, we show in Fig. 7 the beamformer performance using the exact array covariance matrix. All other parameters are chosen as before. Comparing Figs. 6 and 7 shows that the beamformers tested provide a similar performance with and without finite sample errors. In all Monte-Carlo runs of Fig. 7, the norms of the randomly generated signal steering vector estimation errors were less than 1.8. Therefore, Fig. 7 also shows that, even if there are no finite sample errors, the MOPCB achieves its best SINR performance for values of $\epsilon$ that are significantly larger than the actual norms of the signal steering vector estimation errors.

The latter simulation results may suggest that a large value of $\epsilon$ should be chosen for the MOPCB irrespectively of the presumed size of the signal steering vector estimation errors. Fig. 8 depicts the beamformer performance versus the presumed DOA of the desired signal using $\epsilon = 2.5$. All other parameters are chosen as for Fig. 3. In particular, the input SNR is set to $-10$ dB. Comparing Figs. 3 and 8 shows that the MOPCB provides
a significantly better performance when \( \epsilon = 2.5 \) if the desired signal is far from the interferers. At the same time, it can be observed that the larger value of \( \epsilon \) results in a performance degradation of the MOPCB and MRCB if the desired signal is close to the interferer (compare the performance for \( \hat{\theta}_1 = 10^\circ \) in Figs. 3 and 8). Hence, increasing the value of \( \epsilon \) results in a degradation of the spatial resolution capability. Figs. 4–7 show that the MRCB achieves its best SINR performance for significantly smaller values of \( \epsilon \) than the MOPCB, and these smaller values of \( \epsilon \) result in a better spatial resolution capability.

Finally, Fig. 9 depicts beampatterns of the MRCB and MOPCB. The beampatterns have been computed using the exact array covariance matrix. The actual and presumed DOAs of the desired signal are \( \theta_1 = -12^\circ \) and \( \hat{\theta}_1 = -10^\circ \), respectively. The presumed upper bound on the norm of the signal steering vector estimation errors has been set to \( \epsilon = 1 \), and the input SNR is 10 dB. All other parameters are chosen as before. The beamformer weight vectors have been scaled such that \( \mathbf{w}^H \hat{\mathbf{a}}_1 = 1 \). The vertical dashed lines in Fig. 9 show the directions of the desired signal and the two interferers. Fig. 9 demonstrates that the MRCB leads to a wider main beam and lower sidelobes as compared to the MOPCB. The wider main beam results in an improved robustness against look direction errors. Due to its lower sidelobes, it can be expected that the MRCB has a better noise suppression than the MOPCB. However, both beamformers sufficiently suppress the interferers so that the output interference powers are significantly lower than the output noise powers.

If the beamformer weight vectors are scaled such that \( \mathbf{w}^H \hat{\mathbf{a}}_1 = 1 \), then the beamformer sensitivity (14) is proportional to the beamformer output power for spatially white noise. Hence, if \( \mathbf{w}^H \hat{\mathbf{a}}_1 = 1 \), then the output power of the MRCB for spatially white input noise is always less than or equal to that of the MOPCB with the same uncertainty set for the signal steering vector. Further simulation results have shown that the magnitude of the array response towards the desired signal is usually higher for the MRCB as compared to the MOPCB. In other words, the MRCB usually provides a higher output power for the desired signal as compared to the MOPCB. However, it depends on the scenario whether the MRCB or the MOPCB provides a better interference suppression. Consequently, the main advantage of the MRCB as compared to the MOPCB is its improved robustness against model errors, which results in an improved protection of the desired signal component and a better noise suppression.

V. CONCLUSION

We proposed the Capon beamformer with the minimum sensitivity to model errors, considering an uncertainty set for the signal steering vector. An efficient technique to compute the MRCB has been developed. Simulation results suggest that the asymptotic growth rate of the computational complexity of this technique is dominated by the eigendecomposition of the sample covariance matrix.

If the uncertainty set for the signal steering vector is sufficiently large, then the MRCB provides the same SINR performance as the MRB. Moreover, considering the exact array covariance matrix, we showed that the MRCB provides the same SINR performance as the MRB if the actual signal steering vector lies within the interior of the presumed uncertainty set, and if the power of the desired signal is sufficiently high.

Simulation results demonstrate that the MRCB is an attractive alternative to state-of-the-art robust adaptive beamformers. In particular, the MRCB typically provides an improved protection of the desired signal component and a better noise suppression as compared to the MOPCB with the same uncertainty set for the signal steering vector.

APPENDIX A

PROOF OF PROPOSITION 1

To show that (32) and (33) have the same optimum point, we show that, if \( \hat{\mathbf{a}}_1 \) denotes the optimum point of (33), then

\[
\Im \left\{ \mathbf{a}_1^H \hat{\mathbf{R}}_w^{-1} \hat{\mathbf{a}}_1 \right\} = 0 \tag{74}
\]

where \( \Im \{ \} \) denotes the imaginary part operator. For the proof, we assume that (74) is not satisfied and show that this leads to a contradiction. Hence, we assume that \( \mathbf{a}_1^H \hat{\mathbf{R}}_w^{-1} \hat{\mathbf{a}}_1 = 1 + i\xi \),
where \( \xi \) denotes the non-zero imaginary part of \( \hat{a}_1^H \hat{R}_{\xi}^{-1} \hat{a}_1 \), and 
\( i = \sqrt{-1} \). Then, it can be easily verified that \( \hat{a}_1 / (1 - i \xi) \) is a feasible point of (33), which leads to a lower value of the objective function than \( \hat{a}_1 \). However, this contradicts the optimality of \( \hat{a}_1 \). Consequently, (32) and (33) are equivalent.

To show that strong duality holds for (33), let us consider its relaxation

\[
\min_{\tilde{a}_1} \tilde{a}_1^H \hat{R}_z \tilde{a}_1 \\
\text{s.t.} \quad \Re \left\{ \tilde{a}_1^H \hat{R}_z \tilde{a}_1 \right\} \geq 1 \\
\| \nu \|_\infty \leq |\tilde{a}_1^H \hat{a}_1|.
\]  
(75)

Any vector \( \tilde{a}_1 \), which satisfies the first inequality constraint of (75) with strict inequality, cannot be optimum since a lower value of the objective function can be achieved by scaling this vector. Hence, the optimum point of (75) has to satisfy the first inequality constraint of this problem with equality. Therefore, (33) and (75) have the same solution. It has been shown in [32] that strong duality generally holds for optimization problems over the complex plane with a quadratic objective function and up to two quadratic inequality constraints. Thus, strong duality holds for (75). The Lagrange multiplier corresponding to the first constraint of this problem is restricted to be nonnegative. This restriction does not exist for the Lagrange multiplier corresponding to the equality constraint of (33). For this reason, the solution of the dual problem of (33) is larger than or equal to the solution of the dual problem of (75). Therefore, strong duality holds also for (33).

**APPENDIX B**

**PROOF OF PROPOSITION 2**

Equation (46) and the strict monotony of \( f(\lambda) \) imply that \( C_{\lambda} > 0 \) for \( 0 \leq \lambda < \lambda_{\text{max}} \). Consequently, (56) is satisfied for \( 0 \leq \lambda < \lambda_{\text{max}} \).

For \( \lambda = \lambda_{\text{max}} \), the bracketed matrix in (46) is positive semidefinite with a single eigenvalue equal to zero. The eigenvector corresponding to the zero eigenvalue is \( G_{\lambda_{\text{max}}}^{-1/2} \hat{a}_1 \). Hence, also \( C_{\lambda_{\text{max}}}^{-1} \hat{a}_1 \) has exactly one eigenvalue, which is equal to zero, and the corresponding eigenvector is \( G_{\lambda_{\text{max}}}^{-1/2} \hat{a}_1 \). The constraint (56) is satisfied for \( \lambda = \lambda_{\text{max}} \) if and only if \( \hat{R}_z \) is orthogonal to this eigenvector, i.e.,

\[
\hat{a}_1^H G_{\lambda_{\text{max}}}^{-1} \hat{R}_z \hat{a}_1 = 0.
\]  
(76)

However,

\[
G_{\lambda_{\text{max}}}^{-1} \hat{R}_z^{-1} = (\hat{R}_z^{-1} + \lambda_{\text{max}} \nu^2 \hat{R}_z)^{-1}
\]  
(77)

is positive definite. Thus, (56) cannot be satisfied for \( \lambda = \lambda_{\text{max}} \).

**APPENDIX C**

**IMPLEMENTATION OF (57)**

The objective function of (57) and its derivative can be expressed as

\[
h(\lambda) = \hat{a}_1^H \hat{R}_z^{-1} \hat{b}_\lambda
\]  
(78)

\[
h'(\lambda) = -\hat{a}_1^H \hat{b}_\lambda - \nu^2 | \hat{b}_\lambda |^2_2
\]  
(79)

where

\[
\hat{b}_\lambda = C_{\lambda}^{-1} \hat{R}_z^{-1} \hat{a}_1.
\]  
(80)

Using the matrix inversion lemma, we obtain

\[
\hat{b}_\lambda = \left( G_{\lambda}^{-1} + \frac{\lambda G_{\lambda}^{-1} \hat{a}_1 \hat{a}_1^H G_{\lambda}^{-1}}{1 - \lambda \hat{a}_1^H G_{\lambda}^{-1} \hat{a}_1} \right) \hat{R}_z^{-1} \hat{a}_1.
\]  
(81)

The eigendecomposition of \( \hat{R}_z \) allows to express \( \hat{b}_\lambda \) as

\[
\hat{b}_\lambda = U \left( \Psi_{\lambda} \Gamma_\lambda^{-1} + \frac{\lambda \nu^2 \hat{R}_z \hat{a}_1^H \hat{a}_1}{1 - \lambda \nu^2} \Psi_{\lambda} \right) \zeta
\]  
(82)

where

\[
\Psi_{\lambda} \triangleq (\Gamma_{\lambda}^{-2} + \lambda \nu^2 I_N)^{-1}
\]  
(83)

is a diagonal matrix. Substituting (82) in (78) and using the eigendecomposition of \( \hat{R}_z \) gives

\[
h(\lambda) = \zeta^H \Psi_{\lambda} \Gamma_{\lambda}^{-2} \zeta + \frac{\lambda \nu^2 \hat{R}_z \hat{a}_1^H \hat{a}_1 \zeta^H \Psi_{\lambda} \Gamma_{\lambda}^{-1} \zeta^2}{1 - \lambda \nu^2}
\]  
(84)

It can be easily verified that, if \( \Gamma \) and \( \zeta \) are available, then (84) can be evaluated using \( O(N) \) operations. Moreover, since

\[
\hat{a}_1^H \lambda = \zeta^H \Psi_{\lambda} \Gamma_{\lambda}^{-1} \zeta
\]  
(85)

and

\[
\| \hat{b}_\lambda \|_2^2 = \left\| \left( \Psi_{\lambda} \Gamma_\lambda^{-1} + \frac{\lambda \nu^2 \hat{R}_z \hat{a}_1^H \hat{a}_1 \Psi_{\lambda}}{1 - \lambda \nu^2} \Psi_{\lambda} \right) \zeta \right\|_2^2
\]  
(86)

\( h'(\lambda) \) can be evaluated with the same complexity.

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