

A BOUND ON THE SOLUTION TO A STRUCTURED SYLVESTER EQUATION WITH AN APPLICATION TO RELATIVE PERTURBATION THEORY*

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Abstract. Assuming only that the spectra of A and B are disjoint as opposed to the more restrictive assumption previously used, we obtain a bound in all unitarily invariant norms on the solution to the *structured* Sylvester equation $AX - XB = A^{1/2}EB^{1/2}$. This bound is the first of its kind in all unitarily invariant norms under only the disjointness assumption. An application of the bound to the relative perturbation theory for scaled Hermitian eigenvalue problems is given.

Key words. structured Sylvester equation, unitarily invariant norm, relative perturbation theory, eigenspace

AMS subject classifications. 15A18, 15A42, 65F15, 65F35, 65G99

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The operator (matrix) equation $AX - XB = S$, known as the Sylvester equation, appears frequently in the studies of operator theory, stability, and numerical linear algebra; see Bhatia and Rosenthal [6] and references therein for a history of it and its applications. The equation plays a unique role in eigenspace variations as illustrated by Davis and Kahan [7], where it was shown that the singular values of an associated Sylvester equation are the sines of angles between an eigenspace and its perturbation, and consequently, theorems, now known as the Davis–Kahan $\sin \theta$ and $\sin 2\theta$ theorems, were obtained.

Let A and B be Hermitian, and $\lambda(\cdot)$ denote the spectrum of a matrix. It can be proved that for the Frobenius norm $\|X\|_F \stackrel{\text{def}}{=} \sqrt{\sum_{i,j} |(X)_{ij}|^2}$,

$$(1) \quad \|X\|_F \leq \|S\|_F / \delta, \quad \delta \stackrel{\text{def}}{=} \min_{\omega \in \lambda(A), \gamma \in \lambda(B)} |\omega - \gamma|.$$

This holds under the most mild assumption, i.e., $\lambda(A) \cap \lambda(B) = \emptyset$ as in Figure 1. But to derive bounds for other norms, Davis and Kahan [7] imposed a restriction on how $\lambda(A)$ and $\lambda(B)$ are relatively distributed; see Figure 2. With that restriction Davis and Kahan proved that

$$(2) \quad \|X\| \leq \|S\| / \delta$$

for all unitarily invariant norms $\|\cdot\|$ [3, 14], including the spectral (operator) norm $\|\cdot\|_2$. It remained open what kinds of bounds we may have for $\|X\|$ under the situation of Figure 1 until Bhatia, Davis, and McIntosh [4] extended (2) to cover the situation of Figure 1 and proved¹

$$(3) \quad \|X\| \leq (\pi/2) \cdot \|S\| / \delta.$$

(In [4], the constant $\pi/2$ was not given explicitly; see [6] for details.)

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¹Bhatia, Davis, and McIntosh proved a more general result for the case when A and B are normal.

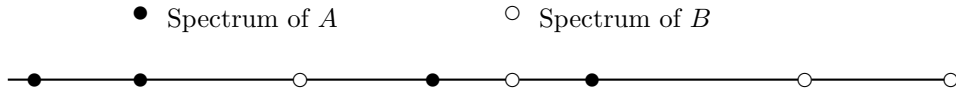


FIG. 1. The spectrum of A and that of B are disjoint.

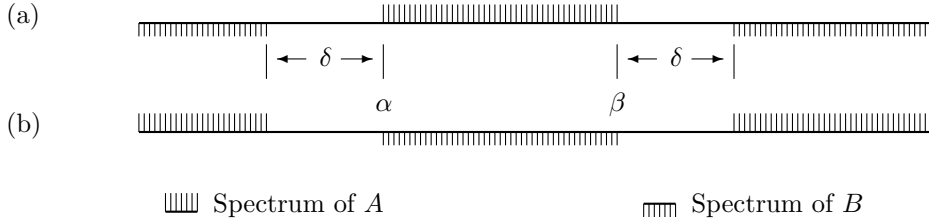


FIG. 2. The spectrum of A and that of B are separated by two intervals.

Interestingly, Li [13], in creating a relative perturbation theory for eigenvectors (see also [10]), showed that the sines of angles between an eigenspace and its perturbation are the singular values of an associated *structured* Sylvester equation $AX - XB = S$ in the sense that S takes one of the following forms:

$$(4) \quad AE, FB, AE + FB, \text{ or } A^{1/2}EB^{1/2},$$

where E and F are some error matrices. Bounds on the norms of X that are inversely proportional to relative gaps between $\lambda(A)$ and $\lambda(B)$ measured by one of 2 ($1 \leq p \leq \infty$)

$$(5) \quad \frac{|\omega - \gamma|}{|\omega|}, \frac{|\omega - \gamma|}{|\gamma|}, \frac{|\omega - \gamma|}{\sqrt[2]{|\omega|^p + |\gamma|^p}}, \text{ or } \frac{|\omega - \gamma|}{\sqrt{|\omega| \cdot |\gamma|}},$$

as opposed to the absolute gap δ measured by $|\omega - \gamma|$, are therefore obtained. (Here $A^{1/2}$ can be taken to be any one of A 's many square roots. Usually for a positive semidefinite A , it is defined to be the unique square root that is also positive semidefinite.) This was done for the Frobenius norm under the situation of Figure 1, but for all other unitarily invariant norms, a restriction similar to Figure 2 was assumed; see Figure 3. Notice the difference between Figure 2 and Figure 3: in Figure 3, either the spectrum of A or that of B has to scatter around the origin. This is for a good reason: the absolute difference is shift invariant while the relative differences are shift varying; see Li [13] for more discussions. For a sample bound, let $S = A^{1/2}EB^{1/2}$, then under the condition of Figure 3,

$$(6) \quad \|X\| \leq \|E\| / \eta, \quad \eta \stackrel{\text{def}}{=} \min_{\omega \in \lambda(A), \gamma \in \lambda(B)} \frac{|\omega - \gamma|}{\sqrt{|\omega| \cdot |\gamma|}} \geq \delta / \sqrt{\alpha(\alpha + \delta)}.$$

A natural question arises: under the condition of Figure 1, can we have, e.g., for $AX - XB = A^{1/2}EB^{1/2}$, something like

$$(7) \quad \|X\| \leq c \|E\| / \eta?$$

²It was proved that $\frac{|\omega - \gamma|}{\sqrt[2]{|\omega|^p + |\gamma|^p}}$ is a metric on the set of real numbers in [12] which also conjectured that it might even be a metric on the set of complex numbers. The conjecture was confirmed by Day [8] for $p = \infty$ and recently by Barrlund [2] for all $1 \leq p \leq \infty$.

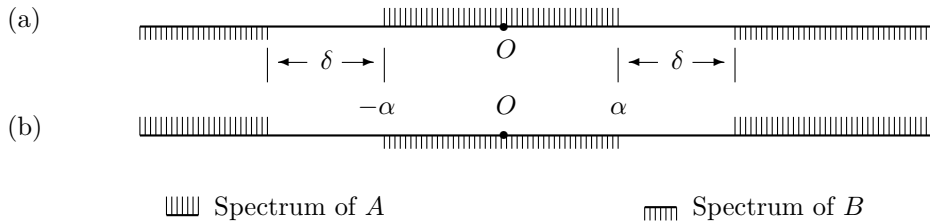


FIG. 3. The spectrum of A and that of B are separated by two intervals, and one of the spectra scatters around the origin.

Here $c \geq 1$ is a constant, and $\eta \stackrel{\text{def}}{=} \min_{\omega \in \lambda(A), \gamma \in \lambda(B)} \frac{|\omega - \gamma|}{\sqrt{|\omega| \cdot |\gamma|}}$. Similar questions with η defined accordingly for S being of any other forms in (4) can be asked.

While we are not able to give a complete answer to this question, in this note we shall prove something that is very much like it. However, we have no answers for S being of any other forms in (4).

THEOREM 1. *Assume that A and B are positive semidefinite and that $\lambda(A) \cap \lambda(B) = \emptyset$. Then $AX - XB = A^{1/2}EB^{1/2}$ has a unique solution X , and moreover for all unitarily invariant norm $\|\cdot\|$,*

$$(8) \quad \|X\| \leq (\pi/2) \cdot \|E\| / \tilde{\eta}, \quad \tilde{\eta} \stackrel{\text{def}}{=} \min_{\omega \in \lambda(A), \gamma \in \lambda(B)} |\ln(\omega/\gamma)|.$$

It has been proved that all relative measures in (5) are topologically equivalent [12], so they are equally good. The first two in (5) are the classical measures usually used in numerical computations, and the last two were introduced later [1, 9, 12, 13] as circumstances called for. What we see now in (8) is yet another relative measure $|\ln(\omega/\gamma)|$. To see this, let us assume ω and γ are close enough such that $|1 - \omega/\gamma| < 1$. (As a matter of fact, $|1 - \omega/\gamma| \leq 1/2$ if the most significant binary digit of ω and that of γ are the same; $|1 - \omega/\gamma| \leq 1/10$ if the most significant decimal digit of ω and that of γ are the same.) Then

$$\begin{aligned} |\ln(\omega/\gamma)| &= |\ln(1 - (1 - \omega/\gamma))| \\ &= |(1 - \omega/\gamma) + (1 - \omega/\gamma)^2/2 + \dots + (1 - \omega/\gamma)^k/k + \dots|, \end{aligned}$$

and thus the topological equivalence of $|\ln(\omega/\gamma)|$ to any of the measures in (5) is easily established.

Proof of Theorem 1. Without loss of generality, we may assume that both A and B are positive definite; we can always handle the singular case by the often used continuity argument. Our proof relies on the following inequality:

$$(9) \quad \|(\ln A)X - X(\ln B)\| \leq \left\| \left\| A^{1/2}XB^{-1/2} - A^{-1/2}XB^{1/2} \right\| \right\|,$$

due to Kosaki [11]; see [5] for another proof. It follows from the equation $AX - XB = A^{1/2}EB^{1/2}$ that $E = A^{1/2}XB^{-1/2} - A^{-1/2}XB^{1/2}$. Thus

$$\|E\| \geq \|(\ln A)X - X(\ln B)\|.$$

Now we apply (3), a result of Bhatia, Davis, and McIntosh [4], to complete the proof. \square

Now let us briefly discuss how Theorem 1 can be applied to the relative perturbation theory. From now on, we shall assume $A = S^*HS$ is an $n \times n$ matrix, perturbed to $\tilde{A} = S^*\tilde{H}S$, where S is a scaling matrix and usually diagonal. But this is not necessary to the theorem below. S^* is the complex conjugate transpose of S . The elements of S can vary wildly. H is nonsingular and usually better conditioned than A itself. Set $\Delta H \stackrel{\text{def}}{=} \tilde{H} - H$, and assume $\|H^{-1}\|_2\|\Delta H\|_2 < 1$. Let A and \tilde{A} admit eigendecompositions

(10)

$$A = (U_1, U_2) \begin{pmatrix} \Lambda_1 & \\ & \Lambda_2 \end{pmatrix} \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} \quad \text{and} \quad \tilde{A} = (\tilde{U}_1, \tilde{U}_2) \begin{pmatrix} \tilde{\Lambda}_1 & \\ & \tilde{\Lambda}_2 \end{pmatrix} \begin{pmatrix} \tilde{U}_1^* \\ \tilde{U}_2^* \end{pmatrix},$$

where $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}$, $\tilde{U} = \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix}$ are unitary, and

(11)
$$\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_k), \quad \Lambda_2 = \text{diag}(\lambda_{k+1}, \dots, \lambda_n),$$

(12)
$$\tilde{\Lambda}_1 = \text{diag}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_k), \quad \tilde{\Lambda}_2 = \text{diag}(\tilde{\lambda}_{k+1}, \dots, \tilde{\lambda}_n).$$

Notice that

$$\begin{aligned} A &= S^*HS = (H^{1/2}S)^* H^{1/2}S, \\ \tilde{A} &= S^*H^{1/2}(I + H^{-1/2}(\Delta H)H^{-1/2})H^{1/2}S \\ &= \left((I + H^{-1/2}(\Delta H)H^{-1/2})^{1/2}H^{1/2}S \right)^* (I + H^{-1/2}(\Delta H)H^{-1/2})^{1/2}H^{1/2}S. \end{aligned}$$

Set $B = S^*H^{1/2}$ and $\tilde{B} = S^*H^{1/2}(I + H^{-1/2}(\Delta H)H^{-1/2})^{1/2} \stackrel{\text{def}}{=} BD$, where $D = (I + H^{-1/2}(\Delta H)H^{-1/2})^{1/2}$. Then $A = BB^*$ and $\tilde{A} = \tilde{B}\tilde{B}^*$. Given the eigendecompositions of A and \tilde{A} as in (10)–(12), it can be seen that B and \tilde{B} admit the following singular value decompositions:

$$B = (U_1, U_2) \begin{pmatrix} \Lambda_1^{1/2} & \\ & \Lambda_2^{1/2} \end{pmatrix} \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} \quad \text{and} \quad \tilde{B} = (\tilde{U}_1, \tilde{U}_2) \begin{pmatrix} \tilde{\Lambda}_1^{1/2} & \\ & \tilde{\Lambda}_2^{1/2} \end{pmatrix} \begin{pmatrix} \tilde{V}_1^* \\ \tilde{V}_2^* \end{pmatrix},$$

where U_i, \tilde{U}_i are the same as in (10), $\begin{pmatrix} V_1 & V_2 \end{pmatrix}$ and $\begin{pmatrix} \tilde{V}_1 & \tilde{V}_2 \end{pmatrix}$ are unitary. We have

$$\tilde{A} - A = \tilde{B}\tilde{B}^* - BB^* = \tilde{B}D^*B^* - \tilde{B}D^{-1}B^* = \tilde{B}(D^* - D^{-1})B^*.$$

Pre- and post-multiply the equations by \tilde{U}_2^* and U_1 , respectively, to get

(13)
$$\tilde{\Lambda}_2\tilde{U}_2^*U_1 - \tilde{U}_2^*U_1\Lambda_1 = \tilde{\Lambda}_2^{1/2}\tilde{V}_2^*(D^* - D^{-1})V_1\Lambda_1^{1/2},$$

a *structured* Sylvester equation for $\tilde{U}_2^*U_1$. Notice that for any unitarily invariant norm

$$\begin{aligned}
\left\| \tilde{V}_2^*(D^* - D^{-1})V_1 \right\| &\leq \left\| D^* - D^{-1} \right\| \\
&= \left\| \left(I + H^{-1/2}(\Delta H)H^{-1/2} \right)^{1/2} - \left(I + H^{-1/2}(\Delta H)H^{-1/2} \right)^{-1/2} \right\| \\
&\leq \left\| \left(I + H^{-1/2}(\Delta H)H^{-1/2} \right)^{-1/2} \right\|_2 \left\| H^{-1/2}(\Delta H)H^{-1/2} \right\| \\
&\leq \frac{\|H^{-1}\|_2 \|\Delta H\|}{\sqrt{1 - \|H^{-1}\|_2 \|\Delta H\|_2}}.
\end{aligned}$$

It is known that $\|\sin \Theta(U_1, \tilde{U}_1)\| \equiv \|\tilde{U}_2^* U_1\|$ for all unitarily invariant norms [14]. The following theorem is now a consequence of (13) and Theorem 1.

THEOREM 2. *Let $A = S^*HS$ and $\tilde{A} = S^*\tilde{H}S$ be two $n \times n$ Hermitian matrices with eigendecompositions (10)–(12), where H is positive definite and $\|H^{-1}\|_2 \|\Delta H\|_2 < 1$. If*

$$\tilde{\eta} \stackrel{\text{def}}{=} \min_{\mu \in \lambda(\Lambda_1), \tilde{\mu} \in \lambda(\tilde{\Lambda}_2)} |\ln(\mu/\tilde{\mu})| > 0,$$

then for any unitarily invariant norm $\|\cdot\|$

$$(14) \quad \left\| \sin \Theta(U_1, \tilde{U}_1) \right\| \leq (\pi/2) \cdot \frac{\|D - D^{-1}\|}{\tilde{\eta}},$$

$$(15) \quad \leq (\pi/2) \cdot \frac{\|H^{-1}\|_2}{\sqrt{1 - \|H^{-1}\|_2 \|\Delta H\|_2}} \frac{\|\Delta H\|}{\tilde{\eta}},$$

where $D = (I + H^{-1/2}(\Delta H)H^{-1/2})^{1/2} = D^*$.

An analogous theorem for all unitarily invariant norms except those under the condition of Figure 3 is given in [13].

We have obtained a bound in all unitarily invariant norms on a *structured* Sylvester equation $AX - XB = A^{1/2}EB^{1/2}$. Though its proof is rather simple once we realize the two crucial results due to Kosaki [11] and to Bhatia, Davis, and McIntosh [4], respectively, it is significant in that for *structured* Sylvester equations it is the first of its kind in all unitarily invariant norms under the weakest assumption on A 's and B 's spectrum distributions. It remains to be seen whether similar results can be proved for other kinds of *structured* Sylvester equations.

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