Group decision making with intuitionistic fuzzy preference relations

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A B S T R A C T
The capability of intuitionistic fuzzy preference relation in representing imprecise or not reliable judgments which exhibit affirmation, negation and hesitation characteristics make it an attractive research area in group decision making. As traditional fuzzy set theory cannot be used to express all the information in a situation as such, its applications are limited. In Zadeh's fuzzy set, the membership degree of an element is defined by a real value, and nonmembership is expressed by a complement of membership. This membership definition actually ignores the decision maker's hesitation in the decision making process. The advantage of Atanassov's intuitionistic fuzzy sets is the capability of representing inevitably imprecise or not totally reliable judgments and the capability of expressing affirmation, negation and hesitation with the help of membership definitions. The consistency of intuitionistic fuzzy preference relations and the priority weights of experts gathered from these preference relations play an important role in group decision making problems in order to reach an accurate decision result. In this paper, we propose a group decision making process with the usage of intuitionistic fuzzy preference relations where we mainly focus our attention on the investigation of consistency of intuitionistic fuzzy preference relations. Initially, we present two different optimization models to minimize the deviations from additive and multiplicative consistency respectively. The optimal deviation values obtained from the model results enable us to improve the consistency of considered preference relations. Then, based on consistent collective preference relations, two mathematical programming models are established to obtain the priority weights, of which the first is a linear programming model considering additive and the second one is a nonlinear model considering multiplicative consistency. Furthermore, a number of numerical illustrations are presented to observe the validity and practicability of the models. Finally, comparative analyses were performed in order to examine the differences between fuzzy and intuitionistic fuzzy preference relations and the results of the analyses showed that the priority vectors and ranking of the alternatives maintained from fuzzy or intuitionistic fuzzy preference relations change significantly.

1. Introduction

Decision making can be considered as the mental processes in which we make a selection among several alternative choices. Making a decision implies that there are alternative choices to be considered, and in such a case we want to choose the one that has the highest probability of success or effectiveness and best fits with our goals, desires, lifestyle or values. In the decision making process, a decision maker (DM) is usually asked to give his/her preferences over alternatives. In this process, preference relations (referred to as pairwise comparison matrices, judgment matrices) help us to explain DM’s preference information in decision making problems of several fields. During the last decades, the concept of preference relations has received an increasing attention and several studies have been developed on this subject. In 2007, Xu presented a comprehensive survey of preference relations [54]. In decision making problems, the experts' preferences on decision alternatives are commonly described by multiplicative preference relations [32,33,62,50,53], fuzzy preference relations [29,37,23,7–12,21,22,15,5,24,28,46] or linguistic preference relations [18,19,50,51,13,40,38].

However, in most real life decision making problems, the DMs may not be able to provide his/her preferences for alternatives to a certain degree due to lack of precise or sufficient level of knowledge related to the problem, or the difficulty in explaining explicitly the degree to which one alternative is better than others. In these situations, there is usually a degree of uncertainty in providing their preferences over the alternatives considered, which makes the result of the preference process exhibit the characteristics of affirmation, negation and hesitation [60]. The voting example is an appropriate example of such a case, where "yes", "no" and
numerical examples are given to illustrate the validity and practicality of the proposed methods. Section 8 provides comparative analyses of fuzzy and intuitionistic fuzzy preference relations and Section 9 concludes this paper.

2. Literature review

In the literature, Atanassov’s intuitionistic fuzzy set theory has been studied by many researchers dealing with decision making concept \([3,25,27,26,61,55,56,42–44,16,17,34,39,6]\). Szmidt and Kacprzyk \([35,36]\) introduced the definition of the intuitionistic fuzzy preference relation (IFPR). In addition, they also studied the consensus reaching process, and analyzed the extent of agreement in a group of experts. Atanassov et al. \([3]\) proposed an algorithm for solving the multi-person multi-attribute decision making problems, in which the attribute weights are given as exact numerical values and the attribute values are expressed in intuitionistic fuzzy numbers. Li \([25]\) investigated multi-attribute decision making with intuitionistic fuzzy information and established several linear programming models to generate optimal weights for attribute. Lin et al. \([27]\) proposed a new method for handling multiple attribute fuzzy decision making problems, where the characteristics of the alternatives are represented by Atanassov’s intuitionistic fuzzy sets. Li et al. \([26]\) presented the fractional programming method for multiple attribute group decision making using Atanassov’s intuitionistic fuzzy sets. Xu \([56]\) investigated group decision making problems based on IFPR and incomplete IFPR. He used averaging operators to aggregate intuitionistic preference information and applied score and accuracy functions for the ranking and selection of alternatives.

Priority weight generation from the preference relations is the main issue of group decision making concept. Preference relation presents a common format which provides the opportunity to explain DM’s preference information in decision making problems by pairwise comparisons \([45]\). However, in the process of decision making it is very difficult for a DM to construct a consistent preference relation. Since an inconsistent preference relation may lead to wrong conclusions, priority weight generation methods should take into consideration the consistency of preference relations. Most of the priority weight generation methods in the fuzzy set theory papers in the literature are based on fuzzy interval preference relations (FIPR), introduced by Xu \([48]\). Xu and Chen \([61]\) proposed a number of linear programming models for deriving the priority weights from various fuzzy interval preference relations considering additive and multiplicative consistency. Genç et al. \([15]\) showed that the consistency and the priority weights can be derived by simple formulas based on interval multiplicative transitivity rather than linear programming models proposed by Xu and Chen \([61]\). Furthermore, these authors proposed two approaches in order to estimate missing values of an incomplete FIPR. Xu \([58]\) investigated the consistency of fuzzy interval preference relations. Initially he established a quadratic programming model to establish the importance weights of experts. He then proposed two approaches to constructing additive and multiplicative consistent fuzzy interval preference relations. Additionally, he showed the relationship between the consistency of individual FIPRs and the consistency of collective FIPR.

Gau and Buehrer \([14]\) introduced the concept of vague sets (interval valued fuzzy sets) and Bustince and Burillo \([4]\) showed that the notion of vague sets is actually that of intuitionistic fuzzy sets. This argument assists researchers to construct priority weight generation methods based on intuitionistic fuzzy preference relations. Xu \([55]\) defined the concept of additive consistent intuitionistic fuzzy preference relation (IFPR) and established a method for estimating criteria weights from intuitionistic fuzzy preference.
relations. In another study, Xu [57] defined the concepts of additive consistent intuitionistic judgment matrix, multiplicative consistent intuitionistic judgment matrix and score matrix of intuitionistic fuzzy decision matrix for the situations where the information on attribute weights is incomplete. He established some simple linear programming models using two transformation functions, from which the attribute weights can be derived. In his study, Wei [42] firstly proposed an optimization model based on the maximizing deviation method to determine the attribute weights which are partially known. He also proposed another optimization model for the special situations in which the information regarding attribute weights is completely unknown. In another study, Wei [43] established an optimization model based on gray relational analyses (GRA) method to determine the attribute weights which are partially known. In a more recent study, Wei [44] established an optimization model based on the basic ideal of traditional gray relational analysis (GRA) method, by which the attribute weights can be determined. For the special situations in which the information about attribute weights is completely unknown, he established another optimization model. Additionally, he extended the results to an interval-valued intuitionistic fuzzy environment, and developed modified GRA method for interval-valued intuitionistic fuzzy multiple attribute decision making with partially known attribute weight information. Gong et al. [16] presented goal programming models to derive the priority vector of the IFPR based on the multiplicative consistent definition of the FIPR. More recently, Gong et al. [17] proposed a least squares method and a goal programming method to determine the priority vector of IFPR based on additive consistency. Su et al. [34] developed an interactive method to solve the dynamic intuitionistic fuzzy multi-attributed group decision making problems. In addition, they measured the consensus level of the group preferences by using spearman correlation coefficient. Chen and Yang [6] established some optimization models to determine the attribute weights for the incomplete attribute weight information. In more recent studies, Wang and Li [41] extended the previous work of Wang et al. [39]. They proposed a framework to manage multi-attributed group decision making problems with incomplete pairwise comparison preference over decision alternatives where qualitative and quantitative attribute values are furnished as linguistic variables and crisp numbers, respectively. They then converted the attribute assessments to interval-valued intuitionistic fuzzy numbers (IVIFNs). In their study, group consistency and inconsistency indices are introduced for incomplete pairwise comparison preference relations on alternatives and an auxiliary linear programming model is developed to obtain unified attribute weights by minimizing the group inconsistency index under certain constraints. Zhang et al. [65] proposed a new type of preference relation which is intuitionistic fuzzy linguistic preference relation. They presented an approach to group decision making based on this new type of preference relation and applied the score and accuracy functions to the ranking and selection of alternatives. Xu [59] investigated the compatibility of intuitionistic preference relations. He proposed some novel compatibility measures of intuitionistic fuzzy information, and used them to derive a consensus reaching procedure in group decision making with intuitionistic preference relations. Paternain et al. [31] presented a construction method for Atanassov’s intuitionistic fuzzy preference relations based on fuzzy preference relations. They considered the ignorance of the expert in the construction of the intuitionistic fuzzy preference relations and they proposed two generalizations of the weighted voting strategy to work with Atanassov’s intuitionistic fuzzy preference relations.

When we investigate the current literature on the fuzzy group decision making concept, we realize that the studies are mostly conducted on fuzzy interval preference relations (FIPR). When, [61,15–17,39,41] since a fuzzy interval preference relation is very suitable for expressing the decision maker’s uncertain preference information. However, as mentioned above, Atanassov’s intuitionistic fuzzy set additionally gives the opportunity of representing inevitably imprecise or not totally reliable judgments and also expressing affirmation, negation and hesitation with the help of membership definitions. Therefore, in this paper we studied intuitionistic fuzzy preference relations (IFPRs) in the decision making concept. In group decision making problems, the decisions are made by a group of experts where the overall decision should be reached from the aggregated values of individual decisions. However, consistency of the preference relations is an important property in group decision making problems in order to draw the correct conclusions. The priority weights of consistent preference relations allow us to determine accurate ranking of the alternatives. A group decision making problem should consider both aggregation of expert decisions and consistency of preference relations.

In this paper, we aim to propose a group decision making model with intuitionistic fuzzy preference relations considering both aggregation of individual preference relations and consistency aspects. To achieve this, we aim to investigate the consequences of additive consistent and multiplicative consistent IFPRs on priority weights. For this reason, we develop a linear programming model considering additive consistency and a nonlinear model considering multiplicative consistency to determine the priority weights. These models also enable us to improve the consistency of considered preference relations whereby we can determine consistent individual preference relations before aggregation.

Furthermore, when we look at the current literature, we notice that the comparison of the usage of fuzzy and intuitionistic fuzzy preference relations in the group decision making problems has not been investigated. The analyses of the ranking of the alternatives in two cases (fuzzy and intuitionistic fuzzy ones) is an interesting study area. In this paper we make this comparison by using the proposed models for both preference relations. We analyze the differences of fuzzy and intuitionistic fuzzy preference relations, priority vectors and ranking of the alternatives obtained from these preference relations.

3. Preliminaries

The notion of intuitionistic fuzzy sets is introduced as [1]:

\[ A = \{ (x_i, \mu_A(x_i), \nu_A(x_i)) \mid x_i \in X \} \]

which is characterized by a membership function \( \mu_A : X \to [0, 1] \) and a nonmembership function \( \nu_A : X \to [0, 1] \) with the condition:

\[ 0 \leq \mu_A(x_i) + \nu_A(x_i) \leq 1 \quad \forall x_i \in X \]

The value, \( \pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i) \) is called the indeterminacy degree or hesitation degree of \( x_i \) to \( A \). If \( 0 \leq \pi_A(x_i) \leq 1 \). Especially, if \( \pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i) = 0 \), \( \forall x_i \in X \) then, the intuitionistic fuzzy set \( A \) is reduced to a common fuzzy set.

**Definition 2.1.** [56]: An intuitionistic fuzzy preference relation \( B \) on \( X \) is defined as a matrix \( B = (b_{ij})_{n \times n} \subset X \times X \) where \( b_{ij} = (\langle x_i, x_j \rangle, \mu(x_i, x_j), \nu(x_i, x_j)) \) for all \( i, j = 1, 2, \ldots, n \). Let \( b_{ij} = (\mu_{ij}, \nu_{ij}) \) is an intuitionistic fuzzy value, composed by the certainty degree \( \mu_{ij} \) to which \( x_i \) is preferred to \( x_j \) and the certainty degree \( \nu_{ij} \) to which \( x_i \) is non-preferred to \( x_j \) and \( \pi_{ij} = 1 - \mu_{ij} - \nu_{ij} \) is interpreted as the hesitation degree to which \( x_i \) is preferred to \( x_j \). Moreover, \( \mu_{ij} \) and \( \nu_{ij} \) satisfy the following condition,
where $\mu_j = (\mu_{ij}, \nu_{ij})$ is a multiplicative consistent preference relation, where $\mu_{ij} \leq \nu_{ij}$ for all $i, j = 1, 2, \ldots, n$. This condition is the same as the condition under which two real numbers form an interval [20]. As a result, an intuitionistic fuzzy preference relation can be transformed into an interval fuzzy preference relation [50].

### Definition 3.2.5. An intuitionistic fuzzy preference relation $B = (b_{ij})_{n \times n}$ is a multiplicative consistent fuzzy interval preference relation where $b_{ij} = (\mu_{ij}, \nu_{ij})$ if there is a vector $w = (w_1, w_2, \ldots, w_n)^T$ such that Eq. (9) holds.

5. Priority weight generation method for additive consistent intuitionistic fuzzy preference relation

Let $B = (b_{ij})_{n \times n}$ be an intuitionistic fuzzy preference relation where $b_{ij} = (\mu_{ij}, \nu_{ij})$. If $B = (b_{ij})_{n \times n}$ is an additive consistent intuitionistic fuzzy preference relation, then a priority vector $w = (w_1, w_2, \ldots, w_n)^T$ of $B$ exists which satisfies Eq. (7). Since the preferences of decision makers are very subjective and depend on personal psychological aspects, this equation does not always hold [58]. In this situation, $B = (b_{ij})_{n \times n}$ will not be an additive consistent intuitionistic fuzzy preference relation, then we relax Eq. (7) by introducing the non-negative deviation variables $d_{ij}^-$ and $d_{ij}^+$, $i = 1, 2, \ldots, n$; $j = i + 1, \ldots, n$.

As the deviation variables $d_{ij}^-$ and $d_{ij}^+$ become smaller, $B$ becomes closer to an additive consistent intuitionistic fuzzy preference relation. As a result, in order to find the smallest deviation variables Xu [55] developed the following linear optimization model:

\[ \delta = \text{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} (d_{ij}^- + d_{ij}^+) \]

\[ \text{s.t.} \ 0.5(w_i - w_j + 1) + d_{ij}^- \geq \mu_{ij} \]

\[ 0.5(w_i - w_j + 1) - d_{ij}^- \leq 1 - \nu_{ij} \]

\[ w_i \geq 0, \ i = 1, 2, \ldots, n; \ \sum_{i=1}^{n} w_i = 1 \]

\[ i = 1, 2, \ldots, n - 1; \ j = i + 1, \ldots, n \]

Solving this model (M-1), we determine the optimal deviation values, $d_{ij}^-$ and $d_{ij}^+$, $i, j = 1, 2, \ldots, n$. Especially, if $\delta = 0$, or in other words, $d_{ij}^- = d_{ij}^+ = 0$, for all $i, j = 1, 2, \ldots, n$, then $B$ is an additive consistent intuitionistic fuzzy preference relation. Otherwise, we may use the nonzero deviation values to improve the additive consistency as;
\[ B = (b_{ij})_{n \times n}, \quad b_{ij} = (\mu_{ij}, v_{ij}), \quad \mu_{ij} = \mu_j - d_{ij}, \quad v_{ij} = v_j - d_{ij}, \quad i,j = 1,2,\ldots,n \]  \hspace{1em} (11)

Here, B is the improved additive consistent intuitionistic fuzzy preference relation. Based on the improved preference relation, we establish the following optimization models to calculate the priority weight vector:

**Model M-2**

\[
\begin{align*}
& w_i = \min w_i \\
& \text{s.t.} \quad 0.5(w_i - w_j + 1) \geq \mu_{ij} \\
& \quad 0.5(w_i - w_j + 1) \leq 1 - v_{ij} \\
& \quad w_i \geq 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1, \\
& \quad i = 1,2,\ldots,n-1; \quad j = i+1,\ldots,n.
\end{align*}
\]

\[ \text{(M-2)} \]

**Model M-3**

\[
\begin{align*}
& w_i = \max w_i \\
& \text{s.t.} \quad 0.5(w_i - w_j + 1) \geq \mu_{ij} \\
& \quad 0.5(w_i - w_j + 1) \leq 1 - v_{ij} \\
& \quad w_i \geq 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1, \\
& \quad i = 1,2,\ldots,n-1; \quad j = i+1,\ldots,n.
\end{align*}
\]

\[ \text{(M-3)} \]

Solving the models (M-2) and (M-3), we determine the weight intervals \([w_i^L, w_i^U]\), if \(w_i^U = w_i^L\) for all \(i\), and we determine a unique priority vector \(w = (w_1, w_2, \ldots, w_n)^T\) from the improved additive consistent intuitionistic fuzzy preference relation.

In the models (M-1)–(M-3) we only consider an individual intuitionistic fuzzy preference relation \(B = (b_{ij})_{n \times n}\); however a decision is usually made by a group of experts, \(E_k (k = 1, 2, \ldots, m)\) with different weights \(w_k = (w_{k1}, w_{k2}, \ldots, w_{kn})^T\) in the decision process. In such cases, the individual preference relations of the experts are aggregated to provide a collective preference relation \(B = (\overline{b}_{ij})_{n \times n}\).

In this situation, the consistency of the collective preference relation, as well as individual relations, need to be checked.

If we aggregate individual preference relations of the experts, then Eq. (10) will be extended by using Eq. (4) as below for the additive consistency of collective intuitionistic fuzzy preference relation:

\[
\sum_{k=1}^{m} \lambda_k w_{ij}^{(k)} - d_{ij} \leq 0.5(w_i - w_j + 1) \\
\leq 1 - \sum_{k=1}^{m} \lambda_k v_{ij}^{(k)} + d_{ij} \quad \text{for all} \quad i = 1,2,\ldots,n-1; \quad j = i+1,\ldots,n; \\
\quad k = 1,2,\ldots,m; \quad w_{ij} \geq 0, \quad \sum_{i=1}^{n} w_i = 1
\]

\[ \text{(12)} \]

As the deviation variables \(d_{ij}\) and \(d_{ij}^*\) become smaller, \(B\) becomes closer to an additive consistent collective intuitionistic fuzzy preference relation. We establish the following model to derive the smallest deviation values:

\[
\delta = \min \left\{ \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left( d_{ij} + d_{ij}^* \right) \right\} \\
\text{s.t.} \quad 0.5(w_i - w_j + 1) + d_{ij} \geq \sum_{k=1}^{m} \lambda_k \mu_{ij}^{(k)} \\
\quad 0.5(w_i - w_j + 1) - d_{ij} \leq 1 - \sum_{k=1}^{m} \lambda_k v_{ij}^{(k)} \\
\quad w_i \geq 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1, \quad d_{ij}, \quad d_{ij}^* \geq 0 \\
\quad i = 1,2,\ldots,n-1; \quad j = i+1,\ldots,n; \quad k = 1,2,\ldots,m.
\]

\[ \text{(M-4)} \]

Solving the model (M-4), we obtain the optimal deviation values. In fact, if all the individual intuitionistic fuzzy preference relations are additive consistent, then the collective preference relation \(B = (b_{ij})_{n \times n}\) will also be additive consistent which means \(\overline{d} = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d_{ij} + d_{ij}^*) = 0\). Based on the improved preference relations, we develop the following optimization model to obtain the priority vector for the collective preference relation:

\[
\begin{align*}
& w_i = \min w_i \quad \text{and} \quad w_j = \max w_i \\
& \text{s.t.} \quad 0.5(w_i - w_j + 1) \geq \sum_{k=1}^{m} \lambda_k \mu_{ij}^{(k)} \\
& \quad 0.5(w_i - w_j + 1) \leq 1 - \sum_{k=1}^{m} \lambda_k v_{ij}^{(k)} \\
& \quad w_i \geq 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1, \\
& \quad i = 1,2,\ldots,n-1; \quad j = i+1,\ldots,n; \quad k = 1,2,\ldots,m.
\end{align*}
\]

\[ \text{(M-5)} \]

The solution of (M-5) determines the priority vector \(\overline{w} = (w_1, w_2, \ldots, w_n)^T\) for the additive consistent collective intuitionistic fuzzy preference relation.

6. Priority weight generation method for multiplicative consistent intuitionistic fuzzy preference relation

We consider \(B = (b_{ij})_{n \times n}\) is an intuitionistic fuzzy preference relation where \(b_{ij} = (\mu_{ij}, v_{ij})\). If \(B = (b_{ij})_{n \times n}\) is a multiplicative consistent intuitionistic fuzzy preference relation, then there is a priority vector \(w = (w_1, w_2, \ldots, w_n)^T\) of B which satisfies Eq. (9). However \(B = (b_{ij})_{n \times n}\) may not be a multiplicative fuzzy intuitionistic preference relation and Eq. (9) does not always hold. In such cases, we introduce non-negative deviation variables \(d_{ij}\) and \(d_{ij}^*\), \(i = 1,2,\ldots,n-1; \quad j = i+1,\ldots,n\) to relax the condition Eq. (9):

\[
\mu_{ij} - d_{ij} \leq \frac{w_i}{w_i + w_j} \leq 1 - v_{ij} + d_{ij}^*, \\
\text{for all} \quad i = 1,2,\ldots,n-1; \quad j = i+1,\ldots,n, \quad w_i \geq 0,
\]

\[ \text{(13)} \]

Considering the smaller the deviation variables \(d_{ij}\) and \(d_{ij}^*\) in Eq. (13), the closer B is to a multiplicative consistent fuzzy intuitionistic preference relation, we develop the nonlinear optimization model:

\[
\begin{align*}
& \gamma = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d_{ij} + d_{ij}^*) \\
& \text{s.t.} \quad \frac{w_i}{w_i + w_j} + d_{ij} \geq \mu_{ij} \\
& \quad \frac{w_i}{w_i + w_j} - d_{ij}^* \leq 1 - v_{ij} \\
& \quad w_i \geq 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1, \quad d_{ij}, \quad d_{ij}^* \geq 0 \\
& \quad i = 1,2,\ldots,n-1; \quad j = i+1,\ldots,n.
\end{align*}
\]

\[ \text{(M-6)} \]

Solving the model (M-6), we determine the optimal deviation values, \(d_{ij}\) and \(d_{ij}^*\), \(i,j = 1,2,\ldots,n\). In particular, if \(\gamma = 0\), i.e., \(d_{ij} = d_{ij}^* = 0\), for all \(i,j = 1,2,\ldots,n\), then B is a multiplicative consistent intuitionistic fuzzy preference relation and we can utilize the nonzero deviation values to improve the multiplicative consistency as
Here, $B$ is the improved multiplicative consistent intuitionistic fuzzy preference relation. In order to derive the priority vector $w = (w_1, w_2, \ldots, w_n)^T$, we establish the nonlinear programming models:

$$w_i = \min w_i \quad \text{s.t.} \quad \frac{w_i}{w_i + w_j} \geq \mu_{ij}$$

$$w_i > 0, \quad i = 1, 2, \ldots, n - 1; \quad j = i + 1, \ldots, n.$$  \hspace{1cm} (M-7)

$$w_i^* = \max w_i \quad \text{s.t.} \quad \frac{w_i}{w_i + w_j} \geq \mu_{ij}$$

$$w_i > 0, \quad i = 1, 2, \ldots, n - 1; \quad j = i + 1, \ldots, n.$$  \hspace{1cm} (M-8)

The solution to the models (M-7) and (M-8), give the weight intervals $[w_i, w_i^*]$, if $w_i^* = w_i^*$ for all $i$, then we determine a unique priority vector $w = (w_1, w_2, \ldots, w_n)^T$ from the improved multiplicative consistent intuitionistic fuzzy preference relation.

The models (M-6)-(M-8) only consider an individual intuitionistic fuzzy preference relation $B = (b_{ij})_{n \times n}$. However, for the decisions made by a group of experts, $E_k (k = 1, 2, \ldots, m)$ with different weights $\omega_k = (\omega_{k1}, \omega_{k2}, \ldots, \omega_{kn})^T$, the individual intuitionistic fuzzy preference relations of the experts are aggregated to obtain a collective preference relation $\overline{B} = (\overline{b}_{ij})_{n \times n}$. If we aggregate individual preference relations of the experts, then Eq. (13) will be extended by using Eq. (4) for the multiplicative consistency of collective intuitionistic fuzzy preference relation as below:

$$\sum_{i=1}^{m} \omega_k b_{ij}^k \leq \frac{w_i}{w_i + w_j} \leq 1 - \sum_{i=1}^{m} \omega_k b_{ij}^k$$

$$\text{for all } i = 1, 2, \ldots, n - 1; \quad j = i + 1, \ldots, n; \quad k = 1, 2, \ldots, m.$$  \hspace{1cm} (15)

Based on Eq. (14) we establish the following model to obtain the smallest deviation values:

$$\overline{y} = \min \sum_{i=1}^{m} \sum_{i=1}^{n} \left( b_{ij}^* + d_{ij}^* \right)$$

$$\text{s.t.} \quad \frac{w_i}{w_i + w_j} \geq \sum_{i=1}^{m} \omega_k b_{ij}^k$$

$$\frac{w_i}{w_i + w_j} \leq 1 - \sum_{i=1}^{m} \omega_k b_{ij}^k$$

$$w_i \geq 0, \quad i = 1, 2, \ldots, n; \quad \sum_{i=1}^{n} w_i = 1; \quad d_{ij}^* \geq 0$$

$$i = 1, 2, \ldots, n - 1; \quad j = i + 1, \ldots, n; \quad k = 1, 2, \ldots, m.$$  \hspace{1cm} (M-9)

Solving this model (M-4), we obtain the optimal deviation values. As in the additive consistency, if all the individual intuitionistic fuzzy preference relations are multiplicative consistent, then the collective preference relation $\overline{B} = (\overline{b}_{ij})_{n \times n}$ will also be multiplicative consistent which means $\overline{y} = \min \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \overline{b}_{ij}^* + d_{ij}^* \right) = 0$. Based on the improved preference relations, we establish the following nonlinear optimization model to obtain the priority vector for the multiplicative consistent collective intuitionistic fuzzy preference relation:

$$\overline{w}_i = \min w_i \quad \text{and} \quad \overline{w}_i^* = \max w_i \quad \text{s.t.} \quad \frac{w_i}{w_i + w_j} \geq \sum_{i=1}^{m} \omega_k b_{ij}^k$$

$$\frac{w_i}{w_i + w_j} \leq 1 - \sum_{i=1}^{m} \omega_k b_{ij}^k$$

$$w_i \geq 0, \quad i = 1, 2, \ldots, n; \quad \sum_{i=1}^{n} w_i = 1; \quad i = 1, 2, \ldots, n - 1; \quad j = i + 1, \ldots, n; \quad k = 1, 2, \ldots, m.$$  \hspace{1cm} (M-10)

Solution of the model (M-10) determines the priority vector $w = (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n)^T$ for the multiplicative consistent collective intuitionistic fuzzy preference relation.

7. Numerical Illustration

A group of decision makers compare alternative facade clothing systems for the surface clothing of a building according to their functional properties. The group involves three experts; $E_1 =$ civil engineer, $E_2 =$ constructor and $E_3 =$ architect, $E_4(k = 1, 2 \text{ and } 3)$ whose weight vector is $\omega = (0.5, 0.2, 0.3)^T$. The experts compare five alternative systems which are $x_1 =$ plastic painting, $x_2 =$ compact laminate clothing, $x_3 =$ wood clothing, $x_4 =$ ceramic clothing and $x_5 =$ natural stone clothing, $x_i (i = 1, 2, 3, 4, 5)$. All the experts individually compare each pair of criteria $x_i$ and $x_j$, and give his/her intuitionistic fuzzy preference value $b_{ij}^k = \left( \mu_{ij}^k, \nu_{ij}^k \right)$, composed by the certainty degree $\mu_{ij}^k$ to which $x_i$ is preferred to $x_j$ and the certainty degree $\nu_{ij}^k$ to which $x_i$ is non-preferred to $x_j$, and then develop the following intuitionistic fuzzy preference relations $B(k) = (b_{ij}^k)_{5 \times 5}:$

$$B^{(1)} = \begin{bmatrix}
(0.5, 0.5) & (0.3, 0.7) & (0.3, 0.6) & (0.5, 0.5) & (0.1, 0.9) \\
(0.7, 0.3) & (0.5, 0.5) & (0.4, 0.6) & (0.6, 0.2) & (0.2, 0.7) \\
(0.6, 0.3) & (0.6, 0.4) & (0.5, 0.5) & (0.7, 0.1) & (0.4, 0.6) \\
(0.5, 0.5) & (0.2, 0.6) & (0.1, 0.7) & (0.5, 0.5) & (0.1, 0.8) \\
(0.9, 0.1) & (0.7, 0.2) & (0.6, 0.4) & (0.8, 0.1) & (0.5, 0.5)
\end{bmatrix}$$

$$B^{(2)} = \begin{bmatrix}
(0.5, 0.5) & (0.6, 0.3) & (0.1, 0.8) & (0.5, 0.5) & (0.3, 0.7) \\
(0.3, 0.6) & (0.5, 0.5) & (0.0, 1.0) & (0.3, 0.7) & (0.1, 0.8) \\
(0.8, 0.1) & (1.0, 0.0) & (0.5, 0.5) & (0.6, 0.3) & (0.5, 0.3) \\
(0.5, 0.5) & (0.7, 0.3) & (0.3, 0.6) & (0.5, 0.5) & (0.3, 0.6) \\
(0.7, 0.3) & (0.8, 0.1) & (0.3, 0.5) & (0.6, 0.3) & (0.5, 0.5)
\end{bmatrix}$$

$$B^{(3)} = \begin{bmatrix}
(0.5, 0.5) & (0.0, 0.9) & (0.1, 0.7) & (0.3, 0.6) & (0.5, 0.5) \\
(0.9, 0.0) & (0.5, 0.5) & (0.6, 0.3) & (0.6, 0.4) & (0.8, 0.2) \\
(0.7, 0.1) & (0.3, 0.6) & (0.5, 0.5) & (0.5, 0.2) & (0.7, 0.2) \\
(0.6, 0.3) & (0.4, 0.6) & (0.2, 0.5) & (0.5, 0.5) & (0.4, 0.5) \\
(0.5, 0.5) & (0.2, 0.8) & (0.2, 0.7) & (0.5, 0.4) & (0.5, 0.5)
\end{bmatrix}$$

Based on $B(k) = (b_{ij}^k)_{5 \times 5}$ we first apply the model (M-1) to the individual intuitionistic fuzzy preference relations in order to check the additive consistency. Applying (M-1), we find the optimal objective
values \(\hat{d}^{(i)}\) as \(\hat{d}^{(1)} = 0.733, \hat{d}^{(2)} = 1.400, \hat{d}^{(3)} = 0.927\) and the optimal nonzero deviation values \(d^{(i)}_y\) and \(d^{(i)}_w\):

\[
\hat{d}^{(1)}_y = \hat{d}^{(2)}_y = 0.133, \quad \hat{d}^{(3)}_y = 0.133, \quad \hat{d}^{(1)}_w = \hat{d}^{(2)}_w = 0.033, \quad \hat{d}^{(3)}_w = \hat{d}^{(3)}_w = 0.033.
\]

\[
\hat{d}^{(1)}_w = \hat{d}^{(2)}_w = 0.100, \quad \hat{d}^{(3)}_w = \hat{d}^{(3)}_w = 0.050, \quad \hat{d}^{(1)}_y = \hat{d}^{(1)}_y = 0.050, \quad \hat{d}^{(2)}_w = \hat{d}^{(2)}_w = 0.250, \quad \hat{d}^{(3)}_y = \hat{d}^{(3)}_y = 0.150, \quad \hat{d}^{(3)}_w = \hat{d}^{(3)}_w = 0.100.
\]

\[
\hat{d}^{(1)}_y = \hat{d}^{(2)}_y = 0.100, \quad \hat{d}^{(3)}_y = \hat{d}^{(3)}_y = 0.100, \quad \hat{d}^{(2)}_w = \hat{d}^{(2)}_w = 0.063, \quad \hat{d}^{(3)}_w = \hat{d}^{(3)}_w = 0.200.
\]

Since, \(\hat{d}^{(i)} \neq 0\), for \(k = 1, 2, 3\) then none of the \(B^{(k)}\) is additive consistent. To improve the additive consistency of \(B^{(k)}\), we use optimal deviation values \(d^{(i)}_y\) and \(d^{(i)}_w\) and construct the improved additive consistent intuitionistic fuzzy preference relations \(\tilde{B}^{(k)}\) by applying Eq. (11):

\[
\tilde{B}^{(1)} = \begin{bmatrix}
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767) \\
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767) \\
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767)
\end{bmatrix}
\]

\[
\tilde{B}^{(2)} = \begin{bmatrix}
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767) \\
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767) \\
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767)
\end{bmatrix}
\]

\[
\tilde{B}^{(3)} = \begin{bmatrix}
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767) \\
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767) \\
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767)
\end{bmatrix}
\]

Since the optimal objective values of the improved preference relations will be equal to zero, \(\hat{d}^{(i)} = 0\) for \(k = 1, 2, 3\), we establish models (M-2) and (M-3) to derive the individual priority weight vector of each expert \(w^{(k)} = (w^{(k)}_1, w^{(k)}_2, w^{(k)}_3, w^{(k)}_4, w^{(k)}_5)\) where \(k = 1, 2, 3\).

The results of the models are summarized in Table 1.

The results shown in Table 1 indicate that \(x_2 =\) natural stone clothing is the best alternative for civil engineer \(x_3 =\) wood clothing is the best for constructor and \(x_2 =\) compact laminate clothing is the best for architect.

<table>
<thead>
<tr>
<th>Experts</th>
<th>Optimal weights</th>
<th>Ranking of alternatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_1)</td>
<td>(w_1^{(1)} = 0, w_2^{(1)} = 0.133, w_3^{(1)} = 0.333, w_4^{(1)} = 0.033)</td>
<td>(x_3 &gt; x_1 &gt; x_2 &gt; x_4 &gt; x_5)</td>
</tr>
<tr>
<td>(E_2)</td>
<td>(w_1^{(2)} = 0, w_2^{(2)} = 0, w_3^{(2)} = 0.5, w_4^{(2)} = 0, w_5^{(2)} = 0.4)</td>
<td>(x_5 &gt; x_3 &gt; x_4 &gt; x_1 &gt; x_2)</td>
</tr>
<tr>
<td>(E_3)</td>
<td>(w_1^{(3)} = 0, w_2^{(3)} = 0, w_3^{(3)} = 0.6, w_4^{(3)} = 0.4, w_5^{(3)} = 0)</td>
<td>(x_2 &gt; x_3 &gt; x_4 &gt; x_5 &gt; x_1)</td>
</tr>
</tbody>
</table>

*Where “>” means “is preferred to” and “<” means “is indifferent to”.

The models (M-1)–(M-3) only consider individual intuitionistic fuzzy preference relations, however, in our group decision making problem, the decision is made by three different experts with the weights \(\lambda = (0.5, 0.2, 0.3)^T\). In such cases, the individual preference relations of the experts should be aggregated to determine a collective preference relation \(\bar{B} = (\bar{b}_{ij})_{n \times n}\). In such a situation, the consistency of the collective preference relation needs to be checked. We aggregate the improved individual preference relations of the experts by using Eq. (4) and apply (M-4) to obtain the smallest deviation values for the collective preference relation. The result of the model (M-4) implies that the optimal objective value is equal to zero, \(\tilde{\pi} = \min \sum_{i=1}^{m} \lambda_i \cdot \tilde{d}_y + \tilde{d}_w = 0\) which means the collective preference relation is additive consistent. This is an expected result since we aggregated the improved individual preference relations to determine the collective preference relation.

Secondly, based on \(\tilde{B}^{(k)} = (\tilde{b}^{(k)})_{n \times n}\), we apply the model (M-6) to the individual intuitionistic fuzzy preference relations to check the multiplicative consistency. Applying (M-6), we find the optimal objective values \((\gamma^{(k)})\) as \(\gamma^{(1)} = 0.390, \gamma^{(2)} = 0.394, \gamma^{(3)} = 0.663\) and the optimal nonzero deviation values \(d^{(i)}_y\) and \(d^{(i)}_w\):

\[
\tilde{d}^{(1)}_y = \tilde{d}^{(2)}_y = 0.060, \quad \tilde{d}^{(3)}_y = \tilde{d}^{(3)}_y = 0.027, \quad \tilde{d}^{(1)}_w = \tilde{d}^{(2)}_w = 0.008.
\]

\[
\tilde{d}^{(2)}_y = \tilde{d}^{(3)}_y = \tilde{d}^{(3)}_y = 0.008, \quad \tilde{d}^{(2)}_w = \tilde{d}^{(2)}_w = 0.008.
\]

\[
\tilde{d}^{(1)}_y = \tilde{d}^{(2)}_y = 0.060, \quad \tilde{d}^{(3)}_y = \tilde{d}^{(3)}_y = 0.032, \quad \tilde{d}^{(3)}_w = \tilde{d}^{(3)}_w = 0.200.
\]

Since, \(\hat{d}^{(i)} \neq 0\), for \(k = 1, 2, 3\) then none of the \(B^{(k)}\) is multiplicative consistent. To improve the additive consistency of \(B^{(k)}\), we use optimal deviation values \(d^{(i)}_y\) and \(d^{(i)}_w\) and develop the improved multiplicative consistent intuitionistic fuzzy preference relations \(\tilde{B}^{(k)}\) by applying Eq. (14):

\[
\tilde{B}^{(1)} = \begin{bmatrix}
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767) \\
(0.640, 0.050, 0.306, 0.050, 0.100, 0.692) \\
(0.640, 0.050, 0.306, 0.050, 0.100, 0.692)
\end{bmatrix}
\]

\[
\tilde{B}^{(2)} = \begin{bmatrix}
(0.505, 0.050, 0.306, 0.050, 0.100, 0.767) \\
(0.640, 0.050, 0.306, 0.050, 0.100, 0.692) \\
(0.640, 0.050, 0.306, 0.050, 0.100, 0.692)
\end{bmatrix}
\]

Based on the improved preference relations, we utilize models (M-7) and (M-8) to find the individual priority weight vector of each
Table 2
Optimal weights based on the improved multiplicative consistent intuitionistic fuzzy preference relations.

<table>
<thead>
<tr>
<th>Experts</th>
<th>Optimal weights</th>
<th>Ranking of alternatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>(w_1^{(1)} = 0.096, w_2^{(1)} = 0.170, w_3^{(1)} = 0.255)</td>
<td>(x_3 &gt; x_1 &gt; x_2 &gt; x_4 &gt; x_1)</td>
</tr>
<tr>
<td>E2</td>
<td>(w_1^{(2)} = 0.114, w_2^{(2)} = 0.049, w_3^{(2)} = 0.456)</td>
<td>(x_3 &gt; x_2 &gt; x_4 &gt; x_1)</td>
</tr>
<tr>
<td>E3</td>
<td>(w_1^{(3)} = 0.091, w_2^{(3)} = 0.364, w_3^{(3)} = 0.242)</td>
<td>(x_2 &gt; x_1 &gt; x_4 &gt; x_1)</td>
</tr>
</tbody>
</table>

\(^{\ast}\) Where \(>\) means “is preferred to” and \(\sim\) means “is indifferent to”.

A fuzzy preference relation \(R\) on the set \(X\) is represented by a complementary matrix \(R = (\mu_y)_{n \times n} \subset X \times X\) with,

\[
\mu_y \geq 0, \quad \mu_y + \mu_{\sim} = 1, \quad \mu_y = 0.5 \quad \text{for all } i, j = 1, 2, \ldots, n
\]

where \(\mu_y\) represents the preference degree of the alternative \(x_i\) over \(x_j\). A fuzzy preference relation \(R = (\mu_y)_{n \times n}\) is referred to as an additive consistent fuzzy preference relation, if it satisfies the following property [47]:

\[
\mu_y = 0.5(w_i - w_j + 1), \quad \text{for all } i, 1, 2, \ldots, n; \quad j = i + 1, \ldots, n,
\]

\[
w_i \geq 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} w_i = 1,
\]

if \(R\) is an additive consistent fuzzy preference relation, then there is a vector \(w = (w_1, w_2, \ldots, w_n)^{T}\) of \(R\) which satisfies Eq. (17). However this property does not always hold. Thus we can introduce a deviation variable \(d_y = (d_y^i, d_y^j)\) to relax Eq. (17).

\[
\mu_y = 0.5(w_i - w_j + 1) + d_y^i - d_y^j, \quad \text{for all } i = 1, 2, \ldots, n; \quad j = i + 1, \ldots, n,
\]

\[
w_i \geq 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} w_i = 1, \quad d_y^i, d_y^j \geq 0
\]

As the deviation variables \(d_y^i, d_y^j\) become smaller, \(R = (\mu_y)_{n \times n}\) becomes closer to an additive consistent fuzzy preference relation. In order to find the smallest deviation variables we can develop the following optimization model:

\[
\delta_y = \min_{\mu_y} \sum_{i=1}^{n} \sum_{j=1}^{n} (d_y^i + d_y^j)
\]

\[
\text{s.t. } 0.5(w_i - w_j + 1) + d_y^i - d_y^j = \mu_y
\]

\[
w_i \geq 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} w_i = 1, \quad d_y^i, d_y^j \geq 0
\]

The solution of this model determines the optimal deviation values \(d_y^i, d_y^j\). If \(\delta_y = 0\) then \(R\) is an additive consistent fuzzy preference relation. Otherwise, the nonzero deviation values may be used to improve the additive consistency as:

\[
\hat{R} = (\hat{\mu}_y)_{n \times n}, \quad \hat{\mu}_y = \mu_y + d_y^i - d_y^j,
\]

\[
i = 1, 2, \ldots, n - 1; \quad j = i + 1, \ldots, n
\]

And a fuzzy preference relation \(R = (\hat{\mu}_y)_{n \times n}\) is referred to as a multiplicative consistent fuzzy preference relation if it satisfies the condition below [49,52]:

\[
\hat{\mu}_y = \frac{w_i}{w_i + w_j}, \quad \text{for all } i, 1, 2, \ldots, n - 1; \quad j = i + 1, \ldots, n
\]

\[
w_i \geq 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} w_i = 1
\]

If \(R\) is a multiplicative consistent fuzzy preference relation, then there is a vector \(w = (w_1, w_2, \ldots, w_n)^{T}\) of \(R\) which satisfies Eq. (20). However this property does not always hold. Thus we can introduce a deviation variable \(d_y = (d_y^i, d_y^j)\) to relax Eq. (20):

\[
\hat{\mu}_y = \frac{w_i}{w_i + w_j} + d_y^i - d_y^j, \quad \text{for all } i, 1, 2, \ldots, n - 1; \quad j = i + 1, \ldots, n
\]

\[
w_i \geq 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} w_i = 1, \quad d_y^i, d_y^j \geq 0
\]

Considering that the smaller the deviation variables \(d_y^i, d_y^j\) in Eq. (21), the closer \(R\) to a multiplicative consistent fuzzy preference relation, we develop the nonlinear optimization model:
\[ \gamma_j = \min \sum_{i=1}^{n} w_i \left( d^+_i - d^-_i \right) \]  

(M-12)

\[ s.t. \quad \frac{w_i}{w_i + d^-_i} + d^+_i - d^-_i = \mu_j \]

\[ w_i > 0, i = 1, 2, \ldots, n; \quad \sum_{i=1}^{n} w_i = 1; \quad d^+_i, d^-_i > 0 \]

\[ i = 1, 2, \ldots, n-1; \quad j = i+1, \ldots, n. \]  

(22)

In fuzzy preference relation, all the information is expressed with only membership functions and nonmembership is represented as the complement of membership which actually ignores the DM's hesitation in the decision making process. However, in intuitionistic fuzzy preference relation simultaneously considers the degrees of membership and nonmembership with hesitation index. These different preference relation definitions would result in different decisions. For comparison of fuzzy and intuitionistic fuzzy preference relations and their varying impacts on experts' decisions, following preference relation matrices are used.

In fuzzy preference relation, the experts individually compare each pair of criteria \( x_i \) and \( x_j \) and provide his/her fuzzy preference value composed only by the certainty degree \( \mu_{ij} \) to which \( x_i \) is preferred to \( x_j \). The value which \( x_j \) is preferred to \( x_i \) \((=\mu_{ji})\) will be the complement of \( \mu_{ij} \) according to Eq. (16). Hence, fuzzy preference relation of the expert will only include the membership values as below:

\[
R = \begin{bmatrix}
(0.5) & (0.3) & (0.2) & (0.5) \\
(0.7) & (0.5) & (0.4) & (0.3) & (0.1) \\
(0.7) & (0.6) & (0.5) & (0.7) & (0.4) \\
(0.8) & (0.7) & (0.3) & (0.5) & (0.1) \\
(0.5) & (0.9) & (0.6) & (0.9) & (0.5)
\end{bmatrix}
\]

Based on this preference relation we first apply the model (M-11) to check the additive consistency. The solution of the model (M-11) gives the optimal objective value as \( \delta^*_j = 2.600 \) and the optimal nonzero deviation values:

\[ d_{14} = d_{23} = d_{24} = d_{34} = 0.2, \quad d_{14} = d_{41} = d_{15} = d_{51} = 0.3, \]

\[ d_{23} = d_{12} = d_{25} = d_{25} = d_{15} = d_{44} = 0.1. \]

Since, \( \delta^*_j = 0 \), then \( R \) is not additive consistent. To improve the additive consistency of \( R \), we use optimal deviation values are used and the improved additive consistent fuzzy preference relation \( \tilde{R} \) is determined by applying Eq. (19):

\[
\tilde{R} = \begin{bmatrix}
(0.5) & (0.5) & (0.3) & (0.5) & (0.2) \\
(0.5) & (0.5) & (0.3) & (0.5) & (0.2) \\
(0.7) & (0.7) & (0.5) & (0.7) & (0.4) \\
(0.5) & (0.5) & (0.3) & (0.5) & (0.2) \\
(0.8) & (0.8) & (0.6) & (0.8) & (0.5)
\end{bmatrix}
\]

Since the optimal objective value of the improved preference relation will be equal to zero we determine the priority weight vector as \( w_1 = 0.149, w_2 = 0.304, w_3 = 0.130, w_4 = 0.457 \) and the ranking of alternatives as \( x_3 > x_5 > x_1 > x_2 > x_4 \).

In intuitionistic preference relation, the experts individually compare each pair of criteria \( x_i \) and \( x_j \), and provide his/her intuitionistic fuzzy preference value \( b_{ij} = (\mu_{ij}, \nu_{ij}) \), composed by the certainty degree \( \mu_{ij} \) to which \( x_i \) is preferred to \( x_j \) and the certainty degree \( \nu_{ij} \) to which \( x_i \) is non-preferred to \( x_j \) and then develop the following intuitionistic fuzzy preference relation \( B = (b_{ij})_{n \times n} \):

\[
B = \begin{bmatrix}
(0.5, 0.05) & (0.03, 0.3) & (0.03, 0.3) & (0.2, 0.5) & (0.5, 0.5) \\
(0.3, 0.3) & (0.05, 0.5) & (0.4, 0.3) & (0.3, 0.2) & (0.1, 0.6) \\
(0.3, 0.3) & (0.3, 0.377) & (0.5, 0.5) & (0.651, 0.0) & (0.4, 0.4) \\
(0.455, 0.2) & (0.2, 0.3) & (0.0, 0.651) & (0.5, 0.5) & (0.1, 0.5) \\
(0.5, 0.417) & (0.6, 0.1) & (0.4, 0.4) & (0.5, 0.1) & (0.5, 0.5)
\end{bmatrix}
\]

Since the optimal objective value of the improved preference relation will be equal to zero, we establish models (M-2) and (M-3) to determine the individual priority weight vector of the expert as \( w_1 = 0.149, w_2 = 0.304, w_3 = 0.457, w_4 = 0.059, w_5 = 0.315 \) and the ranking of alternatives as \( x_3 > x_5 > x_1 > x_2 > x_4 \).

When the results of additive consistent fuzzy and intuitionistic fuzzy priority relations are examined, we realize that the priority vector and ranking of the alternatives change significantly in the two cases. In the intuitionistic fuzzy case, the third alternative is determined as the best choice while in the fuzzy case the fifth alternative is calculated as the best. Furthermore when we examine the rankings of alternatives in two cases, significant differences between the two results are observed.

We can also check the multiplicative consistency of both preference relations and compare the priority vectors. Based on fuzzy preference relation, model (M-12) is applied to check the multiplicative consistency. The solution of the model (M-12) gives the optimal objective value as \( \gamma^*_j = 1.070 \). Since \( \gamma^*_j = 0 \) then \( R \) is not multiplicative consistent. In order to improve the multiplicative consistency of \( R \), optimal deviation values are used and the improved multiplicative consistent intuitionistic fuzzy preference relation \( \tilde{R} \) is determined by applying Eq. (22). The optimal objective value of the improved preference relation will be equal to zero and we calculate the priority weight vector as \( w_1 = 0.033, w_2 = 0.076, w_3 = 0.304, w_4 = 0.130, w_5 = 0.457 \) and the ranking of alternatives as \( x_3 > x_5 > x_4 > x_2 > x_1 \). In the intuitionistic case, we first apply the model (M-6) to the individual intuitionistic fuzzy preference relation to check the multiplicative consistency. Applying (M-6), we find the optimal objective value as \( \gamma^* = 0.385 \) and the optimal nonzero deviation values. Since, \( \gamma^* = 0 \), \( B \) is not multiplicative consistent. In order to improve the multiplicative consistency, optimal deviation values are used and the improved multiplicative consistent intuitionistic fuzzy preference relation is determined by applying Eq. (14). The optimal objective value of the improved
preference relation will be equal to zero and we establish models (M-7) and (M-8) to calculate the individual priority weight vector of the expert as \( w_1 = 0.194, w_2 = 0.129, w_3 = 0.290, w_4 = 0.194, w_5 = 0.194 \) and the ranking of alternatives as \( x_5 > x_3 ~ x_1 ~ x_4 ~ x_2 \).

When we examine the results of multiplicative consistent fuzzy and intuitionistic fuzzy priority relations, we observe that the priority vector and ranking of the alternatives change significantly in two cases as in the additive consistent situation. Although the best alternative does not change in two cases, the ranking of the remaining alternatives changes significantly.

9. Conclusion and further study

In the process of group decision making problems, the decision makers sometimes may not provide their preferences for alternatives to a certain degree and there is usually a degree of uncertainty in providing their preferences over the alternatives considered. Intuitionistic fuzzy preference relations have the capability of representing imprecise or not totally reliable judgments which exhibit affirmation, negation and hesitation characteristics. The consistency of intuitionistic fuzzy preference relations and the priority weights of experts gathered from these preference relations play an important role in group decision making problems in order to reach an accurate decision result. In this study, we have proposed a group decision making process with the usage of intuitionistic fuzzy preference relations. The suggested process is based on the evaluation of the consistency of intuitionistic fuzzy preference relations. We have constructed two different optimization models to minimize the deviations from additive or multiplicative consistency respectively. The optimal deviation values obtained from the model results enable us to improve the consistency of considered preference relations. Following this, we aggregated individual improved (which means consistent) fuzzy intuitionistic preference relations in order to determine a collective consistent fuzzy intuitionistic preference relation. Based on the consistent collective preference relation, we have developed a linear programming model considering additive consistency and a nonlinear model considering multiplicative consistency to obtain the priority weights. The priority weights of the experts also enable us to determine the ranking of the alternatives. Furthermore, we have given some illustrative examples in order to examine the validity and practicality of the developed models. Numerical analyses have shown that although the priority weight vectors of the individual preference relations of the experts differ, the ranking of the individual priority weights do not differ significantly according to the additive or multiplicative consistent intuitionistic fuzzy preference relations. And additionally, if we derive consistent preference relations (additive or multiplicative consistent), the ranking of the alternatives obtained from collective preference relation or aggregated priority vectors will generally be the same.

In the final section, we presented comparative analyses between fuzzy and intuitionistic fuzzy preference relations. When we examine the results of both additive and multiplicative consistent fuzzy and intuitionistic fuzzy priority relations, we realize that the priority vector and ranking of the alternatives change significantly in two cases. Furthermore when we look at the rankings of alternatives in two cases (fuzzy and intuitionistic fuzzy), we observe significant differences between two results. The main reason for this difference is the capability of intuitionistic fuzzy preference relations simultaneously considering the degrees of membership and nonmembership with hesitation index while fuzzy preference relations can only consider membership values. The results of the numerical illustrations showed that intuitionistic fuzzy preference relations provide more accurate priority vectors and rankings of alternatives by taking into consideration the DMs’ affirmation, negation and hesitation with the help of membership definitions.

The proposed group decision making process and models may be used in many real-world applications in which the DMs may not be able to provide his/her preferences for alternatives to a certain degree due to lack of precise or sufficient level of knowledge related to the problem, or the difficulty in explaining explicitly the degree to which one alternative is better than others. Possible application areas may be supply chain management, project evaluation, risk management, pattern recognition, medical diagnosis, investment, personnel examination and military system efficiency evaluation. Although the focus of this study is mainly on the consistency of intuitionistic preference relations; the process developed can be extended to include group consensus and priority weight generation from incomplete intuitionistic preference relations.

References
