Compressive Imaging by Generalized Total Variation Minimization

Jie Yan and Wu-Sheng Lu

Department of Electrical and Computer Engineering
University of Victoria, Victoria, BC, Canada

APCCAS 2014, Okinawa, Japan

UVic

OUTLINE

1. Introduction
2. TV, Generalized TV, and Weighted TV
3. Power-Iterative Reweighting Strategy
4. Solving WTV-Regularized Minimization Problem
5. Performance Evaluation
6. Conclusions

UVic

Introduction

- Compressive imaging (CI) is an important branch of compressive sensing (CS).
- Compressive imaging for magnetic resonance imaging (MRI) by minimizing total variation (TV)

$$\min_{\mathbf{U}} \ \mathrm{TV}(\mathbf{U}) \ \text{ s.t. } \ \|\mathbf{R} \circ (\mathcal{F}\mathbf{U}) - \mathbf{B}\|_{\mathsf{F}}^2 < \sigma^2 \tag{1}$$

  where $\circ$ denotes entrywise product between matrices, $\mathcal{F}$ denotes the 2-D Fourier transform operator, $\mathbf{R}$ represents a random sampling matrix whose entries are either 1 or 0, and $\mathbf{B}$ stores the compressive sampled measurements.
- Problem (1) can be reduced to a sequence of unconstrained problems that can be solved using the Split Bregman technique.
- Images are regarded as matrix variables in this problem.

UVic

Introduction

- Inspired by the relationship between ℓ_p and ℓ_1 norms, we generalize the concept of TV to a pth-power type TV with $0 \leq p \leq 1$.
- The GTV regularizer is applied to the Fourier-based MRI reconstruction problem.
- The algorithm proposed solves GTV-regularized optimization problem that turns out to perform better than existing algorithms in preserving image edges.

UVic

Discrete Anisotropic TV

- Anisotropic total variation (TV) of a digital image $\mathbf{U} \in \mathbb{R}^{n\times n}$ is defined as

$$\begin{aligned}\text{TV}(\mathbf{U}) = &\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(|U_{i,j} - U_{i+1,j}| + |U_{i,j} - U_{i,j+1}|\right) \\ &+ \sum_{i=1}^{n-1}|U_{i,n} - U_{i+1,n}| + \sum_{j=1}^{n-1}|U_{n,j} - U_{n,j+1}|\end{aligned} \tag{2}$$

- Under the periodic boundary condition, TV can be expressed in the form

$$\text{TV}(\mathbf{U}) = \|\mathbf{DU}\|_1 + \|\mathbf{UD}^T\|_1 \tag{3}$$

with $\mathbf{D} \in \mathbb{R}^{n\times n}$ as a circulant matrix with the first row $[1\ \ -1\ 0\ \cdots\ 0]$. The notation $\|\mathbf{X}\|_1$ denotes the sum of magnitudes of all the entries in $\mathbf{X}$, i.e., $\sum|x_{i,j}|$.

UVic

Generalized pth power TV

- We extend the concept of TV by defining a generalized pth power TV (GTV) with $0 \leq p \leq 1$

$$\mathrm{TV}_p(\mathbf{U}) = \|\mathbf{D}\mathbf{U}\|_p + \|\mathbf{U}\mathbf{D}^T\|_p$$

- The newly introduced notation $\|\mathbf{X}\|_p$ resembles an ℓ_p norm as it expresses the sum of pth power magnitudes of all the entries in $\mathbf{X}$, i.e., $\sum |x_{i,j}|^p$.
- With TV_p defined, we consider the optimization model

$$\min_{\mathbf{U}} \ \mathrm{TV}_p(\mathbf{U}) \ \text{ s.t. } \ \|\mathbf{R} \circ (\mathcal{F}\mathbf{U}) - \mathbf{B}\|_{\mathsf{F}}^2 \leq \sigma^2 \tag{4}$$

  which may be regarded as a generalization of model (1) for the reason that the regularizer TV_p promotes a sparser TV when p is less than one.

UVic

Weighted TV

- Optimization of TV_p-related problem is nonconvex.
- The weighted TV (WTV) is introduced to deal with this nonconvexity

$$\mathrm{TV_w}(\mathbf{U}) = \|\mathbf{W}_x \circ (\mathbf{DU})\|_1 + \|\mathbf{W}_y \circ (\mathbf{UD}^T)\|_1$$

  where $\mathbf{W}_x$ and $\mathbf{W}_y$ are weights matrices along the horizontal and vertical direction.
- WTV with properly chosen weights facilitates an excellent convex approximation of the nonconvex objective $\mathrm{TV}_p(\mathbf{U})$.

UVic

Power-Iterative Reweighting Strategy

- In theory, solving (4) with $p = 0$ promotes the solution with the sparsest TV.
- To approach the solution of a TV_0-regularized optimization problem, we adopt a power-iterative strategy by gradually reducing the power p while updating the weights and solving a WTV-regularized problem at each iteration.
- The power-iterative strategy not only properly updates the weights to approach the corresponding TV_p minimization, but also provides convex WTV-regularized problem with a good initial state obtained from the previous iteration.

UVic

Power-Iterative Reweighting Strategy for TV$_p$ Minimization

1. Set $p = 1$, $l = 1$, $\mathbf{W}_x = \mathbf{W}_y = \mathbf{1}$.
2. Solve the WTV-regularized problem for $\mathbf{U}^{(l)}$

$$\min_{\mathbf{U}} \ \mathsf{TV}_\mathsf{w}(\mathbf{U}) \ \text{ s.t. } \ \|\mathbf{R} \circ (\mathcal{F}\mathbf{U}) - \mathbf{B}\|_\mathsf{F}^2 < \sigma^2 \tag{5}$$

3. Terminate if $p = 0$; otherwise, set $p = p - 0.1$ and update the weights $\mathbf{W}_x$ and $\mathbf{W}_y$ as

$$\mathbf{W}_x = |\mathbf{D}\mathbf{U}^{(l)} + \epsilon|.^{p-1}, \ \mathbf{W}_y = |\mathbf{U}^{(l)}\mathbf{D}^T + \epsilon|.^{p-1} \tag{6}$$

Then set $l = l + 1$ and repeat from step 2.

UVic

Power-Iterative Reweighting Strategy

- Note that by Eq. (6), $\mathrm{TV}_{\mathrm{w}}(\mathbf{U})$ essentially becomes $\mathrm{TV}_p(\mathbf{U})$ for $\mathbf{U}$ in a neighborhood of iterate $\mathbf{U}^{(l)}$.
- The parameter ϵ in (6) is a small constant to prevent the weights from being zero.
- In this way, nonconvex minimization of $\mathrm{TV}_p(\mathbf{U})$ can practically be achieved by a series of convex minimization of $\mathrm{TV}_{\mathrm{w}}(\mathbf{U})$.

UVic

WTV-Regularized Minimization

- The analysis has led to the WTV-regularized problem (5).
- We propose to solve the problem using a Split Bregman approach but with important changes.
- The entire analysis presented below is carried out in terms of matrix operations.
- Using Bregman iteration, we reduce the problem to

$$\mathbf{U}^{(k+1)} = \underset{\mathbf{U}}{\operatorname{argmin}} \, \mathsf{TV}_{\mathsf{w}}(\mathbf{U}) + \frac{\mu}{2} \|\mathbf{R} \circ (\mathcal{F}\mathbf{U}) - \mathbf{B}^{(k)}\|_{\mathsf{F}}^2 \tag{7a}$$

$$\mathbf{B}^{(k+1)} = \mathbf{B}^{(k)} + \mathbf{B} - \mathbf{R} \circ (\mathcal{F}\mathbf{U}^{(k+1)}) \tag{7b}$$

UVic

WTV-Regularized Minimization

- A Split Bregman strategy applied to (7a) leads to the formulation

$$\min_{\mathbf{U}} \|\mathbf{W}_x \circ \mathbf{D}_x\|_1 + \|\mathbf{W}_y \circ \mathbf{D}_y\|_1 + \frac{\mu}{2}\|\mathbf{R} \circ (\mathcal{F}\mathbf{U}) - \mathbf{B}^{(k)}\|_{\mathsf{F}}^2 \tag{8a}$$

$$\text{s.t. } \mathbf{D}_x = \mathbf{D}\mathbf{V}, \mathbf{D}_y = \mathbf{V}\mathbf{D}^T, \mathbf{U} = \mathbf{V} \tag{8b}$$

  where we split $\mathbf{D}_x = \mathbf{D}\mathbf{V}$, $\mathbf{D}_y = \mathbf{V}\mathbf{D}^T$, and introduce an additional split as $\mathbf{U} = \mathbf{V}$.
- Such a split allows us to decompose the most expensive step of the algorithm into two much simpler steps.

UVic

WTV-Regularized Minimization

- Applying Bregman method to (8) to enforce constraints in (8b), we are led to the following unconstrained problem with respect to $\{\mathbf{U}, \mathbf{V}, \mathbf{D}_x, \mathbf{D}_y\}$

$$\begin{aligned} \min\ & \|\mathbf{W}_x \circ \mathbf{D}_x\|_1 + \|\mathbf{W}_y \circ \mathbf{D}_y\|_1 + \frac{\mu}{2}\|\mathbf{R} \circ (\mathcal{F}\mathbf{U}) - \mathbf{B}^{(k)}\|_\mathsf{F}^2 \\ & + \frac{\lambda}{2}\|\mathbf{D}_x - \mathbf{D}\mathbf{V} - \mathbf{E}_x^{(h)}\|_\mathsf{F}^2 + \frac{\lambda}{2}\|\mathbf{D}_y - \mathbf{V}\mathbf{D}^T - \mathbf{E}_y^{(h)}\|_\mathsf{F}^2 \\ & + \frac{\nu}{2}\|\mathbf{U} - \mathbf{V} - \mathbf{G}^{(h)}\|_\mathsf{F}^2 \end{aligned}$$

where $\mathbf{E}_x^{(h)}$, $\mathbf{E}_y^{(h)}$ and $\mathbf{G}^{(h)}$ are updated by Bregman iterations.

WTV-Regularized Minimization

In the hth iteration we solve four subproblems

$$\begin{aligned}\mathbf{U}^{(h+1)} = \underset{\mathbf{U}}{\operatorname{argmin}} \frac{\mu}{2}\|\mathbf{R}\circ(\mathcal{F}\mathbf{U}) - \mathbf{B}^{(k)}\|_{\mathsf{F}}^2 \\ + \frac{\nu}{2}\|\mathbf{U} - \mathbf{V}^{(h)} - \mathbf{G}^{(h)}\|_{\mathsf{F}}^2\end{aligned} \tag{9}$$

$$\begin{aligned}\mathbf{V}^{(h+1)} = \underset{\mathbf{V}}{\operatorname{argmin}} \frac{\nu}{2}\|\mathbf{V} - \mathbf{U}^{(h+1)} + \mathbf{G}^{(h)}\|_{\mathsf{F}}^2 \\ + \frac{\lambda}{2}\|\mathbf{D}\mathbf{V} + \mathbf{E}_x^{(h)} - \mathbf{D}_x^{(h)}\|_{\mathsf{F}}^2 + \frac{\lambda}{2}\|\mathbf{V}\mathbf{D}^T + \mathbf{E}_y^{(h)} - \mathbf{D}_y^{(h)}\|_{\mathsf{F}}^2\end{aligned} \tag{10}$$

$$\mathbf{D}_x^{(h+1)} = \underset{\mathbf{D}_x}{\operatorname{argmin}} \|\mathbf{W}_x\circ\mathbf{D}_x\|_1 + \frac{\lambda}{2}\|\mathbf{D}_x - \mathbf{D}\mathbf{V}^{(h+1)} - \mathbf{E}_x^{(h)}\|_{\mathsf{F}}^2 \tag{11a}$$

$$\mathbf{D}_y^{(h+1)} = \underset{\mathbf{D}_y}{\operatorname{argmin}} \|\mathbf{W}_y\circ\mathbf{D}_y\|_1 + \frac{\lambda}{2}\|\mathbf{D}_y - \mathbf{V}^{(h+1)}\mathbf{D}^T - \mathbf{E}_y^{(h)}\|_{\mathsf{F}}^2 \tag{11b}$$

UVic

WTV-Regularized Minimization

- The problems in (11a) and (11b) can be solved simply by soft shrinkage as the unknowns are separate from each other

$$\mathbf{D}_x^{(h+1)} = \mathcal{T}_{\mathbf{W}_x/\lambda}(\mathbf{D}\mathbf{V}^{(h+1)} + \mathbf{E}_x^{(h)}) \tag{12a}$$

$$\mathbf{D}_y^{(h+1)} = \mathcal{T}_{\mathbf{W}_y/\lambda}(\mathbf{V}^{(h+1)}\mathbf{D}^T + \mathbf{E}_y^{(h)}) \tag{12b}$$

  where soft shrinkage operator $\mathcal{T}$ applies pointwisely as

$$\mathcal{T}_{w_{i,j}/\lambda}(z) = \text{sgn}(z) \cdot \max\{|z| - w_{i,j}/\lambda, 0\} \tag{13}$$

- Solving problems (9) and (10) are far from trivial.

UVic

WTV-Regularized Minimization

- We write first-order optimality condition of (9) as

$$\mu\mathcal{F}^T\mathbf{R}\circ\mathcal{F}\mathbf{U}+\nu\mathbf{U}=\mu\mathcal{F}^T\mathbf{R}\circ\mathbf{B}^k+\nu(\mathbf{V}^{(h)}+\mathbf{G}^{(h)}) \tag{14}$$

- Multiplying both sides of (14) by $\mathcal{F}$ and applying the orthogonality of Fourier transform, we obtain solution of (9) as

$$\mathbf{U}^{(h+1)}=\mathcal{F}^T\left\{\left[\mu\mathbf{R}\circ\mathbf{B}^k+\nu\mathcal{F}(\mathbf{V}^{(h)}+\mathbf{G}^{(h)})\right]\circ/(\mu\mathbf{R}+\nu)\right\} \tag{15}$$

where $\circ/$ denotes the pointwise division.

UVic

WTV-Regularized Minimization

- To solve (10), we use matrix calculus to write its first-order optimality condition as

$$\nu \mathbf{V} + \lambda \mathbf{D}^T \mathbf{D} \mathbf{V} + \lambda \mathbf{V} \mathbf{D}^T \mathbf{D} = \mathbf{C}^{(h)} \tag{16}$$

  where

$$\begin{aligned} \mathbf{C}^{(h)} &= \nu(\mathbf{U}^{(h+1)} - \mathbf{G}^{(h)}) \\ &\quad + \lambda \mathbf{D}^T (\mathbf{D}_x^{(h)} - \mathbf{E}_x^{(h)}) + \lambda (\mathbf{D}_y^{(h)} - \mathbf{E}_y^{(h)}) \mathbf{D} \end{aligned} \tag{17}$$

- Circulant matrix $\mathbf{D}$ can be diagonalized by the 2-D Fourier transform $\mathcal{F}$ as $\mathbf{D} = \mathcal{F}^T \mathbf{\Lambda} \mathcal{F}$.

UVic

WTV-Regularized Minimization

- Substituting $\mathbf{D} = \mathcal{F}^T \mathbf{\Lambda} \mathcal{F}$ into (16), we reduce the equation to

$$\nu \tilde{\mathbf{V}} + \lambda(\mathbf{T}\tilde{\mathbf{V}} + \tilde{\mathbf{V}}\mathbf{T}) = \mathcal{F}\mathbf{C}^{(h)}\mathcal{F}^T \tag{18}$$

  where $\tilde{\mathbf{V}} = \mathcal{F}\mathbf{V}\mathcal{F}^T$ and $\mathbf{T} = \mathbf{\Lambda}^* \mathbf{\Lambda}$.
- As $\mathbf{T}$ is a diagonal matrix, we can further express (18) as

$$(\nu + \lambda \mathbf{T}_r + \lambda \mathbf{T}_c) \circ \tilde{\mathbf{V}} = \mathcal{F}\mathbf{C}^{(h)}\mathcal{F}^T \tag{19}$$

  where $\mathbf{T}_r$ has each element in its ith row as $T_{i,i}$ and $\mathbf{T}_c$ has each element in its ith column as $T_{i,i}$.
- In consequence, we obtain solution of (10) as

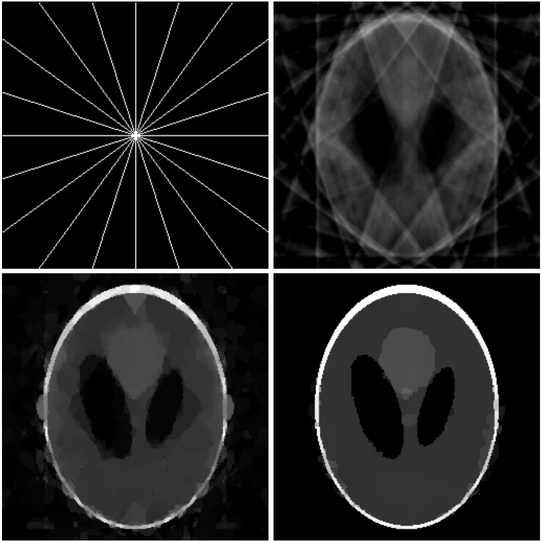
$$\mathbf{V}^{(h+1)} = \mathcal{F}^T \left\{ (\mathcal{F}\mathbf{C}^{(h)}\mathcal{F}^T) \circ / (\nu + \lambda \mathbf{T}_r + \lambda \mathbf{T}_c) \right\} \mathcal{F} \tag{20}$$

UVic

Performance Evaluation: MRI of the Shepp-Logan Phantom

- A normalized Shepp-Logan phantom of size 256×256, was measured at 2521 locations (as low as 3.85%) in the 2D Fourier plane (k-space).
- The sampling pattern was a star-shaped pattern consisting of only 10 radial lines.
- Recover the image based on the 2521 star-shaped 2D Fourier samples.

MRI of the Shepp-Logan Phantom



(a) Star-shaped sampling pattern (b) Minimum energy reconstruction (c) Minimum TV reconstruction (d) Minimum GTV reconstruction with $p = 0$

UVic

Recovered Shepp-Logan Phantom

Minimum GTV reconstruction for $p = 0$ with inner and outter iterations $H = 10$ and $K = 100$

- The signal to noise (SNR) ratio: 16.3 dB.
- Computation time on a PC laptop with a 2.67 GHz Intel quad-core processor: 770.7 seconds.

Minimum TV reconstruction with $K = 1100$ iterations

- The signal to noise (SNR) ratio: 8.8 dB.
- Computation time on a PC laptop with a 2.67 GHz Intel quad-core processor: 756.8 seconds.

UVic

Performance evaluation: compressive imaging of natural images

- Each test image of size 256 by 256 was measured at 13107 random locations (20%) in the 2D Fourier plane.

Images	`cameraman`	`jet`	`building`	`milk`
GTV (dB)	19.5	18.3	18.3	14.5
TV (dB)	14.3	16.2	15.2	12.1

- The images reconstructed using the proposed GTV minimization method possess consistently higher SNRs than those from the conventional minimum TV reconstruction.

Left: minimum TV reconstruction
Right: minimum GTV reconstruction with $p = 0$

Conclusions

- An algorithm for generalized TV minimization for compressive imaging has been proposed.
- A weighted TV-regularized problem has been solved in the Split Bregman framework with additional splitting technique.
- A power-iterative strategy is utilized by gradually reducing the power p from 1 to 0.
- The algorithm is found to outperform the conventional TV minimization method on reconstructing a variety of medical and natural images.