Structural damage identification via a combination of blind feature extraction and sparse representation classification

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\textbf{Abstract}

This paper addresses two problems in structural damage identification: locating damage and assessing damage severity, which are incorporated into the classification framework based on the theory of sparse representation (SR) and compressed sensing (CS). The sparsity nature implied in the classification problem itself is exploited, establishing a sparse representation framework for damage identification. Specifically, the proposed method consists of two steps: feature extraction and classification. In the feature extraction step, the modal features of both the test structure and the reference structure model are first blindly extracted by the unsupervised complexity pursuit (CP) algorithm. Then in the classification step, expressing the test modal feature as a linear combination of the bases of the over-complete reference feature dictionary—constructed by concatenating all modal features of all candidate damage classes—builds a highly underdetermined linear system of equations with an underlying sparse representation, which can be correctly recovered by $\ell_1$-minimization; the non-zero entry in the recovered sparse representation directly assigns the damage class which the test structure (feature) belongs to. The two-step CP–SR damage identification method alleviates the training process required by traditional pattern recognition based methods. In addition, the reference feature dictionary can be of small size by formulating the issues of locating damage and assessing damage extent as a two-stage procedure and by taking advantage of the robustness of the SR framework. Numerical simulations and experimental study are conducted to verify the developed CP–SR method. The problems of identifying multiple damage, using limited sensors and partial features, and the performance under heavy noise and random excitation are investigated, and promising results are obtained.

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1. Introduction

The damage identification problem in structural health monitoring (SHM) commonly includes four levels \cite{1}: (1) detecting the presence of damage; (2) locating damage; (3) assessing damage severity; (4) predicting the structural remaining service life. Vibration-based damage identification techniques \cite{2} have been extensively studied in the literatures \cite{3}. In the earlier years, much attention was focused on those methods based on the change of structural modal parameters as damage signature; specifically, the frequency tends to reflect global damage information while the modeshape may additionally provide spatial information of damage. However, researchers have pointed out that modal information alone is
not sensitive to local damage and that its capability as direct damage indicators is easily influenced by noise. Nevertheless, researches do converge to support the conclusion that structural damage information is hidden in modal features, which need to be further processed for successful damage identification.

Novel signal processing techniques, on the other hand, have been successfully applied in the recent years to detect and locate structural damage, such as wavelet transform [4–6], Hilbert–Huang transform [7,8], time–frequency analysis [9,10], fractal dimension [11,12], cyclostationarity [13], and blind source separation [14–16]. The signal-based methods are typically non-parametric and enjoy efficient implementation; they are mostly limited up to the level 2, however.

With a structural model (e.g., a finite element model) available as reference information, it is possible to develop damage identification methods that can address even the problem of level 3 [3], that is, the quantification of damage severity. Some researchers used the model updating method [17,18] to address this problem, by comparing the undamaged (reference) and candidate (test) model (physical or modal) matrices [3]. These methods are essentially parametric; as such, they are prone to model error.

More recently, the damage identification problem including that of level 3 has been treated as a pattern recognition issue [14,19–27]. The classification-based methods involve three steps: feature extraction, training, and classification. For damage identification, the extracted features from various predefined or reference damage classes, including different damage locations and damage extents, are used as inputs to train the classifiers, which can then identify the damage class of the test feature representing the current state of the structure. Successful examples include those based on artificial neural networks (ANNs) [14,20], support vector machines (SVMs) [22–25], nearest neighbor [26], and Markov observers [27].

Several factors, however, could influence the performance of these classification-based damage identification methods that are mostly dependent on the training process of the classifiers. In the ANN-based methods, for example, the number of input and hidden nodes in the network could affect its accuracy [14,20], and the global convergence of the algorithm is not guaranteed [19]. Compared to ANN, the multi-class SVM-based methods have advantages when the sample numbers are small [21,23]; nevertheless, their success depends on the choice of the algorithm parameters, i.e., the kernel function selection and its associated parameters [21,22,24]. Although optimal choice may be obtained through trial and error or optimization algorithms, and such an approach increases computational burden and needs the skill of an experienced practitioner; hence is not preferable in some situations, e.g., online monitoring.

Attempting to address the aforementioned concerns, this paper proposes a new algorithm in the classification framework for both locating and assessing structural damage, using the recent theory from blind source separation (BSS) [28] and sparse representation (SR) [29] along with compressed sensing (CS) [30,31]. The proposed damage identification method consists of two steps: feature extraction and classification.

Specifically, in the feature extraction step, a BSS method termed complexity pursuit (CP) [32] is used to extract the modal features of the structure. As an unsupervised learning algorithm, BSS is able to recover the hidden sources using only the measured mixture signals. Recently, two BSS techniques— independent component analysis (ICA) [33] and second order blind identification (SOBI) [34]—have been studied extensively in structural dynamics and found to be efficient in output-only modal identification [35–42]. However, it is to be noted that ICA is only suitable for lightly-damped structures [36,41], while SOBI overcomes this difficulty, yet it has difficulty in handling cases with close modes and non-stationary vibrations [39,40]. Time–frequency ICA [41] and modified SOBI methods [39,40] have been proposed to address these drawbacks; especially, the extended SOBI method recently proposed in Ref. [42] has tackled quite a few earlier limitations in SOBI based methods.

Another BSS technique, CP, recently explored by the authors [43], has been found to be a useful alternative to efficiently perform output-only system identification of many structures with highly-damped, closely-spaced, and non-proportionally damped modes requiring limited parameter adjustments. In the first step of the proposed method, CP serves to blindly extract the structural modal features, which are subsequently used by the classification framework for damage identification; its performance is also compared with SOBI.

In the following classification step for both locating damage and identifying damage extent, the sparsity nature of the classification itself is exploited. An SR framework is developed based on the theory of SR and CS [29–31], due to their recent success on sparse MRI [44], robust SR face recognition [45], and more lately on SHM [46–49]. The key idea is that the damage class of the test structure naturally belongs to only one unique class of the predefined reference feature space: (1) an over-complete reference feature dictionary is built by concatenating all the modal features of all candidate damage classes; (2) the test modal feature is most sparsely represented as a linear combination of the bases of this reference dictionary, activating only the relevant feature in the same damage class. This establishes a highly underdetermined linear system of equations with an underlying sparse representation that directly dictates the damage class of the test structure. The theory of SR and CS enables one to find the correct sparse solution of such a highly underdetermined linear system of equations using the efficient $\ell_1$-minimization technique, leading to the test structure’s class of damage location and damage severity.

Numerical simulations and experimental study are conducted to verify the proposed CP–SR method. Results show that it can accurately and efficiently identify the damage locations and damage extents. In addition, several problems of identifying multiple damage, using limited sensors and partial features, and in the presence of heavy noise and random excitation are also presented.

The remainder of this paper is organized as follows. Section 2 introduces the theory of BSS and CP for the feature extraction step, and Section 3 presents the SR classification framework for damage identification. Section 4 summarizes the
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$$y_i(t) = \mathbf{w}_i^T x(t)$$


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has the least complexity and thus estimates the simplest source signal, where the complexity of a signal \( y_i(t) \) is measured by temporal predictability \([32]\),

\[
F(y_i) = \text{log } \frac{V(y_i)}{U(y_i)} = \log \left( \frac{\sum_{t} (y_i(t) - \hat{y}_i(t))^2}{\sum_{t} (\hat{y}_i(t) - \hat{y}_i(t))^2} \right)
\]

in which the long-term predictor \( \hat{y}_i(t) \) and the short-term predictor \( \hat{y}_i(t) \) are defined by

\[
y_i(t) = \lambda L y(t) + (1 - \lambda L) y_i(t-1) \quad 0 \leq \lambda L \leq 1
\]

\[
\hat{y}_i(t) = \lambda S \hat{y}(t) + (1 - \lambda S) \hat{y}_i(t-1) \quad 0 \leq \lambda S \leq 1
\]

where

\[
\lambda = 2^{-1/h}
\]

with \( h_S = 1 \) as determined and \( h_L \) is arbitrarily set as long as \( h_L \gg h_S \) \([32]\) (hence “long-term”; \( h_L \approx 900.000 \) is used as default setting in this study). Instead of extracting the source one by one, Stone formulated the BSS problem based on the contrast function Eq. (5) to a generalized eigenproblem \([32]\), which computes the short-term and long-term covariance matrices using the fast convolution operations, thereby simultaneously estimating all sources hidden in the mixtures

\[
s(t) = y(t) = Wx(t)
\]

where \( y(t) = [y_1(t), ..., y_n(t)]^T \) approaches \( s(t) \). According to the correspondence between the BSS model and modal expansion described in Section 2.1, \( q(t) = s(t) = y(t) \) and the mode matrix

\[
\Phi = A - W^{-1}
\]

The frequency and damping ratio can then be estimated from \( q(t) \) using Fourier transform and logarithm decrement technique; but they are not used in the proposed damage identification method. More details on how Stone’s algorithm is cast to the modal identification are referred to Ref. \([43]\). The modal features \( \Phi \in \mathbb{R}^{n \times n} \) containing the state of the structure are subsequently used by the classification framework for damage identification, as detailed in the ensuing section.

3. Sparse representation classification for damage identification

With extracting modal features by CP being the first step, the second step of the proposed method is classification. In this section, a classification framework based on sparse representation for damage identification is developed by exploiting the sparsity nature implied in the classification problem itself.

3.1. Classification for damage identification

Given a test feature, the objective of classification is to identify which class it pertains to, within a predefined reference feature space containing a large set of candidate classes. For the damage identification issue, this is equivalent to determining which damage pattern the feature of the test structure belongs to, from a predefined feature class space with various damage classes (i.e., cases with different damage locations or damage severity). This predefined reference feature space typically comprises those features extracted from a structural model (e.g., an FEM model) simulating various damage classes. In the proposed method, the test feature and the reference feature space are obtained by the CP method in the first step, blindly extracting the modal features from the structural responses of the test structure and the reference FEM model, respectively.

3.2. Classification with SR

Suppose there are \( N \) distinct predefined candidate damage classes of an \( n \)-DOF structure (simulated by its FEM model), then for the \( j \)th class \((j = 1, ..., N)\), its structural mode matrix \( \Phi_j \in \mathbb{R}^{n \times n} \) consists of \( n \) modal feature columns \( \Phi_j = [\phi_{j,1}, ..., \phi_{j,i}, ..., \phi_{j,n}] \) \((i = 1, ..., n)\), where \( \phi_{j,i} \in \mathbb{R}^n \) is its \( i \)th column. For \( N \) reference damage classes (simulated from the structural FEM model), the predefined reference feature space is the reference modal feature space (dictionary) \( \Psi \in \mathbb{R}^{n \times w} \) consists of \( w = n \times N \) concatenated modal feature columns,

\[
\Psi = [\Phi_1 \ldots, \Phi_j \ldots, \Phi_N] = [\phi_{1,1}, ..., \phi_{j,i}, ..., \phi_{N,n}]
\]

The key idea of sparse representation classification is that the damage class of the test structure (represented by the test modal feature columns) can only belong to one of the predefined reference damage classes (represented by the reference modal feature matrix or dictionary); as such, it can be formulated as a sparse representation classification problem.

Specifically, the test structure with a (test) modal feature matrix \( \Phi = [\phi_1, ..., \phi_n] \in \mathbb{R}^{n \times n} \) (representing its current state) coincides with one of the \( N \) predefined candidate damage classes, say, the \( j \)th class; then any column of \( \Phi \), say, the \( i \)th column \( \phi_i \in \mathbb{R}^n \) \((i = 1, ..., n)\), if not normalized, will approximately have only a scale difference \( \alpha_{ji} \) from \( \phi_{j,i} \in \mathbb{R}^n \), which is the
ith column of \( \Phi_i \) representing the predefined \( j \)-th damage class, i.e.,

\[
\hat{\phi}_i = \alpha_j \phi_{ij},
\]

Then expressing \( \hat{\phi}_i \in \mathbb{R}^n \) in terms of the whole reference feature space \( \Psi \in \mathbb{R}^{n \times w} \), it can be sparsely represented as a linear combination of the bases (the feature columns) of \( \Psi \in \mathbb{R}^{n \times w} \),

\[
\hat{\phi}_i = \Psi \alpha_i = \sum_{k=1}^{N} \sum_{j=1}^{n} \alpha_{kj} \phi_{kj},
\]

where \( \alpha_i \in \{0, ..., 0, \alpha_{ji}, 0, ..., 0\}^T \in \mathbb{R}^w \) is an underlying sparse vector, in which the location of the non-zero entry \( \alpha_{ji} \) naturally assigns the damage class the given test feature \( \hat{\phi}_i \) falls into within the \( N \) predefined reference damage classes (also see Fig. 1 for illustrations).

The formulated classification problem for damage identification with a sparse representation therefore centers around the issue of recovering the underlying sparse \( \alpha_i \in \mathbb{R}^w \) from the knowledge of the predefined reference matrix \( \Psi \in \mathbb{R}^{n \times w} \) and the test feature \( \hat{\phi}_i \in \mathbb{R}^n \).

### 3.3. Sparse solution via \( \ell_1 \)-minimization

The reference feature matrix \( \Psi \in \mathbb{R}^{n \times w} \) typically consists of a large amount of modal feature columns that correspond to vast candidate damage classes that the structure may possibly incur; consequently, \( n \ll w \) and \( \Psi \in \mathbb{R}^{n \times w} \) is an over-complete reference modal feature dictionary such that the sparse representation classification problem Eq. (12) is an (highly) underdetermined linear system of equations, which is ill-posed: there exist infinite feasible solutions.

Implied in the nature of the sparse representation classification problem, the sparsest solution to Eq. (12), \( \alpha_i^* \in \mathbb{R}^w \) (\( i = 1, ..., n \)) with only one non-zero entry \( \alpha_{ji} \), is needed to determine the identity of the test feature, and can be exactly found by the well-known sparsity optimization \( \ell_0 \)-minimization program \( (P_0) \) [29],

\[
(P_0): \quad \alpha_i^* = \arg \min \|\alpha_i\|_{\ell_0} \text{ subject to } \Psi \alpha_i = \hat{\phi}_i
\]

or a stable version to account for possible noise or errors

\[
(P_1): \quad \alpha_i^* = \arg \min \|\alpha_i\|_{\ell_1} \text{ subject to } \Psi \alpha_i - \hat{\phi}_i \leq \delta
\]

in which the \( \ell_1 \)-norm is defined by \( \|\alpha_i\|_{\ell_1} = \sum_{q=1}^{w} |\alpha_{iq}| \) and the \( \ell_2 \)-norm is defined by \( \|\alpha_i\|_{\ell_2} = \sqrt{\sum_{q=1}^{w} |\alpha_{iq}|^2} \); \( \delta \) is associated with the noise level. \( (P_1) \) and \( (P_1') \) can be solved very efficiently via linear programming and convex quadratic programming techniques [52,53], respectively.

Since the underlying \( \alpha_i \in \mathbb{R}^w \) is very sparse with only one (or few) non-zero entries, it is guaranteed to be correctly recovered by \( (P_1) \) (or \( (P_1') \)) from the knowledge of the test feature \( \hat{\phi}_i \in \mathbb{R}^n \) and the predefined reference matrix \( \Psi \in \mathbb{R}^{n \times w} \).
3.4. Robust damage identification index based on SR classification

Theoretically, only one column (say, the \( i \)th column \( \hat{\phi}_i \in \mathbb{R}^n \)) of the test structure’s modal feature matrix \( \hat{\Phi} \in \mathbb{R}^{n \times n} \) is needed to set up the underdetermined sparse representation classification problem in Eq. (12), \( \hat{\phi}_i = \Psi \alpha_i \); whose, sparse solution sought by \( \ell_1 \)-minimization \( \alpha_i^* \in \mathbb{R}^n \) has only one non-zero entry, say, \( \alpha_{ij} \), indicating the test structure belongs to the \( j \)th predefined damage class. Nevertheless, in practice, noise and other factors can affect \( \alpha_{ij}^* \) (\( i = 1, \ldots, n \)), wherein other entries may only be approximately zero. Also, because damage information may be distributed among all test modal feature columns, it would be more robust to combine more sparse solutions using different test modal feature columns.

With \( n \) modal feature columns accompanying each test \( n \)-DOF structure and constructing \( n \) underdetermined linear systems of equations, there are \( n \) recovered sparse solutions \( \alpha_i^* \)’s (\( i = 1, \ldots, n \)). Alternatively a simpler and robust damage identification index can be used,

\[
e_j = \sum_{i=1}^n \| \hat{\phi}_i - \Phi_j(\alpha_i^*) \|_{\ell_2}
\]

(16)

where \( (\alpha_i^*)_j \in \mathbb{R}^n \) (\( j = 1, \ldots, N \)) is the \( j \)th segment of \( \alpha_i^* \in \mathbb{R}^n \), only consisting of \( n \) entries associated with the predesigned \( j \)th class; \( e_j \) is the recovery error associated with the \( j \)th class, summating over the whole \( n \) test modal features. This index evaluates how well the partial solution \( (\alpha_i^*)_j \) associated with the \( j \)th class reconstructs the test feature \( \hat{\phi}_i \in \mathbb{R}^n \) (\( i = 1, \ldots, n \)); obviously the smallest \( e \) among \( j = 1, \ldots, N \) determines which predefined class the test features belong to (i.e., if the test features indeed belongs to, say, the \( j \)th class, then its associated recovery error \( e_j \) would naturally be the smallest), such that the damage class of the test structure can be accordingly assigned.

3.5. Robustness of the SR classification

In practice, the test feature column may not exactly correspond to any one of the predefined reference feature columns; this may be due to model errors or simply because the reference dictionary contains no such damage class with a particular feature column. However, within this reference feature dictionary, \( \ell_1 \)-minimization can still recover the sparsest solution to the classification problem, picking the most relevant damage class and rejecting all other possible but less relevant damage classes. Illustrations will be shown in Section 5.

In case the damage class of the test structure reside outside the \( N \) individual predefined candidate damage classes, yet correspond to some combination of these individual candidate classes, \( \alpha_i \) will also be a sparse vector with a few non-zero entries, each of which also locates the identity of the individual damage class in the combination within the reference feature dictionary. In the proposed damage identification framework, this situation arises in identifying multiple damage, as will be illustrated in Section 5.1.2.3.

3.6. Limited modal feature columns and limited sensors

For the \( n \)-DOF structure (with \( n \) modes), in practice, many modes may not be excited and thus may be absent in the structural responses; as a result, CP may not extract these mode feature columns. However, as mentioned in Section 3.4, not all modal feature columns are needed for the sparse representation classification method; if only \( p \) (it can be \( p < n \)) mode feature columns are available, then in Eq. (16), \( e \) will only summate over the \( p \) modes. The performance of CP–SR under this circumstance will be shown in Sections 5.1.3, 5.2, and 6.

Another common situation is that only limited \( m \) sensors are available. If \( m < n \) or \( m \) is less than the active mode number, the modal identification problem in Eq. (3) becomes underdetermined; where, CP will “blindly” extract \( m \) mode feature columns \( \hat{\phi}_i \in \mathbb{R}^m \) (\( i = 1, \ldots, m \)) that do not correspond to the theoretical structural modeshapes—CP still functions as a dimensionality reduction tool. In this situation, the sparse classification framework still holds provided that the reference matrix is also set up along the same lines (applying CP on \( m \) sensors’ structural responses from the FEM model), i.e., \( \Psi \in \mathbb{R}^{m \times w}, \ w = m \times N \), since \( \hat{\phi}_i \in \mathbb{R}^m \) will still correspond to one of the \( \Psi \in \mathbb{R}^{m \times w} \)’s columns. However, if \( m \) is too small then the spatial resolution would be very poor for locating damage; for example, one may not expect to identify the damage location of a large-scale structure using only one or two sensors. The limited sensor situation will also be illustrated in Section 5.

4. Damage identification algorithm procedure

The proposed damage identification based on SR in the classification framework consists of two stages: locate damage (Stage 1) and assess damage severity (Stage 2). Each stage undergoes the two-step CP–SR procedure: modal feature extraction by CP (Step 1) and SR classification (Step 2); Fig. 2 shows the flowchart of the CP–SR procedure, and the whole task is implemented as follows:

Stage 1. Identification of damage location
(1a) Predefine damage classes of the reference structural FEM model with all possible damage locations (e.g., a damage class is defined as the structure damaged at one location with 50% stiffness reduction). For each damage class, simulate
the structural responses from the corresponding FEM model, and then use CP to extract the modal feature columns of this damage class. Concatenate all the modal feature columns of all candidate damage classes for the reference feature matrix or dictionary.

(2) For each test modal feature column, use \( \ell_1 \)-minimization to solve (16) and obtain the sparse solution. Calculate the recovery error of each damage class using Eq. (16). The damage class with the smallest recovery error is the one that the test structure belongs to.

Stage 2. Identification of damage extent

Repeat Stage 1, except that the predefined damage classes only consider the reference structural FEM model damaged at the identified location with all possible damage extents.

5. Numerical simulations

This section presents two numerical structure examples to investigate the ability of the developed CP–SR damage identification method; one is a discrete mass-spring damped model, and the other is a distributed-parameter fixed beam model.

5.1. Damage identification of a discrete system

The proposed CP–SR damage identification method is first verified by numerical simulations on an \( n=12 \)-DOF linear time-invariant spring-mass damped model (Fig. 3). The parameters are set as follows: the masses are \( m_1 = 2, m_2 = m_3 = \ldots = m_{11} = 1, m_{12} = 3 \), the spring stiffness are \( k_1 = k_2 = \ldots = k_{13} = 1 \), and proportional damping is added as \( C = 0.03M \). The first four natural frequencies of the structure are 0.0378, 0.0716, 0.0990, 0.1250 Hz. Damage is modeled by stiffness reduction of the spring. Impact or Gaussian white noise excitation is induced at the 6th DOF (mass) and the 200-s structural responses are obtained by Newmark–Beta algorithm with a sampling frequency of 10 Hz.
This section first shows that CP can efficiently provide reliable structural modal information in the feature extraction step. The correlation between the modal feature columns extracted by CP from the structural responses using \( m = n = 12 \) sensors and the theoretical modeshapes are measured by the modal assurance criterion (MAC). For the CP method, \( h_S = 1 \) is chosen, and \( h_L \gg h_S \) is required. The MAC results using a wide range of values of \( h_L \) between 100 and 900,000 are presented in Fig. 4, as well as the corresponding computational time. For random vibration, the results are averaged over 100 tests.

It is seen both for free or random vibration, the MAC values are very high and the computational time is quite little, both of which remain stable with varying \( h_L \) values, indicating that CP efficiently provides reliable modal features (modeshapes) without adjusting the parameters. In all damage identification tasks in this paper, \( h_L = 900,000 \) is chosen and remains unchanged. Note that the lower MAC of the 11th mode (shown as green star marker in Fig. 4) is because it is not well

![Fig. 4. The CP performance with varying long-term parameter \( h_L \) in (a) free vibration and (b) random vibration. (The star markers are the MAC values of 12 modes using the left y-axis and the circle markers are computational time using the right y-axis.)](image-url)
excited, as it is rarely present in the structural responses. The frequency and damping ratio can also be accurately identified from the simultaneously recovered time-domain modal responses [43]; they are not presented here, however, since CP–SR algorithm only needs the modal feature columns.

![Figure 5](image.png)

**Fig. 5.** The SOBI performance with varying lag parameters in (a) free vibration and (b) random vibration. (The star markers are the MAC values of 12 modes using the left y-axis and the circle markers are computational time using the right y-axis.).

<table>
<thead>
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<th>Mode</th>
<th>Free vibration</th>
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<th>Random vibration</th>
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<tr>
<td></td>
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<td>SOBI (50 lags)</td>
<td>SOBI (200 lags)</td>
<td>CP ($\nu L = 900,000$)</td>
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<td>0.9778</td>
<td>0.9939</td>
<td>0.9919</td>
<td>0.9098</td>
<td>0.9813</td>
</tr>
<tr>
<td>10</td>
<td>0.9891</td>
<td>0.9943</td>
<td>0.9985</td>
<td>0.9987</td>
<td>0.9780</td>
<td>0.9922</td>
</tr>
<tr>
<td>11</td>
<td>0.6541</td>
<td>0.6724</td>
<td>0.7988</td>
<td>0.6706</td>
<td>0.3322</td>
<td>0.4130</td>
</tr>
<tr>
<td>12</td>
<td>0.9970</td>
<td>0.9981</td>
<td>0.9990</td>
<td>0.9993</td>
<td>0.9839</td>
<td>0.9984</td>
</tr>
<tr>
<td>Computational time (s)</td>
<td>0.0427</td>
<td>0.0731</td>
<td>0.1256</td>
<td>0.0047</td>
<td>0.1684</td>
<td>0.0028</td>
</tr>
</tbody>
</table>
Comparisons with SOBI are also conducted on the same set of structural responses, and its performance with different lag parameters is shown in Fig. 5. It is seen that SOBI’s accuracy depends on the lags, and the computational time increases linearly with increasing lags. Table 1 lists some of the MAC results by CP and compared with the SOBI method.

5.1.2. Noise-free damage identification

5.1.2.1. Damage localization. The first stage is identifying damage location. Since there are 13 candidate damage locations in the structural model, \(N=13\) distinct predefined damage classes are simulated, respectively, each has 50% stiffness reduction at single location (e.g., class 1 is defined as 50% stiffness reduction at the Spring 1). The detailed predefined reference damage classes are listed in Table 2. For each of the \(N=13\) predefined damage classes, \(p = n = 12\) modal feature columns are extracted by the CP algorithm from the structural responses measured at the \(m = 12\) sensors of the structural FEM model. Concatenating all the modal feature columns of the 13 damage classes, the reference feature matrix \(\Psi_1 \in \mathbb{R}^{1 \times 156}\) (the subscript here means for Stage 1 of locating damage) is set up with \(n = 12\) and \(w = 12 \times 13 = 156\) columns and is thus of size \(12 \times 156\) (Table 2); accordingly, for example, the 1st to 12th columns in \(\Psi_1\) is \(\Phi_1 \in \mathbb{R}^{12 \times 12}\) that belongs to damage class 1, and so on. Note that \(\Psi_1\) only needs to be computed once.

Table 3 lists the test structure cases with different damage locations of different damage extents. For each test case, \(p = n = 12\) test modal feature columns (e.g., the \(i\)th column is \(\Phi_i \in \mathbb{R}^{12}\)) are also extracted by CP from the structural responses; combining with \(\Psi_1 \in \mathbb{R}^{1 \times 156}\), 12 underdetermined linear systems of equations \(\Phi_i = \Psi_1 \alpha_i\) \((i = 1, \ldots, 12)\) are established, each of which is then solved by \(\ell_1\)-minimization to obtain \(\alpha_i^\star \in \mathbb{R}^{156}\) \((i = 1, \ldots, 12)\), and the damage index is calculated using Eq. (16).

Test Cases 1–6 of single damage are considered in this subsection. Fig. 6 presents one specific example of the sparse solution to the SR classification framework: the test modal feature is the 12th column \(\hat{\Phi}_{12} \in \mathbb{R}^{12}\) of test structure of Test Case

### Table 2
Predefined damage classes of the 12-DOF structure for Stage 1.

<table>
<thead>
<tr>
<th>Class</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damage location</td>
<td>Spring 1</td>
<td>Spring 2</td>
<td>Spring 3</td>
<td>...</td>
<td>Spring 11</td>
<td>Spring 12</td>
<td>Spring 13</td>
</tr>
<tr>
<td>Damage severity</td>
<td>50%</td>
<td>50%</td>
<td>50%</td>
<td>...</td>
<td>50%</td>
<td>50%</td>
<td>50%</td>
</tr>
<tr>
<td>Obtained features</td>
<td>(\Phi_1 \in \mathbb{R}^{12 \times 12})</td>
<td>(\Phi_2 \in \mathbb{R}^{12 \times 12})</td>
<td>(\Phi_3 \in \mathbb{R}^{12 \times 12})</td>
<td>...</td>
<td>(\Phi_{11} \in \mathbb{R}^{12 \times 12})</td>
<td>(\Phi_{12} \in \mathbb{R}^{12 \times 12})</td>
<td>(\Phi_{13} \in \mathbb{R}^{12 \times 12})</td>
</tr>
<tr>
<td>Reference matrix</td>
<td>(\Psi_1 = [\Phi_1, \ldots, \Phi_3, \ldots, \Phi_{13}] \in \mathbb{R}^{1 \times 156})</td>
<td>(\Psi_1 = [\Phi_1, \ldots, \Phi_3, \ldots, \Phi_{13}] \in \mathbb{R}^{1 \times 156})</td>
<td>(\Psi_1 = [\Phi_1, \ldots, \Phi_3, \ldots, \Phi_{13}] \in \mathbb{R}^{1 \times 156})</td>
<td>...</td>
<td>(\Psi_1 = [\Phi_1, \ldots, \Phi_3, \ldots, \Phi_{13}] \in \mathbb{R}^{1 \times 156})</td>
<td>(\Psi_1 = [\Phi_1, \ldots, \Phi_3, \ldots, \Phi_{13}] \in \mathbb{R}^{1 \times 156})</td>
<td>(\Psi_1 = [\Phi_1, \ldots, \Phi_3, \ldots, \Phi_{13}] \in \mathbb{R}^{1 \times 156})</td>
</tr>
</tbody>
</table>

### Table 3
Test cases of the 12-DOF structure.

<table>
<thead>
<tr>
<th>Test case</th>
<th>Single damage</th>
<th>Multiple damage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Damage location</td>
<td>Spring 6</td>
<td>Spring 6</td>
</tr>
<tr>
<td>Damage severity</td>
<td>90%</td>
<td>50%</td>
</tr>
<tr>
<td>Extracted features</td>
<td>Each test case: (\Phi = [\Phi_1, \ldots, \Phi_3, \ldots, \Phi_{13}] \in \mathbb{R}^{12 \times 12})</td>
<td>(\Phi_6 \in \mathbb{R}^{12 \times 12})</td>
</tr>
</tbody>
</table>

![Fig. 6](image-url) The sparse solution \(\alpha_i^\star \in \mathbb{R}^{156}\) to Eq. (12) sought by \(\ell_1\)-minimization. The test feature column \(\hat{\Phi}_{12} \in \mathbb{R}^{12}\) is the 12th column (hence the subscript of \(\alpha_i^\star\) and \(\Phi_{12}^\star\)) from Test Case 2 extracted by CP, and the reference feature matrix is \(\Psi \in \mathbb{R}^{1 \times 156}\) defined in Table 2. The significant non-zero entry is at the 72nd location, which exactly corresponds to the predefined 6th damage class whose feature columns \(\Phi_6 \in \mathbb{R}^{12 \times 12}\) range from the 61st–72nd locations in the reference feature dictionary \(\Psi_1 \in \mathbb{R}^{1 \times 156}\).
and the recovered sparse solution \( \alpha_{n}^{*} \in \mathbb{R}^{156} \) to the underdetermined problem \( \hat{\Phi}_{12} = \Psi_{1} \alpha_{12} \) has a significant non-zero entry exactly at the 72nd location; as the 61st to 72nd columns of the reference feature dictionary \( \Psi_{1} \in \mathbb{R}^{12 \times 156} \) is \( \Phi_{6} \in \mathbb{R}^{12 \times 12} \) which belongs to the predefined damage class 6, then this non-zero \( \alpha_{6,12}^{*} \) directly indicates the test structure to be damage class 6 (damaged at the 6th spring, see Table 2), which exactly agrees with Test Case 2.

As mentioned, it may be more robust to use the recovery error damage index (Eq. (16)) involving more test modal features; take Test Case 2 for example, there are \( p = n = 12 \) test modal feature columns available, use each of them to obtain 12 sparse solutions along with Eq. (16) to calculate the recovery error associated with each predefined damage class. Clearly in Figs. 7 and 8(a) for Test Cases 1–6, damage is accurately located.

It is worth noting that the relaxation of the SR classification framework discussed in Section 3.5 is indicated here. Take Test Case 1 (90% damage at the 6th spring) for example, the reference feature dictionary matrix \( \Psi_{1} \) contains no exactly such a damage class (Table 2); however, SR automatically picks the most relevant class in the dictionary (50% damage at the 6th spring), rejecting all other possible but less irrelevant classes (damage at other springs). This is sufficient for locating damage while keeping the reference feature dictionary not too large by avoiding vast damage classes with a combination of different damage locations and extent; such a strategy is further beneficial to a computationally efficient identification. As expected, the recovery error (damage index) at the damage location for Test Case 2 is the smallest among Test Cases 1–3 (similarly Test Case 5 is the smallest among Test Cases 4–6), since it has exact corresponding damage class in the reference feature dictionary (Class 6 in Table 2).

![Identification results by CP-SR of (a) damage location and (b) damage extent of Test Cases 1–3 with single damage.](image)
5.1.2.2 Identification of damage severity. After damage is located, its severity can be conveniently identified as Stage 2 with similar scheme. Take Test Cases 1–3 for example, since damage has been already identified at the 6th spring in Stage 1, the predefined damage classes can only consider different damage extents all at this location, as listed in Table 4. Here, \( N = 5 \) candidate damage extents are considered, that is 10\%, 20\%, 50\%, 70\%, 90\% (arbitrarily finer levels can also be predesigned), and the reference feature matrix \( \Psi_2 \) (Stage 2) therefore comprises \( w = 12 \times 5 = 60 \) modal feature columns and is of size \( 12 \times 60 \). Same procedures are for Test Cases 4–6.

CP–SR is applied and the identification results are presented in Fig. 7(b) for Test Cases 1–3 and Fig. 8(b) for Test Cases 4–6, which indicate accurate identification of damage extent of the test structure. This example illustrates the convenience

### Table 4
Predefined damage classes of the 12-DOF structure for Stage 2 of Test Case 1–3.

<table>
<thead>
<tr>
<th>Class</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damage Location</td>
<td>Spring 6</td>
<td>Spring 6</td>
<td>Spring 6</td>
<td>Spring 6</td>
<td>Spring 6</td>
</tr>
<tr>
<td>Damage Severity</td>
<td>10%</td>
<td>20%</td>
<td>50%</td>
<td>70%</td>
<td>90%</td>
</tr>
<tr>
<td>Obtained Features</td>
<td>( \Phi_1 \in \mathbb{R}^{12 \times 12} )</td>
<td>( \Phi_2 \in \mathbb{R}^{12 \times 12} )</td>
<td>( \Phi_3 \in \mathbb{R}^{12 \times 12} )</td>
<td>( \Phi_4 \in \mathbb{R}^{12 \times 12} )</td>
<td>( \Phi_5 \in \mathbb{R}^{12 \times 12} )</td>
</tr>
<tr>
<td>Reference Matrix</td>
<td>( \Psi_2 = [\Phi_1, \ldots, \Phi_5] \in \mathbb{R}^{12 \times 60} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
of the two-stage identification method: by taking advantage of the identified damage location information, the reference
dictionary $\Psi_2$ for identification of damage extent can be dramatically small with few relevant damage classes at the
identified damage location, making the procedure more efficient.

5.1.2.3. Identification of multiple damage. In case the test structure suffers damage at multiple locations, it is supposed to set
up a new reference feature dictionary comprising all the features of various candidate damage classes, each of which
simulates damage at multiple locations. This, unfortunately, will result in an exponential increase of the reference feature
matrix size, since there are numerous possible combinations of the damage locations.

In the SR classification method, the multiple-damage class can be approximately treated as a linear combination of the
 corresponding single-damage classes. Such a strategy, as mentioned in Section 3.5, induces the sparse solution to yield
multiple non-zero entries, each identifies the individual damage class (location). Following this analysis, with the same
reference matrix $\Psi_1$ consisting of only predefined classes each with single damage location, the proposed method is able to
identify multiple damage.

Test Cases 7–8 of multiple damage (Table 3) are considered. For Stage 1 of locating damage, the above same reference
matrix $\Psi_1 \in \mathbb{R}^{12 \times 156}$ is used. After conducting CP–SR, the calculated recovery errors are presented in Fig. 9(a), indicating
successful locating multiple damage. Take Test Case 7 (damage at 3rd and 6th springs) for example, there are two
significantly smaller recovery errors as a combination of two individual damage locations: one is at the 3rd (pointing to the
3rd class) and the other at the 6th (pointing to the 6th class).

![Fig. 9. Identification results by CP–SR of (a) damage location and (b) damage extent of Test Cases 7 and 8 with multiple damage.](image)
With the identified information of damage locations, a new reference matrix $\Psi_2$ is accordingly set up for identifying the damage extent. Here, three predefined extents (20%, 50%, and 90%) are used, though finer levels can always be predesigned. Again take Test Case 7 for example, the predefined classes only consist of different damage extents at the identified damage locations at 3rd and 6th spring; this results in $3^2 = 9$ distinct combinations and thus $N=9$ reference damage classes to simulate. Therefore, $\Psi_2$ has $w = 12 \times 9 = 108$ modal feature columns and is of size $12 \times 108$. As can be seen in Fig. 9(b), the damage extents at the couple of damage locations are also successfully identified, matching the test structure cases.

5.1.2.4. Random vibration. The effectiveness of the proposed method in random vibration is also studied. Test Cases 1–3 are considered, where the test structure is subject to zero-mean Gaussian white noise excitation at the 6th DOF, and CP are used to extract the test modal feature columns from the random responses of the test structure. As mentioned, the reference feature matrix $\Psi_1$ remains unchanged, and $\Psi_2$ is identical to that in Section 5.1.2.2.

The CP–SR identification procedures are conducted. Fig. 10 shows that the proposed method shows no degradation in random vibration. This is because the CP algorithm is robust even in random vibration to extract those modal features (modeshapes), as shown in Section 5.1.1.

![Identification results by CP–SR of (a) damage location and (b) damage extent of Test Cases 1–3 with single damage in random vibration.](image)
5.1.3. Damage identification with limited features in noisy environment

Real-world structures are subject to harsh service environment, where noise is present in the measurements. To be effective for practical applications, the robustness of the proposed CP–SR method against noise must be investigated. This sub-section studies its performance in noisy environment, taking Test Case 2 for instance. Zero-mean Gaussian white noise is added to the structural responses measured from the test structure with various signal-to-noise-ratio (SNR) levels, respectively; SNR is defined as

\[
\text{SNR} = 20 \log_{10} \frac{\text{RMS (signal)}}{\text{RMS (noise)}}
\]  

(17)

The reference feature dictionary for the two stages has already been set up above. Same procedures of CP–SR are performed and the identification results are presented in Fig. 11. It is seen that the method accurately locates damage at SNR as low as 15 dB (17.8% RMS noise level), while identification of the damage severity fails at this noise level. This is because with significant noise, the modal feature columns extracted by CP from noisy structural responses can no longer reflect the structural damage features; see for example the lower mode feature columns (Fig. 12(a)) and the high mode feature (Fig. 12(b)), which have been affected by heavy noise.

However, such a problem may be solved by using limited modal feature columns in the CP–SR procedure. As the structural responses are typically dominated by lower-mode components which contain most of the damage information, it is natural to drop those higher-mode components that have been largely contaminated by noise—these are termed “noise

![Fig. 11.](image) Identification results by CP–SR of (a) damage location and (b) damage extent of Test Case 2 with single damage under different noise levels.
features”. This can be implemented automatically since the modal feature columns extracted by the CP algorithm are approximately in frequency sequence; see Ref. [43] for more details.

Therefore in the following heavily noisy environment, only the $p = 3$ lower-mode feature columns of the test structure are used and the CP–SR procedures are implemented. The results shown in Fig. 13 support such a strategy in noisy environment. There, when the noise is as heavy as comparable to the structural response (SNR = 0 dB or 100% RMS noise), CP–SR can still locate damage and assess the damage level. However, as mentioned in Section 3.4, it is usually more robust to include more modal feature columns of the test structure when noise is not heavy; Fig. 14 shows the identification results using only $p = 3$ lower-mode feature columns of the test structure in Test Cases 1–3 without noise. Clearly Test Case 3 (20% damage) is not accurately identified; contrarily, when using $p = n = 12$ test modal feature columns, this small damage can be accurately identified, as already shown in Fig. 7 and Section 5.1.2.

This occurs primarily because small damage may cause little variation of test modal features $\Phi \in \mathbb{R}^{12 \times 12}$ that is distributed among all its 12 columns. As such, it is necessary to use all its modal feature columns for the method to identify this little variation (damage)—this is exactly the original strategy of the method: the damage index (recovery error) (Eq. (16)) is summed over all modal feature columns—instead of dropping many columns which may cause loss of variation information. Therefore, in heavily noisy environment, this method with partial features ($p < n$) is incapable of identifying small damage.

5.1.4. Damage identification with limited sensors

This subsection studies the performance of CP–SR when sensors are inadequate compared to the active modes or the DOFs of the structure, i.e., $m < n$. In this situation as mentioned in Section 3.6, the system identification problem Eq. (3) becomes underdetermined, where CP extracts $p = m < n$ “blind” feature columns that do not exactly correspond to the modeshapes; most of all, the modal feature columns are degenerate for limited spatial resolution.

For illustrations, Test Cases 1–3 are used for example with the same above settings, but using only $m = 6$ sensors. Then CP extracts six modal feature columns for the test structure as well as for each damage class of the structural FEM model, respectively; i.e., $\Phi_i \in \mathbb{R}^{6 \times 12}$, $w = 6 \times 13 = 78$ for Stage 1 ($w = 6 \times 5 = 30$ for Stage 2), and the reference feature matrix $\Psi_i \in \mathbb{R}^{6 \times 78}$ in Stage 1 ($\Psi_i \in \mathbb{R}^{6 \times 30}$ for Stage 2). Fig. 15 shows that the CP–SR successfully identified the damage pattern of Test Cases 1–2 while failed on the small damage case. This indicates that CP–SR holds as long as the “blind” features reflect the structural variation due to damage as discussed in Section 3.6, and its failure on small damage case is mostly because the degenerate feature columns with inherently poor spatial resolution do not contain sufficient structural variation due to small damage.
5.2. Damage identification of a distributed-parameter beam

This section considers applying CP–SR on a two-dimensional distributed-parameter fixed beam model (Fig. 16). The parameters of the beam are set similar with those presented in Kerschen et al. [36]: Young's modulus is 200 GPa, the density is 7800 kg/m², the cross section dimension is 0.014 x 0.014 m, the length of the beam is 0.7 m, and the damping matrix is set as $C = 2M + 2 \times 10^{-6}K$. The beam is modeled by the finite element method, divided into seven elements, and each node has three DOFs: axial, vertical, and rotational, resulting in $n = 3 \times 6 = 18$ DOFs of the structure. The first four natural frequency of the structure are 148.74, 410.38, 806.85, and 1342.12 Hz. A vertical velocity in the 2nd node is induced and the structural responses only at $m = 6$ vertical sensors are computed with a sampling frequency of 10,000 Hz. The recorded time history is 1 s.

5.2.1. CP and SOBI performance

In the feature extraction step, CP extracts $p = m = 6$ modal feature columns from the structural responses. Table 5 shows that the CP mode features match the theoretical modeshapes very well. It is seen that SOBI also provides reasonable identification. The lag parameter affects the accuracy of SOBI. CP yields good accuracy and computational efficiency.

![Fig. 13. Identification results by CP–SR using only three lower-mode feature columns of (a) damage location and (b) damage extent of Test Case 2 with single damage under different noise levels.](image)
5.2.2. Damage identification via CP–SR

Damage is simulated by reducing the moment of inertia of the cross section. The predefined damage classes are introduced along the same line with those in Section 5.1: for Stage 1, class 1 with 50% damage at element 1, class 2 with 50% damage at element 2, and so on, resulting in $N = 7$ damage classes predefined for Stage 1. For each class, CP extracts six modal feature columns from the structural responses, i.e., $p = 6$, $w = 6 \times 7 = 42$, and $\Psi_1 \in \mathbb{R}^{6 \times 42}$ for Stage 1.

Three Test Cases 1–3 are considered similar with those in Section 5.1: 90%, 50%, 20% damage at Element 4, respectively. Fig. 17(a) presents the identification results of the damage location; clearly CP–SR is accurate for all the test cases. After damage location is identified at the 4th element, Stage 2 for identifying damage extents is conducted. Also $N = 5$ damage extents are considered, that is 10%, 20%, 50%, 70%, and 90%, thus $w = 6 \times 5 = 30$ and the reference feature matrix for Stage 2 is set up $\Psi_2 \in \mathbb{R}^{6 \times 30}$. As seen in Fig. 17(b), CP–SR is also able to identify the damage extents in each test case. This example shows that CP–SR is also applicable on continuous structures. In addition, the method performs well using limited modal feature columns ($p < n$).

6. Experimental study

The proposed CP–SR damage identification method is applied on an experimental building structure (Fig. 18(a)). It is a three-story structure with dominant mass on each floor. Originally four accelerometers are attached to record the structural
responses while it was subject to band-limited white noise excitation at the base: three were on the right side of each floor and one on the left side of the 1st floor. The sampling frequency was set at 200 Hz. During the test, the left column between the base and the 1st floor suddenly fractured due to damage at the weld right below the front beam of the 1st floor. However, an FEM model of the healthy structure has already been set up before the test.

A segment of 20-s recorded structural responses (damage occurred at the 10th second) from $m=3$ accelerometers on the right side and their (normalized) power spectra density (PSD) before (0–10 s) and after damage (10–20 s) are presented in Fig. 18(b). CP and SOBI are applied to extract three mode feature columns from the structural responses before and after damage, respectively. It is seen in Fig. 19 that the 1st and 2nd modeshapes before damage extracted by CP and SOBI match those by FEM.
well, and after damage they extract the 1st mode reasonably. The degradation of other mode feature columns is most probably because they were not well-excited (see the PSD of the structural responses in Fig. 18(b)). Nevertheless, these “blind” feature columns contain the structural damage information and can be used for the following step of the proposed method.

### Table 5
MAC results by SOBI and CP of the beam model in free vibration.

<table>
<thead>
<tr>
<th>Mode</th>
<th>SOBI (20 lags)</th>
<th>SOBI (50 lags)</th>
<th>SOBI (200 lags)</th>
<th>SOBI (900 lags)</th>
<th>CP ($\bar{h}_1 = 900,000$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>2</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9999</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>3</td>
<td>1.0000</td>
<td>0.9996</td>
<td>0.9994</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>4</td>
<td>0.9607</td>
<td>0.7555</td>
<td>0.7836</td>
<td>0.9842</td>
<td>0.9980</td>
</tr>
<tr>
<td>5</td>
<td>0.9939</td>
<td>0.9881</td>
<td>0.9809</td>
<td>0.9985</td>
<td>0.9993</td>
</tr>
<tr>
<td>6</td>
<td>0.9828</td>
<td>0.9663</td>
<td>0.9160</td>
<td>0.9972</td>
<td>0.9992</td>
</tr>
<tr>
<td></td>
<td>0.3891</td>
<td>0.9050</td>
<td>3.4754</td>
<td>15.1780</td>
<td>0.1607</td>
</tr>
</tbody>
</table>

**Fig. 17.** Identification results by CP–SR of the distributed-parameter beam of (a) damage location and (b) damage extent.
In building the reference feature matrix using the FEM model, damage is simulated by reducing the lateral stiffness of each floor. For Stage 1 of locating damage, $N = 3$ candidate damage locations are thus considered: class 1 with $\sim 50\%$ lateral stiffness reduction of the columns between the base and the 1st floor, and so on for classes 2 and 3. Therefore, $w = 3 \times 3 = 9$

Fig. 18. (a) The three-story experimental structure and (b) the recorded accelerations (damage at the 10th second) and their normalized power spectra density (PSD) before and after damage.

Fig. 19. The modeshapes (modal feature columns) extracted by CP compared with those by SOBI and eigenvectors of FEM: (a) before damage and (b) after damage.
and $\Psi_1 \in \mathbb{R}^{3 \times 9}$. The test feature columns $\Phi_i \in \mathbb{R}^3 \ (i = 1, 2, 3$, shown in Fig. 19(b)) extracted by CP from the real-recorded post-damage (10–20 s) are used. The identification results using $p = 3$ (including the “blind” features) or $p = 1$ (only the 1st mode) feature columns are shown in Fig. 20(a), which indicates that damage is accurately located by CP–SR on the first floor (the column between the base and the 1st floor).

After damage is located, the reference feature matrix in Stage 2 is set up, simulating $N = 5$ damage classes with different damage extents at the identified damage location (i.e., classes 1–5 are defined by lateral stiffness reduction $\sim 10\%$, $\sim 30\%$, $\sim 50\%$, $\sim 70\%$, and $\sim 90\%$, respectively, all between the base and the 1st floor). $w = 3 \times 5 = 15$ and $\Psi_2 \in \mathbb{R}^{3 \times 15}$. As seen in Fig. 20(b), with both $p = 3$ and limited $p = 1$ feature columns, the CP–SR method identifies the structure to belong to the most serious damage class (the 5th damage class), which quite matches the actual situation—the damaged column was seriously fractured.

7. Conclusions

This study develops a new two-step damage identification method via a combination of blind feature extraction and sparse representation classification framework for identification of both structural damage location and severity. In the feature extraction step, the modal features are blindly extracted by the unsupervised CP system identification algorithm. Then in the classification step, the sparsity nature implied in the classification problem itself is exploited: expressing the modal feature column of the test structure as a linear combination of the bases of the over-complete reference feature dictionary builds an underdetermined linear system of equations, whose underlying sparse representation is correctly recovered by $\ell_1$–minimization, directly assigning the most relevant damage class of the test feature (and rejecting all other possible but less relevant damage classes) so as to realize damage identification.

The capability of CP–SR is first verified by numerical simulations. Results illustrate that CP–SR is suitable for identification of small and severe single or multiple damage. Attribute to the capability of the CP algorithm, the method is also robust to random excitation.

Its effectiveness in noisy environment is also studied. It is found that using only few lower-mode features enhances the robustness of the method against heavy noise, which, in turn, is compromised in identification of very small damage. It is also found that CP–SR is incapable of identifying small damage with very limited sensors or poor spatial resolution.

CP–SR is successfully applied on an experimental building structure in both identifying damage location and estimating damage extent. The method is shown to have straightforward implementation and efficient computation as well as robustness in identification of structural damage.

References


