Cuspon and regular gap solitons between three dispersion curves

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ABSTRACT

A general model is introduced which describes a system with cubic nonlinearities, in a situation when the linear dispersion relation has three branches which nearly intersect. The system consists of two waves with a strong linear coupling between them, to which a third wave is coupled. This model has are two gaps in its linear spectrum. A nonlinear analysis is performed for zero-velocity solitons, disregarding self-phase-modulation. We find that there simultaneously exist two different families of generic gap soliton solutions: regular solitons, which may be regarded as smooth deformations of the usual gap solitons, and a family of *cuspons*, which have finite amplitude and energy, but a singularity in the first derivative at their center. Even in the limit when the linear coupling of the third waves to the first two vanishes, the family of solitons remains drastically different from that in the uncoupled system: regular solitons whose amplitude exceeds a critical value are replaced by cuspons, which in this limit take the form of *peakons*.

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Gap solitons (GS) is a common name for solitary waves in nonlinear models which feature one or more gaps in their linear spectrum [1]. A soliton may exist if its frequency belongs to the gap, as then it does not decay into linear waves.

Gaps in the linear spectrum are a generic phenomenon in two- or multicomponent systems, as intersection of dispersion curves belonging to different components is prevented by an arbitrarily weak linear coupling between the components. This alters the spectrum so that, instead of the intersection, a gap opens. Approximating the two dispersion curves, that would intersect in the absence of the coupling, by straight lines, and assuming a generic cubic nonlinearity, one arrives at a generalized massive Thirring (GMT) model, which has a family of exact GS solutions that completely fill the gap [2]. The model has a direct application to nonlinear optics, describing co-propagation of left- and right-traveling electromagnetic waves in a fiber with a resonant Bragg grating (BG). The gap solitons, first predicted theoretically, were observed in experiments with light pulses launched into a short piece of the BG-equipped fiber [3]. Gap solitons are also known in other physical situations, for instance, in stratified fluid flows, where dispersion curves pertaining to different internal-wave modes can readily intersect. Taking into regard the nonlinearity, one can easily predict the occurrence of GS in stratified fluids [4].

In this work, we aim to consider GS that may exist in a situation when the underlying system consists of three components, and there is a parametric region in which the corresponding *three* dispersion curves are close to intersection at a single point, unless linear couplings are taken into regard. Of course, the situation with three curves passing through a single point is degenerate. Our objective is to investigate GS not for this special case, but instead in its vicinity in the parameter space.

Situations of this type can be expected to occur in the above-mentioned stratified flows in some parameter regions [5], and are also known in optics. For instance, this case takes place in a *resonantly absorbing* BG, which are arranged as a system of thin (~ 100 nm) parallel layers of two-level atoms, with the spacing between them equal half the wavelength of light. This system combines the resonant Bragg reflection and self-induced transparency (SIT), see [6] and the references therein. A model describing the BG-SIT system indeed has a linear spectrum with three dispersion curves close to intersecting at one point, so that *two* gaps open in the system's spectrum. Another realization of gaps between three dispersion curves is possible in terms of stationary optical fields in a planar nonlinear waveguide equipped with BG in the form of parallel scores [7]. In this case, the resonant Bragg reflection linearly couples waves propagating in two different directions. Without going into details, we mention that the corresponding system of equations for the spatial evolution of the three fields can also provide for a situation close to the intersection of three dispersion lines at a point.

Thus, it is relevant to introduce a generic model describing a nonlinear system of three waves with linear couplings between them. We assume that the system can be derived from a corresponding Hamiltonian and confine ourselves to the case of cubic nonlinearities. Taking into regard the above restrictions, and making use of scaling invariances to diminish the number of free parameters, we arrive at a system

$$i(\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x}) + u_2 + \kappa u_3 + \alpha\left(\alpha\sigma_1|u_1|^2 + \alpha|u_2|^2 + |u_3|^2\right)u_1 = 0\,, \tag{1}$$

$$i(\frac{\partial u_2}{\partial t} - \frac{\partial u_2}{\partial x}) + u_1 + \kappa u_3 + \alpha\left(\alpha\sigma_1|u_2|^2 + \alpha|u_1|^2 + |u_3|^2\right)u_2 = 0\,, \tag{2}$$

$$i\frac{\partial u_3}{\partial t} + \kappa\,(u_1 + u_2) + \left(\sigma_3|u_3|^2 + \alpha|u_1|^2 + \alpha|u_2|^2\right)u_3 = \omega_0 u_3\,. \tag{3}$$

We use a reference frame in which the third wave u_3 has zero group velocity, and we assume full symmetry between the two waves $u_{1,2}$, following the pattern of the GMT model; in particular, the group velocities of these waves are normalized to be ± 1. Note that we assume the symmetry of the system's dispersion law $\omega = \omega(k)$ with respect to the sign of k, but not of ω. To this end, we have the parameter ω_0 in Eq. (3) which breaks the "ω-symmetry", that, unlike the "k-symmetry", does not have any natural cause to exist.

In Eqs. (1 -3), the coefficient of the linear coupling between the first two waves is normalized to be 1, while κ accounts for their coupling to the third wave, and it may always be defined to be positive. The coefficients $\sigma_{1,3}$ and α account for the nonlinear self-phase modulation (SPM) and cross-phase modulation (XPM), respectively. We keep only the most natural nonlinear SPM and XPM terms of the "optical" type, i.e., essentially the same as in the standard GMT model. In particular, possible nonlinear corrections to the linear coupling terms are known to be negligible in models of optical media.

The first step in the investigation of the system is to understand its linear spectrum. Substituting $u_{1,2,3} \sim \exp(ikx - i\omega t)$ into Eqs. (1 -3), and omitting nonlinear terms, we obtain a dispersion equation in the form

$$(\omega^2 - k^2 - 1)(\omega - \omega_0) \;=\; 2\kappa^2(\omega - 1). \tag{4}$$

If $\kappa = 0$, the third wave decouples, and the coupling between the first two waves produces a commonly known gap, so that the solutions to Eq. (4) are $\omega_{1,2} = \pm\sqrt{1+k^2}$ and $\omega_3 = \omega_0$. If $\kappa \neq 0$, the spectrum can be easily understood by treating κ as a small parameter. However, the following analysis is valid for all values in the range $0 < \kappa^2 < 1$.

First, consider the situation when $k = 0$. The solutions of Eq. (4) are then

$$\omega = 1,\, \omega \,=\, \omega_\pm \equiv\, (\omega_0 - 1)/2 \pm \sqrt{(\omega_0+1)^2/4 + 2\kappa^2}. \tag{5}$$

It is readily shown that $\omega_- < \min\{\omega_0, -1\} \le \max\{\omega_0, -1\} < \omega_+$, so that one always has $\omega_- < -1$, while $\omega_+ \lesseqgtr 1$ according as $1 - \omega_0 \lesseqgtr \kappa^2$. Next, it is readily seen that, as $k^2 \to \infty$, either $\omega^2 \sim k^2$, or $\omega \sim \omega_0$. It can also be shown that each branch of the dispersion relation generated by Eq. (4) is a monotonic function of k^2. The spectrum is shown in Fig. 1, where the panels (a) and (b) pertain, respectively, the cases $\omega_0 < 1 - \kappa^2$ with $\omega_+ < 1$, and $\omega_0 > 1$ with $\omega_+ > 1$. The intermediate case, $1 - \kappa^2 < \omega_0 < 1$, is similar to that shown in panel (a), but with the points ω_+ and 1 at $k = 0$ interchanged. When $\omega_0 < 1$, the upper gap in the spectrum is $\min\{\omega_+, 1\} < \omega < \max\{\omega_+, 1\}$, while the lower gap is $\omega_- < \omega < \omega_0$ When $\omega_0 > 1$, the upper gap is $\omega_0 < \omega < \omega_+$ and the lower one is $\omega_- < \omega < 1$.

The next step is to search for GS solutions to the full nonlinear system. In this work, we confine ourselves to the case of zero-velocity GS, substituting into Eqs. (1 - 3)

$$u_n(x,t) \;=\; U_n(x)\exp(-i\omega t)\,, \tag{6}$$

where $n = 1, 2, 3$, and it is assumed that the soliton's frequency ω belongs to one of the gaps. In fact, even the description of zero-velocity solitons is quite complicated (however, if one sets $\kappa = 0$ keeping nonlinear XPM couplings between the first two and third waves, the gap which exists in the GMT model remains unchanged, and the GS family does not suffer any conspicuous modification, in accord with the principle that nonlinear couplings cannot alter gaps or open a new one if the linear coupling is absent [8]). Explicit, and quite nontrivial results can be obtained in the case $\sigma_1 = \sigma_3 = 0$, i.e., if the SPM terms are dropped in Eqs. (1 - 3), while the XPM ones are kept. In the case of the standard two-wave GS model, this special case corresponds to the massive Thirring model proper, which is exactly integrable by means of the inverse scattering transform [9]. Although the SPM coefficient does not usually vanish in optical media, it is very plausible that this analytically tractable case correctly represents new essential features of GS in the general case.

If SPM terms are absent, $U_3(x)$ can be eliminated from Eq. (3) after the substitution of Eq. (6),

$$U_3 = \frac{\kappa\,(U_1+U_2)}{\omega_0-\omega-\alpha\,(|U_1|^2+|U_2|^2)}\,, \tag{7}$$

and then the substitution of this expression into the other two equations leads to a reduction to a system of two ordinary differential equations for $U_1(x)$ and $U_2(x)$,

$$-iU_1' = \omega U_1+U_2+\alpha^2|U_2|^2U_1+\frac{\kappa^2\,(U_1+U_2)}{\omega_0-\omega-\alpha\,(|U_1|^2+|U_2|^2)}+\frac{\alpha\kappa^2\left[\left(|U_1|^2+|U_2|^2\right)U_1+|U_1|^2U_2+U_1^2U_2^*\right]}{\left[\omega_0-\omega-\alpha\,(|U_1|^2+|U_2|^2)\right]^2}, \tag{8}$$

$$iU_2' = \omega U_2+U_1+\alpha^2|U_1|^2U_2+\frac{\kappa^2\,(U_1+U_2)}{\omega_0-\omega-\alpha\,(|U_1|^2+|U_2|^2)}+\frac{\alpha\kappa^2\left[\left(|U_1|^2+|U_2|^2\right)U_2+|U_2|^2U_1+U_2^2U_1^*\right]}{\left[\omega_0-\omega-\alpha\,(|U_1|^2+|U_2|^2)\right]^2}. \tag{9}$$

Here the prime represents d/dx. To solve these equations, we substitute $U_{1,2}=A_{1,2}(x)\exp\left(i\phi_{1,2}(x)\right)$ with real A_n and ϕ_n. After simple manipulations, it is established that $\left(A_1^2-A_2^2\right)'=0$ and $(\phi_1+\phi_2)'=0$. With regard to the condition that the soliton fields vanish at infinity, we immediately conclude that $A_1(x)=A_2(x)=\sqrt{S(x)}$. After this, there remain two equations for S and $\phi(x)=\phi_1(x)-\phi_2(x)$,

$$\frac{d\phi}{dx} = -2\left(\omega+\cos\phi+\alpha^2S+\frac{\kappa^2\,(\omega_0-\omega)\,(1+\cos\phi)}{\left[\omega_0-\omega-2\alpha S\right]^2}\right)\,, \tag{10}$$

$$\frac{dS}{dx} = -2S\sin\phi\left[1+\frac{\kappa^2}{(\omega_0-\omega-2\alpha S)}\right]\,. \tag{11}$$

These equations are Hamiltonian, as they can be represented in the form $dS/dx=\partial H/\partial\phi$, $d\phi/dx=-\partial H/\partial S$, with

$$H = 2S\cos\phi+\alpha^2S^2+2\omega S+\frac{2\kappa^2S(1+\cos\phi)}{(\omega_0-\omega-2\alpha S)}\,, \tag{12}$$

which is precisely a reduction of the Hamiltonian of the original system (1-3). For a soliton solution, $H=0$, so that the solution can be obtained in an implicit form,

$$2\cos\phi+\alpha^2S+2\omega+\frac{2\kappa^2(1+\cos\phi)}{(\omega_0-\omega-2\alpha S)} = 0. \tag{13}$$

Further, one can eliminate ϕ from Eq. (11) to obtain a single equation for S,

$$(dS/dx)^2 = 4S^2F(S), \tag{14}$$

$$F(S)=(1-\omega-\frac{1}{2}\alpha^2S)\left[2\left(1+\frac{\kappa^2}{\omega_0-\omega-2\alpha S}\right)-(1-\omega-\frac{1}{2}\alpha^2S)\right]. \tag{15}$$

For a soliton solution of (14), we need first that $F(0)>0$, which can be shown to be equivalent to requiring that ω belongs to the upper or lower gap of the linear spectrum. We note that the coupling to the third wave gives rise to the rational nonlinearity in Eq. (15), despite the fact that the underlying system (1 - 3) has only cubic nonlinear terms. Even if the coupling constant κ is small, it is clear that the rational nonlinearity may produce a strong effect in a vicinity of a critical value of the squared amplitude,

$$S_{\mathrm{cr}} = (\omega_0 - \omega)/2\alpha, \tag{16}$$

where we must have $\alpha(\omega_0 - \omega) > 0$ for $S_{\mathrm{cr}} > 0$. At $S = S_{\mathrm{cr}}$, $F(S)$ becomes singular.

The structure of the soliton depends on whether, with the increase of S, the function $F(S)$ defined by Eq. (15) first reaches zero at some $S = S_0$, or instead reaches the *singularity* at $S = S_{\mathrm{cr}}$, i.e., whether $0 < S_0 < S_{\mathrm{cr}}$, or $0 < S_{\mathrm{cr}} < S_0$. In the former case, the soliton is regular, having the amplitude $\sqrt{S_0}$, and can be obtained by a smooth deformation from the usual one known at $\kappa = 0$. In the latter case, the soliton has the amplitude $\sqrt{S_{\mathrm{cr}}}$ and is singular (it is a *cuspon*, see details below), but, nevertheless, it is an absolutely relevant solution. If $S_{\mathrm{cr}} < 0$ and $S_0 > 0$ or vice versa, then the soliton is, respectively, regular or singular, and no soliton exists if both S and S_{cr} are negative. Note that the function $F(S)$ defined by Eq. (15) has either one or three real zeros. One is $S_{01} = 2(1-\omega)/\alpha^2$, and the remaining two are roots of the quadratic equation, $(2 + 2\omega + \alpha^2 S_0)(\omega_0 - \omega - 2\alpha S_0) + 4\kappa^2 = 0$. Only the smallest positive real root, to be denoted S_{02} (if such exists), is relevant here. Note, incidentally, that $S \equiv S_0$ may be a solution to Eq. (14) only if S_0 is a double zero of $F(S)$; however it can be shown that this possibility does not occur.

It is now necessary to determine which parameter combinations in the set $(\omega, \omega_0, \alpha)$ permit the various options described above. The most interesting case occurs when $\omega_0 > \omega$ (so that ω must lie in the lower gap, see Fig. 1) and $\alpha > 0$ (the latter condition always holds in nonlinear optics). In this case it can be shown that the possible root S_{02} is not relevant, and the options are determined by a competition between S_{01} and S_{cr}. The soliton is a cuspon ($0 < S_{\mathrm{cr}} < S_{01}$) if

$$\alpha(\omega_0 - \omega) < 4(1 - \omega). \tag{17}$$

In effect, the condition (17) sets an upper bound on of α for given ω_0 and ω. It is always satisfied if $0 < \alpha < 4$. If, on the other hand, (17) is not satisfied (i.e., $0 < S_{01} < S_{\mathrm{cr}}$), then we obtain regular soliton. In a less physically relevant case, when again $\omega_0 > \omega$ but $\alpha < 0$, cuspons cannot occur (as this time $S_{\mathrm{cr}} < 0$, see Eq. (16)), and only regular solitons may exist.

Now we proceed to the case $\omega_0 < \omega$, so that ω is located in the upper gap. For $\alpha > 0$, we have $S_{\rm cr} < 0$, hence only regular solitons may occur, and indeed it can be shown that there is always at least one positive root S_0, so a regular soliton does indeed exist. If $\alpha < 0$, then we have $S_{\rm cr} > 0$, but it can be shown that, if $\omega_0 < 1 - \kappa^2$ (when also $\omega < 1$), there is at least one positive root $S_0 < S_{\rm cr}$; thus, only a regular soliton can exist in this case. On the other hand, if $\alpha < 0$ but $\omega_0 > 1 - \kappa^2$ (and then $\omega > 1$), there are no positive roots S_0, and so only cuspons occur.

Next we turn to the local structure of a cuspon near its center, when S is close to $S_{\rm cr}$. From the analysis above, we see that cuspons occur whenever ω lies in the lower gap with $\omega_0 > \omega$ and $\alpha > 0$, so that the criterion (17) is satisfied, or when ω lies in the upper gap with $1 - \kappa^2 < \omega_0 < \omega$ and $\alpha < 0$. To analyze the structure of the cuspon, we first note from Eq. (13) that, when $S = S_{\rm cr}$, one has $\cos\phi = -1$, which suggest to set

$$S_{\rm cr} - S = \delta\,\kappa^2 R, \qquad 1 + \cos\phi = \delta\,\rho, \tag{18}$$

where δ is a small positive parameter, and the stretched variables R and ρ are positive. It then follows from Eq. (13) that, to leading order in δ, $\rho = \rho_0 R$, with $\rho_0 = 2\alpha(1-\omega) - (1/2)\alpha^2(\omega_0 - \omega)$ (which is always positive for a cuspon, as follows from the above analysis). We also stretch the spatial coordinate, defining $x = \delta^{3/2}\kappa^2 y$, the soliton center being at $x = 0$. Then, on substituting the first relation from Eq. (18) into Eq. (14), we get, to the leading order in δ, an equation $R\,(dR/dy)^2 = \rho_0 S_{\rm cr}^2/\alpha^2 = K^2$, a solution to which is

$$R = (3K|y|/2)^{2/3} \tag{19}$$

Note that in the original unstretched variables, the relation (19) shows that near the cusp $S_{\rm cr} - S \approx (3K|\kappa x|/2)^{2/3}$, hence $|dS/dx| \approx (2/3)^{1/3}\,(K|\kappa|)^{2/3}/|x|^{1/3}$. Thus, as κ^2 decreases, the cusp gets localized in a narrow region where $|x| \sim \kappa^2$.

Although the first derivative in the cuspon is singular at its center, see Fig. 2(a), it is easily verified that it has a finite value of the Hamiltonian of Eqs. (1-3) (different from the reduced Hamiltonian (12)). These solitons are similar to cuspons found as exact solutions to the Camassa-Holm (CH) equation [11], which have a singularity of the type $|x|^{1/3}$ or $|x|^{2/3}$ as $|x| \to 0$. The CH equation is integrable, and is degenerate in the sense that its right-hand side has no linear terms (this makes the existence of the solution with a cusp possible). Our underlying three-component model is not degenerate in that sense;

nevertheless, the cuspon solitons are possible in it because of the model's multicomponent structure: the elimination of the third component generates a nonpolynomial nonlinearity, which gives rise to the cusp. It is noteworthy that, as well as the CH model, ours gives rise to two different families of solitons, viz., regular ones and the cuspons.

In the special case $\kappa \ll 1$, when the third component is weakly coupled to the first two ones, a straightforward perturbation analysis shows that our cuspons look like "quasi-peakons"; that is, except for the above-mentioned narrow region of width $|x| \sim \kappa^2$, where the cusp is located, they have the shape of a soliton with a discontinuity in the first derivative of $S(x)$ and a jump in the phase $\phi(x)$, which are the defining features of "peakons" ([12]). An important result of our analysis is that the family of solitons obtained in the limit $\kappa \to 0$ is drastically different from that in the model where $\kappa = 0$ from the very beginning. In the most important case with $\omega_0 > \omega$ and $\alpha > 0$, the family corresponding to $\kappa \to 0$ contains regular solitons whose amplitude is smaller than $\sqrt{S_{\mathrm{cr}}}$; however, the solitons whose amplitude at $\kappa = 0$ is larger than$\sqrt{S_{\mathrm{cr}}}$, i.e., the ones whose frequencies belong to the range (17) (note that the definition of S_{cr} does not depend on κ at all, see Eq. (16)), are replaced by these quasi-peakons which are constructed in a very simple way; drop the part of the usual soliton above the critical level $S = S_{\mathrm{cr}}$, and bring together the two symmetric parts, which remain below the critical level, see Fig. 2(b). It is interesting that peakons are known as exact solutions to a version of the CH equation slightly different from that which gives rise to the cuspons, and, in that equation, the peakons coexist with regular solitons [12].

As concerns the stability of both the regular solitons and cuspons, a very sophisticated numerical analysis is necessary, as may be conjectured on the basis of known rigorous results for the gap solitons in the GMT model [13]. However, following the analogy with the same model, it seems very plausible that in direct simulations solutions of both types will seem stable, which is sufficient to expect experimental observation of these solitons in optics and stratified flows.

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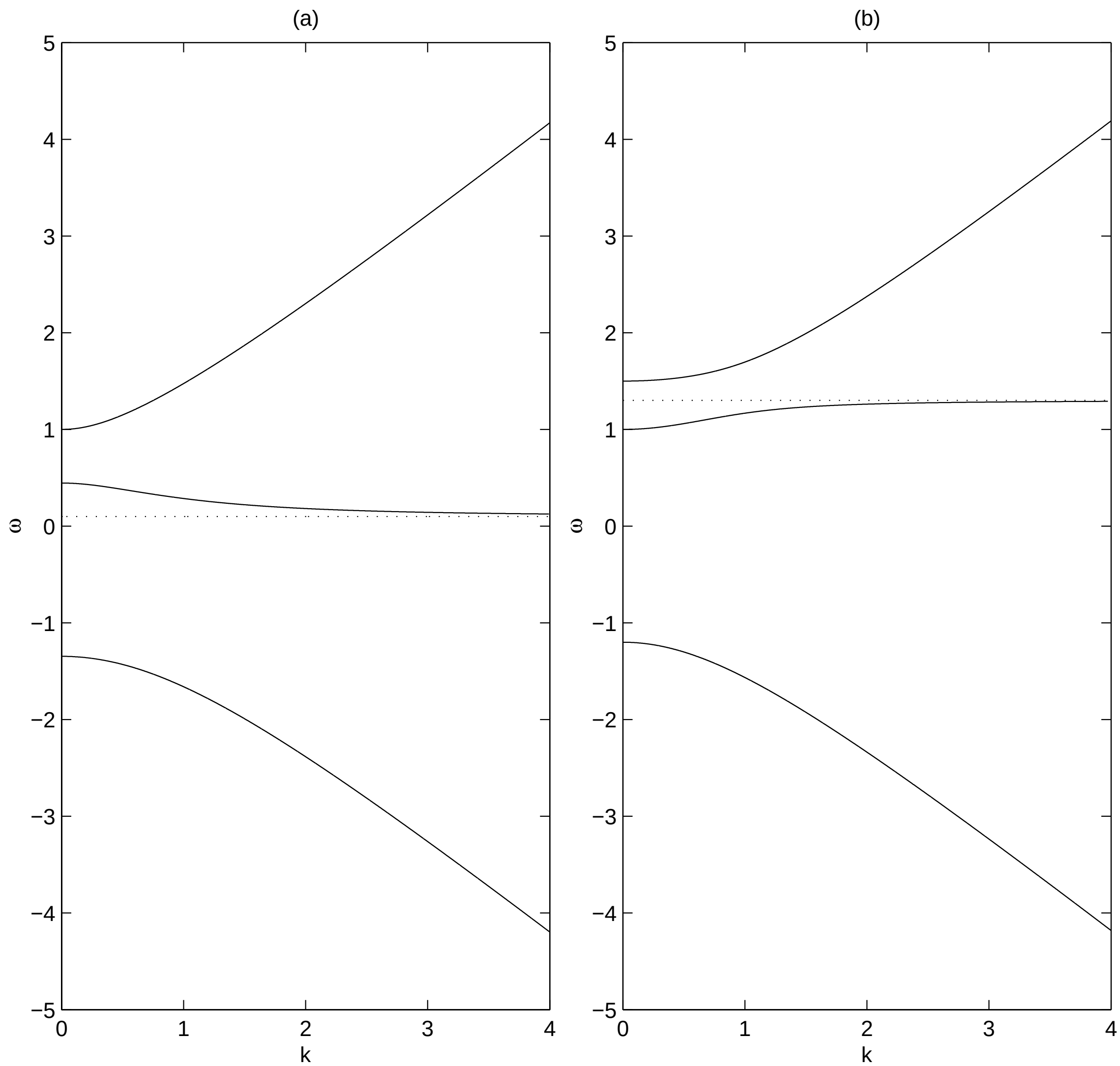

Figure 1: Dispersion curves produced by Eq. (4) in the case $\omega_0 = 0.1$ and $\kappa = 0.5$: (a) $\omega_0 < 1 - \kappa^2$; (b) $\omega_0 > 1$. The dashed line in each panel is $\omega = \omega_0$. The case with $1 - \kappa^2 < \omega_0 < 1$ is similar to case (a) but with the points ω_+ and 1 at $k = 0$ interchanged.

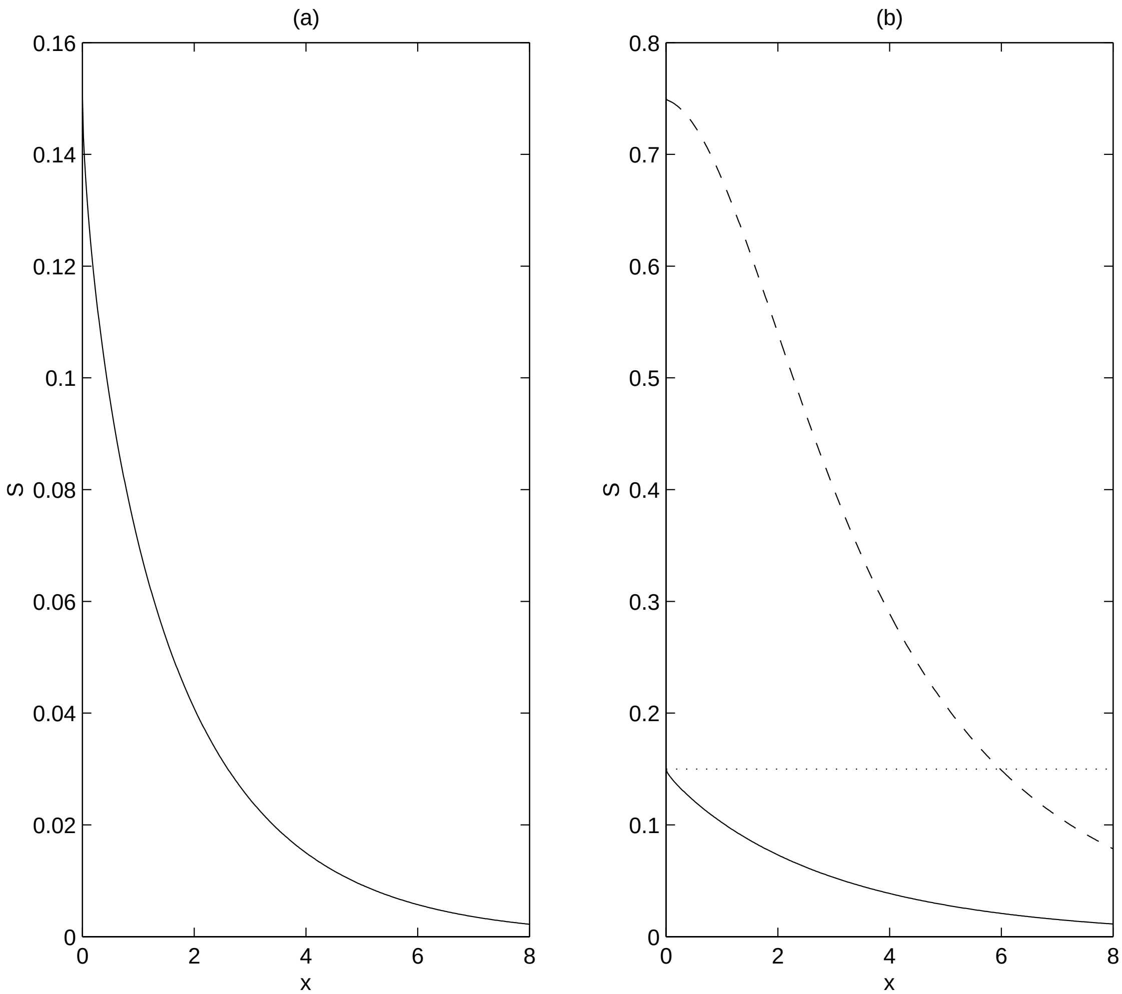

Figure 2: The shape of the cuspon at $\alpha = 2.0$, $\omega_0 = 0.1$, $\omega = -0.5$, and (a) $\kappa = 0.5$, i.e., in the general case, and (b) $\kappa = 0.1$, i.e., at small κ. In case (b) we also show the usual gap soliton (by the dashed line), for which the part above the critical value $S = S_{\rm cr}$ (shown by the dotted line) should be removed and the remaining parts brought together to form the "quasi-peakon" for $\kappa \to 0$. Note that only the region $x \geq 0$ is shown as the solutions are symmetric in x.