

# Clifford algebra: use and abuse in physics and engineering

Peter Renaud

Department of Mathematics and Statistics  
University of Canterbury  
Christchurch, New Zealand

11 January 2011

# Assumptions and Axioms: Mathematical Structures to Describe the Physics of Rigid Bodies

(arXiv:1005.0669v1 [math-ph])

Philip Butler, Niels Gresnigt, Peter Renaud  
Department of Physics and Astronomy  
Department of Mathematics and Statistics  
University of Canterbury  
Christchurch, New Zealand

- The Foundations of Physics I: From Homogeneity to Vector Spaces
- The Foundations of Physics II: Pythagoras and Clifford
- The Foundations of Physics III: Motion in 3 Spacial Dimensions
- The Foundations of Physics IV: Time as a Fourth Dimension
- The Foundations of Physics V: Algebraic Structure of  $Cl(1, 3)$

# Warning to the listener

This talk contains material of an explicit philosophical and mathematical nature.

Parental guidance is advised.

From time to time I have tried to read current work on the philosophy of science. Some of it I found to be written in a jargon so impenetrable that I can only think it aimed at impressing those who confound obscurity with profundity.

I find no help in professional philosophy. I am not alone in this; I know of *no one* who has participated actively in the advance of physics in the postwar period whose research has been significantly helped by the work of philosophers.

Stephen Weinberg “Dreams of a Final Theory” (Pantheon Books, 1992)

(But that doesn't mean that we shouldn't think about what we're doing . . .)

# Some remarks on the unreasonable effectiveness of physics in a mathematical theory.

(With apologies to Eugene Wigner.)

Problem: What constitutes a good mathematical theory?

# An example: quantum mechanics

Physical theory	Mathematical theory
System	(Complex) Hilbert space
(Pure) state	Unit vector
Observable	Hermitian operator
Transition probability	$ \langle \phi, \psi \rangle ^2$
...	...

Next problem: What is wrong with this approach?

(Hint: What are the natural operations on vectors, Hermitian operators...? What do they correspond to physically?)

# Should this be our working hypothesis?

- The natural operations on the mathematical objects **MUST** correspond to natural physical ideas.
- Any important theorem **MUST** correspond to an equally important physical result.  
(E.g. the spectral theorem.)
- Get the mathematics right and follow it to the end of the world. If it clashes with experimental results, ask the experimentalists to check their equipment.



# The real world of Clifford algebras

Clifford algebra:  
use and abuse in  
physics and  
engineering

Peter Renaud

Ignore complex algebras - they are the work of the devil.

# The Clifford algebra $Cl(1, 3)$

Begin with Minkowski space  $R^{1,3}$  with signature  $(+, -, -, -)$ . Choose an orthonormal basis  $\{e_0, e_1, e_2, e_3\}$  and denote by  $Cl(1, 3)$  the (associative) Clifford algebra constructed on  $R^{1,3}$  satisfying the conditions

$$e_0^2 = 1, \quad e_1^2 = e_2^2 = e_3^2 = -1,$$
$$e_\mu e_\nu + e_\nu e_\mu = 0, \quad (\mu \neq \nu).$$

We write  $e_{\mu\nu}$  for  $e_\mu e_\nu$  etc.

Write also the unit pseudo-scalar,  $e = e_{0123}$ . Then

$$e^2 = -1.$$

$Cl(1, 3)$  is then 16-dimensional and can be concretely represented as the algebra of all  $2 \times 2$  quaternion matrices.

$Cl(1, 3)$  splits conveniently into a direct sum of subspaces which are

- the scalars or 0-vectors (a 1-dimensional subspace),
- the 1-vectors, spanned by the basis elements  $\{e_0, e_1, e_2, e_3\}$  (4-dimensional),
- the 2-vectors (or bivectors), spanned by  $e_{\mu\nu}$ , ( $\mu < \nu$ ) (6-dimensional),
- the 3-vectors (trivectors) spanned by  $e_{\mu\nu\lambda}$ , ( $\mu < \nu < \lambda$ ) (4-dimensional),
- the pseudoscalars, spanned by  $e = e_{0123}$  (1-dimensional).

We also use the notation

$$\text{Cl}(1, 3) = \text{Cl}(1, 3)^+ \oplus \text{Cl}(1, 3)^-$$

for the decomposition of elements in  $\text{Cl}(1, 3)$  as a sum of even and odd terms.

The subspace  $\text{Cl}(1, 3)^+$  has a further decomposition which we will find useful. By considering its basis elements we see that

$$\text{Cl}(1, 3)^+ = H \oplus He$$

where  $H$  is the (4-dimensional) subspace spanned by  $1, e_{23}, e_{31}, e_{12}$ .

$H$  is isomorphic to the algebra of quaternions.

# Is signature important? $Cl(1,3)$ or $Cl(3,1)$ ?

Two answers

- no (most of the world)
- yes ...

BUT "...the existence of two Pin groups [one for each signature] is relevant to physics."

(Berg, DeWitt-Morette, Gwo and Kramer, "The Pin groups in Physics, C, P and T", Rev in Math Phys 13 (8), 2001, 953-1034)

Although the *vector spaces*  $R^{1,3}$  and  $R^{3,1}$  are equivalent, the *algebras*  $Cl(1, 3)$  and  $Cl(3, 1)$  are not.

# Some geometry in $C/1, 3$

- $e_1, e_2, e_3$  is a basis for (ordinary) vectors.
- Question. What do bivectors such as  $e_{23}$  represent?
- Answer. Oriented planes, e.g.  $e_{23}$  represents the  $y - z$  plane oriented by the direction of rotation which takes the  $y$  axis to the  $z$  axis.

(So  $e_{32}$  would be the same plane but with opposite orientation. This is why (mathematically) we define  $e_{32} = -e_{23}$ .)

- $e_{123}$  represents an oriented cube.

# Warning! Life is not quite so simple.

Question. If  $x$ ,  $y$  are 1-vectors which are not necessarily orthogonal, what is  $xy$  geometrically?

Answer.  $xy$  is scalar plus a bivector, with the scalar giving the angle between  $x$  and  $y$ .

The mathematics of Clifford algebras is telling you (and you'd better be listening) that mixing elements of different ranks (0- and 2-vectors here) is not just natural, but to be encouraged.

Later we'll see that adding 1-vectors and 3-vectors is also natural and that it is required to describe spin  $1/2$ .

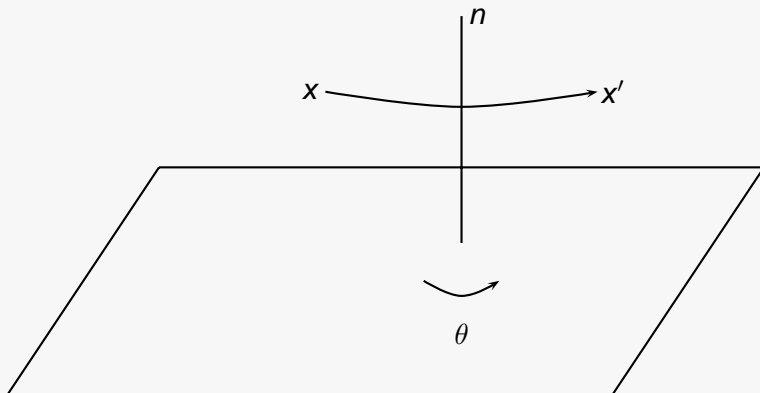
But, nature seems to draw the line at combining objects of different parities. Parity seems more important than rank.



# Two areas where Clifford algebras are powerful allies

- rotations and spin
- some fundamental equations of relativistic physics: Maxwell, Proca, Dirac and Klein-Gordon

# Rotations in $R^3$ — mathematics of spin-1



$$n = n_1 e_{23} + n_2 e_{31} + n_3 e_{12} \quad (\text{unit normal})$$

$$x' = qxq^{-1}$$

$$\text{where } q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n$$

# How then to describe spin 1/2?

## Relevant questions

- think of the irreducible representations of  $SU(2)$ .  
What are their dimensions?
- How would two spin 1/2 transformations give you spin 1?
- What about the bit that's left over?

Remember! Get the mathematical ideas right and let the mathematics do all the hard work.

# Some fundamental equations — basic ideas

- To write the Klein-Gordon, Maxwell and Proca equations in a geometric, Clifford algebra setting.
- To use the similarity between these equations as motivation for deriving the Dirac equation from the same standpoint.
- To use this common geometric language to describe solutions as well as consequences (such as conservation laws).
- To explore the idea that the various spin representations of the Lorentz group are perhaps best described in terms of this algebra and that they illustrate not only why these equations are so similar, but where they might be expected to differ.

# The Klein-Gordon equation

With respect to a fixed orthonormal basis for  $R^{1,3}$ , the four-gradient or Dirac operator is defined by

$$d = e_0 \partial_0 - e_1 \partial_1 - e_2 \partial_2 - e_3 \partial_3$$

where

$$\partial_\mu = \frac{\partial}{\partial x^\mu}.$$

Since (with the obvious notation)

$$d^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 = \square \quad (\text{the d'Alembertian operator})$$

the Klein-Gordon equation for a particle of mass  $m$  can then be written as

$$(d^2 + m^2)\psi = 0.$$

# Maxwell's equations

The free space Maxwell's equations can be expressed in the one equation

$$dF = j$$

where  $F$  is the 2-vector

$$F = E_1 e_{01} + E_2 e_{02} + E_3 e_{03} + B_1 e_{23} + B_2 e_{31} + B_3 e_{12}$$

and  $\underline{E} = (E_1, E_2, E_3)$  and  $\underline{B} = (B_1, B_2, B_3)$  are the electric and magnetic fields respectively.

$j = j_0 e_0 + j_1 e_1 + j_2 e_2 + j_3 e_3$  is the source 1-vector.

We may introduce the scalar and vector potentials via a 1-vector

$$\alpha = A_0 \mathbf{e}_0 + A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3$$

or  $(A_0, \underline{A})$ .

If we assume the Lorenz condition

$$L = \partial_0 A_0 + \operatorname{div} \underline{A} = 0$$

(which is equivalent to the requirement that  $d\alpha$  is a bivector, i.e. has no scalar part) Maxwell's equations become

$$d\alpha = -F$$

$$dF = j.$$

# Proca equations

Staying with the notation for Maxwell's equations, we have a 1-vector source term  $j$ , a 1-vector potential  $\alpha$  and a bivector  $F$  such that  $d\alpha = -F$  (for simplicity we still assume the Lorenz condition  $L = 0$ .) The Proca equations for a particle of mass  $m$  can then be written as

$$\begin{aligned}d\alpha &= -F \\dF &= m^2\alpha + j.\end{aligned}$$

This formulation immediately illustrates two ideas. The first is that the mass of a particle couples with the potential to behave like an additional source term. The second is that gauge invariance is broken by the mass term. Now  $F$  determines  $\alpha$  completely.



# The Dirac equation

We want a first order equation to “square” to the Klein-Gordon equation. As in the Maxwell and Proca cases, we choose our solutions to lie in  $\text{Cl}(1, 3)^+$ . The simplest assumption we can make, is an equation of the form

$$d\phi = \phi\lambda$$

where this constant  $\lambda$ , is a linear combination of a 1-vector and a 3-vector.

Note then that

$$\square\phi = d^2\phi = d(d\phi) = d(\phi\lambda) = (d\phi)\lambda = \phi\lambda^2$$

so that if  $\phi$  is to satisfy the Klein-Gordon equation we need  $\lambda^2 = -m^2$ .

The fact that  $\lambda^2$  is real imposes restrictions on its 1- and 3- parts. If we put

$$\lambda = \alpha + \beta \mathbf{e}$$

where  $\alpha$  and  $\beta$  are 1-vectors, a simple calculation shows that  $\alpha$  and  $\beta$  must be (real) linearly dependent. We can then write

$$\lambda = m(\cos \theta + \mathbf{e} \sin \theta)\mu$$

where  $\theta$  is real parameter,  $\mu$  is a 1-vector with  $\mu^2 = -1$ .

The choice of the 1-vector  $\mu$  is probably not too critical. For if  $\mu'$  is another 1-vector satisfying  $(\mu')^2 = -1$  there is a Lorentz transformation  $\Lambda$  (perhaps improper though) such that

$$\Lambda^{-1}\mu\Lambda = \mu'.$$

If we write  $\phi' = \phi\Lambda$  and  $\lambda' = m(\cos\theta + e\sin\theta)\mu'$  then an easy calculation shows that

$$d\phi' = \phi'\lambda'.$$

This means the the choice of  $\lambda$  is, *to that extent*, arbitrary. But a Lorentz transformation will not change the parameter  $\theta$  (and it is  $\theta$  which distinguishes particles from anti-particles).

# Dirac particle at rest

In the standard theory the Dirac equation describing spin  $\frac{1}{2}$  particles has two positive and two negative energy solutions represented by  $e^{\pm imt}$ .

In the Clifford algebra formulation, we start with  $\phi = \phi(t)$  satisfying

$$d\phi = \phi\lambda, \quad \lambda^2 = -m^2$$

and look for solutions of the form

$$\phi = \phi_0 \exp(umt)$$

where  $\phi_0$  is constant and  $u$  is even with  $u^2 = -1$ .

For simplicity, assume that  $\lambda$  is a 1-vector and write  $\lambda = m\nu$ , so that  $\nu^2 = -1$ .

Assume also (still to keep things simple) that  $\nu$  has no time component. Then we obtain the solution

$$u = \pm t e_{123} \nu$$

and

$$\phi_0 = q(1 \pm e) \quad q \in H.$$

The spin-1/2 nature of these solutions is now clear. For a fixed choice of  $u$ , the solutions are invariant under left multiplication by elements of  $H$ , and so, form a (real) 4-dimensional space invariant under the group action of  $H^1$ . This provides the natural geometric setting for describing the spin-1/2 representation of  $SO(3)$ .

The Clifford algebra approach is that the two solutions  $\phi = \phi_0 \exp(umt)$  come about from the two possible choices for  $u$ . Geometrically,  $u$  represents an oriented plane in which case  $-u$  represents the same plane but with opposite orientation. One choice (arbitrary no doubt) represents a particle and the other, its anti-particle. There is no need to introduce the idea of negative energy which is confusing at best.

More precisely, suppose as above, that  $\phi_0$  is a solution of the Dirac equation in the rest frame, corresponding to a particular choice  $u$ , i.e.

$$\phi_0 u = e_0 \phi_0 v.$$

Then it is easy to verify that  $\phi_0 e$  is also a solution but now corresponding to  $-u$ .

This argument can be extended to the general (not just rest frame) case and shows that  $\phi \rightarrow \phi e$  represents a change from a particle to its anti-particle. In fact if we have a solution  $\phi$  to the general Dirac equation  $d\phi = \phi\lambda$ , then

$$d(\phi e) = (d\phi)e = (\phi\lambda)e = (\phi e)(-\lambda)$$

so that  $\phi e$  is a solution corresponding to  $-\lambda$ .

# Thanks for listening

Clifford algebra:  
use and abuse in  
physics and  
engineering

Peter Renaud