Nonlinearity **9** (1996) 703–737. Printed in the UK

From attractor to chaotic saddle: a tale of transverse instability

Peter Ashwin†‡, Jorge Buescu‡§ and Ian Stewart

Nonlinear Systems Laboratory, Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

Received 20 October 1994
Recommended by J-P Eckmann

Abstract. Suppose that a dynamical system possesses an invariant submanifold, and the restriction of the system to this submanifold has a chaotic attractor A. Under which conditions is A an attractor for the original system, and in what sense?

We characterize the transverse dynamics near A in terms of the normal Liapunov spectrum of A. In particular, we emphasize the role of invariant measures on A. Our results identify the points at which A: (1) ceases to be asymptotically stable, possibly developing a locally riddled basin; (2) ceases to be an attractor; (3) becomes a transversely repelling chaotic saddle. We show, in the context of what we call 'normal parameters' how these transitions can be viewed as being robust. Finally, we discuss some numerical examples displaying these transitions.

AMS classification scheme numbers: 34D35, 58F12

1. Introduction

Although not a generic situation in the broadest sense, many smooth dynamical systems defined by flows or maps leave certain submanifolds of the phase space invariant. This situation can become generic if appropriate constraints are imposed—for instance, if the system has a symmetry. However, there are many other situations in which this is the case, such as in evolutionary dynamics, synchronisation of chaotic oscillators and systems with hidden symmetries.

To be specific, suppose the phase space M is a smooth finite-dimensional manifold and $f : M \to M$ is a smooth map leaving a lower-dimensional submanifold N invariant—that is, $f(N) \subset N$. The restriction $g = f_{|N} : N \to N$ determines a discrete dynamical system in its own right. We address the following problem. Suppose $A \subset N$ is an attractor for g. Under what conditions is A an attractor for f on M?

Liapunov exponents [33] provide useful quantitative indicators of asymptotic expansion and contraction rates in a dynamical system. They therefore have a fundamental bearing upon questions of stability and bifurcation. In particular, local stability of fixed points or periodic orbits is determined by whether the real parts of the eigenvalues are inside the

† Present address: Institut Non-Linéaire de Nice, 1361 route des Lucioles, 06560 Valbonne, France.
‡ Permanent address: Department of Mathematical and Computing Sciences, University of Surrey, Guilford GU2 5XH, UK.
§ Permanent address: Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1100 Lisboa, Portugal.

0951-7715/96/030703+35$19.50

unit circle or not, or equivalently by whether the corresponding Liapunov exponents are negative or not. Suppose that a hyperbolic periodic orbit P lies on an invariant submanifold N as above, and that P is attracting for $g = f_{|N}$—that is, all its Liapunov exponents are negative regarding it as an object of N. In this case, asymptotic stability of P in the global phase space M is determined by the remaining Liapunov exponents as noted in [2, 18]. Specifically, if all the remaining exponents are negative, P is asymptotically stable, whereas if there is at least one positive exponent, it cannot be asymptotically stable.

For invariant sets with more complex dynamics, the situation is much more subtle, but the considerations above provide useful guidelines for their analysis. One of the main goals of the present work is to show that local dynamic stability of chaotic attractors in invariant submanifolds may be described in terms of their *normal* Liapunov exponents—that is, the extra exponents that are introduced when considering the attractor as a subset of the global phase space M instead of just the invariant manifold N.

A semi-rigorous approach to this question is given by the following argument, expounded by the authors in [4]. Suppose $\dim M = m$, $\dim N = n < m$. Given any ergodic invariant measure μ supported in A, then for μ-almost all $x \in A$ there exist $m - n$ *normal Liapunov exponents* $\lambda_\perp^i(x)$ (that is, Liapunov exponents whose corresponding eigenspaces are not tangent to N). Indeed by ergodicity they are constant μ-a.e., allowing us to drop the dependence on x. If all $\lambda_\perp^i < 0$, then by standard results [47] we know that there exists a set $B_\mu \subset A$ of full μ-measure such that for all $x \in B_\mu$ there exists an $m-n$-dimensional local stable manifold $W^s_{loc}(x)$ which is transverse to N. Therefore, if we consider a neighbourhood U of A in M, points lying in the intersection $\bigcup_{x\in B_\mu} W^s_{loc}(x) \cap U$ will be forward-asymptotic to A. On the other hand, if at least one of these normal Liapunov exponents is positive, then for all $x \in B_\mu$ there exists a corresponding unstable manifold and A must be Liapunov unstable.

Invariant measures on A are, however, often not unique; for instance, associated to each (unstable) periodic orbit contained in an attractor is a Dirac ergodic measure whose support is the orbit. Ergodic measures are thus not unique if A contains more than one periodic orbit—as is the case with most chaotic attractors, including all (non-trivial) Axiom A attractors. Each ergodic measure carries its own Liapunov exponents, so the question of stability in transverse directions arises independently for *every* ergodic measure supported in A. For example, two different periodic orbits in A may have normal exponents of different signs. If there exists a 'natural' measure on A, then Lebesgue-almost all points have corresponding normal exponents and manifolds, but there can still be a dense set in A with the opposite behaviour.

Liapunov exponents only give a linearized picture of stability. The 'global' stability of an attractor will be typically determined by dynamics far from A, and cannot be found by a local analysis of higher order terms in some Taylor expansion. As noticed by Ott and Sommerer [36], there are at least two different types of global dynamics: intermittency, where the local unstable manifolds fold back on A (giving rise to the occurrence of 'fluctuations' away from A that are forced back again by global dynamics) or riddled basins (see Alexander *et al.* [2]), where the dense set of unstable manifolds are contained in the basin of a second, distinct attractor. We generalize these concepts by defining a 'locally riddled basin', which includes the case where the basin of A is open but local normal unstable manifolds exist in a dense set in A. As a result, we are able to put much of the discussion in [4] onto a firm theoretical footing.

The paper is organized as follows. In subsection 1.1 we introduce some definitions and terminology. section 2 describes the local theory for a manifold M with an embedded invariant submanifold N on which $f_{|N}$ has an asymptotically stable attractor A. We give a characterisation of the local normal stability of A in terms of the spectrum of normal Liapunov exponents. In section 3 we develop an appropriate bifurcation theory and show that the relevant bifurcations arise in a persistent way under additional assumptions on the dynamics on A. section 4 considers two numerical examples showing the same local behaviour but contrasting global behaviour; these examples illustrate much of the theory in sections 2 and 3. Finally, in section 5 we present some conclusions, discuss the effects of low levels of noise on the attractors and indicate some further directions of study.

1.1. Topological and measure-theoretic attractors

Many of the definitions in this section have appeared before or are standard, but we include them for completeness. Let M be an m-dimensional manifold and let $f : M \to M$ be a smooth map. We say a compact invariant set $A \subset M$ is *topologically transitive* if there exists $x \in A$ such that $\omega(x) = A$, where $\omega(x)$ is the set of limit points of the orbit $\{f^n(x)\}_{n\geqslant 0}$. Throughout the paper we always suppose A to be a compact transitive set (invariance follows from transitivity). We say A is *Liapunov stable* if for every neighbourhood U of A there exists a neighbourhood V of A such that $f^n(V) \subset U$ for all $n \in \mathbf{N}$. We say the compact invariant set is *chaotic* if A supports an ergodic measure but is not uniquely ergodic (this is a very weak definition).

Suppose M is an m-dimensional Riemannian manifold. In what follows we denote Lebesgue measure on M—which may, for instance, be derived from a volume form—by $\ell(\cdot)$.

Definition 1.1. *The* **basin of attraction** $\mathcal{B}(A)$ *is the set of points whose* ω*-limit set is contained in* A.

For non-empty A the basin $\mathcal{B}(A)$ is always non-empty because it includes A. For A to be an attractor, we require that $\mathcal{B}(A)$ is large in the appropriate sense.

Definition 1.2. A *is an* asymptotically stable attractor *if it is Liapunov stable and* $\mathcal{B}(A)$ *contains a neighbourhood of* A.

It can happen, however, that even though $\mathcal{B}(A)$ contains no neighbourhood of A (so that, in particular, $\mathcal{B}(A)$ is not open), it is still large in a measure-theoretic sense, meaning that an initial condition taken at random in a small neighbourhood of A still has a positive probability of being attracted to A. This motivates the following weaker definition (see Milnor [28]).

Definition 1.3. A *is a* **Milnor attractor** *if* $\mathcal{B}(A)$ *has non-zero Lebesgue measure and there is no compact proper subset* A' *of* A *whose basin coincides with* $\mathcal{B}(A)$ *up to a set of zero measure.*

In fact, any measure equivalent to Lebesgue is suitable for the purposes of definition 1.3 as this only distinguishes between sets of zero and non-zero measure. The next definition (cf Melbourne [24]) is a stronger version of a Milnor attractor:

Definition 1.4. A *is an* **essential attractor** *if*

$$\lim_{\delta\to 0} \frac{\ell(B_\delta(A) \cap \mathcal{B}(A))}{\ell(B_\delta(A))} = 1,$$

where $\ell(.)$ *is a Lebesgue measure on* M.

Here $B_\delta(A)$ is a δ-neighbourhood of A in M. A well known example of the distinction between asymptotically stable and Milnor attractors is the Cantor set of the logistic map at the Feigenbaum point: although Liapunov stable and attracting a set of full Lebesgue measure, it is not an asymptotically stable attractor since it is contained in the closure of the unstable periodic points. (See Buescu and Stewart [10] for a full discussion of this behaviour and its implications.)

An interesting case occurs when $\mathcal{B}(A)$ has a positive Lebesgue measure but contains no open sets, as introduced by Alexander *et al* [2].

Definition 1.5. *A Milnor attractor A has a* **riddled basin** *if for all $x \in \mathcal{B}(A)$ and $\delta > 0$ we have*

$$\ell\left(B_\delta(x) \cap \mathcal{B}(A)\right) \ell\left(B_\delta(x) \cap \mathcal{B}(A)^c\right) > 0. \tag{1}$$

If there is another Milnor attractor C such that $\mathcal{B}(A)^c$ in equation 1 may be replaced with $\mathcal{B}(C)$, then we say that the basin of A is riddled with the basin of C. If $\mathcal{B}(A)$ and $\mathcal{B}(C)$ are riddled with each other, we say they are **intertwined***.*

We next offer a variation on this definition which allows for the possibility of A having an open basin but *not* being Liapunov stable. Given a Milnor attractor A and an open neighbourhood V of A, set $U(V) = \bigcap_{n\geqslant 0} f^{-n}(V)$; that is, $U(V)$ is the set of points in V whose iterates always remain in V.

Definition 1.6. *A Milnor attractor A has a* **locally riddled basin** *if there exists a neighbourhood V of A such that, for all $x \in A$ and $\delta > 0$,*

$$\ell\left(B_\delta(x) \cap U(V)^c\right) > 0.$$

This definition states that an arbitrarily small ball centered around *any* point of the attractor contains a set of positive measure which eventually leaves V under iteration. It is a broader definition than that of a riddled basin since it allows the set which is locally repelled from A to eventually return to A. Thus it may happen in this case that $\mathcal{B}(A)$ contains an open neighbourhood of A. Note, however, that an attractor with a locally riddled basin is *never* Liapunov stable. Thus even in the case where $\mathcal{B}(A)$ is open, A is not asymptotically stable.

A chaotic invariant transitive set A is a *chaotic saddle* (Nusse and Yorke [32]) if there is a neighbourhood U of A such that $\mathcal{B}(A) \cap U \neq A$ but $\ell(\mathcal{B}(A)) = 0$. In the next definition we suppose that $N \supset A$ is an invariant n-dimensional submanifold of M, where $n < m$.

Definition 1.7. *We say that A is a* **normally repelling chaotic saddle** *if $\mathcal{B}(A) \neq A$ and $\mathcal{B}(A) \subset N$.*

Just as an asymptotically stable attractor is a special case of a Milnor attractor, a normally repelling chaotic saddle is a special case of a chaotic saddle.

In section 2.5 we shall consider the case where A has a natural invariant measure—ideally a Sinai–Bowen–Ruelle (SBR) measure [15]) for $f_{|N}$, that is, one which is absolutely continuous on unstable manifolds in N.

Throughout this paper we always work with Borel measures, that is, measures defined on the Borel σ-algebra. Given a non-empty compact set A invariant under a continuous map f, we denote by $\mathcal{M}_f(A)$ and $Erg_f(A)$ respectively the sets of invariant probability measures and ergodic measures supported in A; it is a standard fact that both $\mathcal{M}_f(A)$ and $\mathrm{Erg}_f(A)$ are non-empty (see Walters [52]).

2. Normal Liapunov exponents and stability indices

We define the Liapunov exponents (also known as characteristic exponents, see Ruelle [48]) and discuss the role of the normal Liapunov exponents in determining the stability of the attractor A for $f_{|N}$ with respect to perturbations outside N. These exponents are a.e. constant for each ergodic measure supported on A, and we refer to the set of all normal Liapunov exponents as the *normal spectrum* of A. Lemma 2.8 shows, under an appropriate assumption, that the upper and lower limits of the normal spectrum can be characterized by real numbers $\lambda_{\min}$ and $\Lambda_{\max}$. Theorem 2.12 shows that A is asymptotically stable for $\Lambda_{\max} < 0$ and Liapunov unstable for $\Lambda_{\max} > 0$. The normal spectrum also determines whether a chaotic saddle is normally repelling or not (see theorem 2.16).

For many systems of interest there is a special invariant measure supported on A that is natural with respect to Lebesgue measure on initial conditions. This is the Sinai–Bowen–Ruelle (SBR) measure introduced in section 2.5. The normal Liapunov exponents of this measure determine whether A is a Milnor attractor or a chaotic saddle (see Theorems 2.19 and 2.20).

The results of this section are collected together in proposition 2.21 which classifies the stability or instability of A according to its normal spectrum.

2.1. Normal Liapunov exponents

Let M be an m-dimensional Riemannian manifold and let $f : M \to M$ be a $C^{1+\alpha}$ map. Suppose that f leaves an n-dimensional embedded submanifold $N \subset M$ invariant, where $n < m$—that is, $f(N) \subseteq N$. It follows that for $p \in N$

$$d_p f(T_p N) \subseteq T_{f(p)} N. \tag{2}$$

Define a smooth splitting of the tangent bundle $TM = \bigcup_{p\in M} T_p M$ in a neighbourhood of N which coincides with $T_p M = T_p N \oplus (T_p N)^{\perp}$ when $p \in N$, the orthogonal complement $(T_p N)^{\perp}$ being taken with respect to the Riemannian structure in M. This is possible because N is an embedded submanifold.

Throughout this paper we suppose that $A \subset N$ is an asymptotically stable attractor for $f_{|N}$.

For $p \in A$, $0 \neq v \in T_p N$, the *Liapunov exponent* $\lambda(p, v)$ *at the point* p *in the direction of* v is defined to be

$$\lambda(p, v) = \lim_{n\to\infty} \frac{1}{n} \log \| d_p f^n(v) \|_{T_{f^n(p)} N} \tag{3}$$

if this limit exists.

Given an f-invariant measure $\mu \in \mathcal{M}_f(A)$, the multiplicative ergodic theorem of Oseledec [33] implies that the limit in (3) exists for μ–a.a. $p \in N$ and every $v \in T_p N$.

In what follows, p will denote an arbitrary point of N. We write

$$TM_n = T_{f^n(p)} M \tag{4}$$

to simplify the notation.

Given the splitting $TM_n = TN_n \oplus (TN_n)^{\perp}$, the derivative $d_p f : TM_0 \to TM_1$ block-decomposes in matrix form with respect to these subspaces as

$$\begin{pmatrix} d_p f \circ \Pi_{TN_0} & \Pi_{TN_1} \circ d_p f \circ \Pi_{(TN_0)^{\perp}} \\ 0 & \Pi_{(TN_1)^{\perp}} \circ d_p f \circ \Pi_{(TN_0)^{\perp}} \end{pmatrix} \tag{5}$$

where Π_V denotes the orthogonal projection onto the vector subspace V and use is made of (2).

Lemma 2.1. *With the above notation,*

$$\Pi_{(TN_n)^\perp} \circ d_p f^n \circ \Pi_{(TN_0)^\perp}$$
$$= (\Pi_{(TN_n)^\perp} d_{f^{n-1}(p)} f) \circ (\Pi_{(TN_{n-1})^\perp} d_{f^{n-2}(p)} f) \circ \ldots$$
$$\circ (\Pi_{(TN_1)^\perp} d_p f \circ \Pi_{(TN_0)^\perp}).$$

Proof. This results by a straightforward computation using (5) and the chain rule. □

Definition 2.2. *Consider* $p \in A,\ v \in T_pM$. *Define the* **tangential Liapunov exponent at** p **in the direction of** v *to be*

$$\lambda_\parallel(p, v) = \lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{(TN_n)} \circ d_p f^n \circ \Pi_{TN_0}(v)\|_{TM_n}. \tag{6}$$

Similarly define the **normal Liapunov exponent at** p **in the direction of** v *to be*

$$\lambda_\perp(p, v) = \lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{(TN_n)^\perp} d_p f^n \circ \Pi_{(TN_0)^\perp}(v)\|_{TM_n}. \tag{7}$$

Both definitions apply only when the limit exists.

Note that the projection Π_{TN_n} in (6) is in fact the identity operator in view of (2), and could therefore be omitted. It is included to stress the similarity of the two definitions.

Theorem 2.3. *Let* μ *be an* f*-invariant measure supported in* A*. Then, for* μ*-a.a.* $p \in A$ *and every* $v \in T_pM$*, the following hold:*

(a) $\lambda_\parallel(p, v)$ *and* $\lambda_\perp(p, v)$ *exist.*
(b) If $w = \Pi_{T_pN}(v) \neq 0$*, then* $\lambda_\parallel(p, v)$ *equals the Liapunov exponent* $\lambda(p, w)$ *for* A *considered as an attractor for* $f_{|N} : N \to N$.
(c) If $0 \neq \Pi_{(T_pN)^\perp}v$*, then there exists* $s(p) \leqslant m - n$ *such that* $\lambda_\perp(p, v)$ *takes one of the values*

$$\lambda_\perp^1(p) < \lambda_\perp^2(p) < \ldots < \lambda_\perp^{s(p)}(p).$$

(d) The function $p \mapsto s(p)$ *is* f*-invariant and* μ *-measurable.*
(e) There exists a filtration

$$\{0\} = V^0(p) \subset V^1(p) \subset \ldots \subset V^{s(p)}(p) = (T_pN)^\perp$$

with

$$\Pi_{(T_pN)^\perp} \circ d_p f(V^i(p)) \subset V^i(f(p))$$

such that $\lambda_\perp(p, v) = \lambda_\perp^k(p)$ *for* $v \in V^k(p) \setminus V^{k-1}(p)$.

Proof. By (2)

$$\Pi_{TN_n} \circ d_p f^n \circ \Pi_{TN_0}(v) \equiv d_p f^n \circ \Pi_{TN_0}(v).$$

The first statement of (a) then follows (with the proviso $\lambda_\parallel = -\infty$ if $v \in \ker \Pi_{TN_0}$) because definitions (6) and (3) coincide: note however that, by definition of induced metric in a Riemannian submanifold, $\|.\|_{T_pM} \equiv \|.\|_{T_pN}$ if $v \in T_pN$. (b) follows since the projection $\Pi_{TN_0} : TM_0 \to TN_0$ is surjective.

The second statement in (a), as well as those in (c), (d), (e), are proved in the following way. If $v \in \ker \Pi_{(TN_0)^\perp}$ then $\lambda_\perp(p, v)$ exists trivially. Otherwise, set $w = \Pi_{(TN_0)^\perp}v$. Define a linear map $L_p : (TN_0)^\perp \to (TN_1)^\perp$ by

$$L_p(w) = \left(\Pi_{(TN_1)^\perp} \circ d_p f\right)(w), \qquad w \in (TN_0)^\perp$$

and set

$$L_p^n(w) = L_{f^{n-1}(p)} \circ L_{f^{n-2}(p)} \circ \ldots \circ L_p(w).$$

By lemma 2.1 we have

$$L_p^n(w) = \left(\Pi_{(TN_n)^\perp} \circ d_p f^n\right)(w).$$

By Oseledec's multiplicative ergodic theorem the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|L_p^n(w)\|$$

exists for μ-a.a. $p \in A$ and all $w \neq 0$ in $(TN_0)^\perp$ (or equivalently all $v \notin \ker \Pi_{TN_0}$ in TM_0), and it takes one of the values $\lambda_\perp^1(p) < \ldots < \lambda_\perp^{s(p)}(p)$. Statements ($c$), ($d$) and ($e$) are now immediate consequences of Oseledec's theorem. □

If we take μ ergodic in theorem 2.3, the invariant functions $\{\lambda_\perp^i(p), s(p)\}$ are μ-a.e. constant. The normal Liapunov exponents for $v \notin \ker(TN_0)^\perp \equiv TN_0$ can only take the $s \leqslant m - n$ values $\lambda_\perp^1 < \ldots < \lambda_\perp^s$ which are independent of p (and the corresponding filtrations invariant under $d_p f$). In this case we denote these normal Liapunov exponents by $\lambda_\perp^1(\mu) < \ldots < \lambda_\perp^s(\mu)$.

2.2. *The Normal Liapunov spectrum*

Definition 2.4. *The* **measurable normal Liapunov spectrum** $S_n(A)$ **of** A *is*

$$S_n(A) = \bigcup_{\mu\in \mathrm{Erg}_f(A)} \{\lambda_\perp^i(\mu)\},$$

where $\mathrm{Erg}_f(A)$ *is the set of all ergodic invariant probability measures supported in* A*.*

Remark 2.5. *We call the spectrum 'measurable' because in general the limit in the definition of Liapunov exponents exists only in a set* $B \subset A$ *such that* $\mu(B) = 1$ *for all invariant measures* $\mu \in \mathcal{M}_f(A)$*. In a general chaotic attractor this set may be topologically small; see, however, remark 2.10.*

In the rest of this paper we refer to $S_n(A)$ *as the normal spectrum.*

Lemma 2.6. *The normal spectrum* $S_n(A)$ *is bounded above.*

Proof. For all $p \in A$, all $v \in TM_0$, lemma 2.1 implies that

$$\begin{aligned}\frac{1}{n} &\log \|\Pi_{(TN_n)^\perp} \circ d_p f^n \circ \Pi_{(TN_0)^\perp}(v)\| \\ &= \frac{1}{n} \log \|\Pi_{(TN_n)^\perp} d_{f^{n-1}(p)} f \circ \ldots \circ \Pi_{(TN_1)^\perp} d_p f\, \Pi_{(TN_0)^\perp(v)}\| \\ &\leqslant \frac{1}{n} \log \|\Pi_{(TN_n)^\perp} d_{f^{n-1}(p)} f\, \Pi_{(TN_{n-1})^\perp}\| \cdot \ldots \cdot \|\Pi_{(TN_1)^\perp} d_p f\, \Pi_{(TN_0)^\perp}\| \cdot \|v\| \\ &\leqslant \frac{1}{n} \log(\gamma^n \|v\|)\end{aligned}$$

where $\gamma = \max_{p\in A\,:\,|p|=1} \|\Pi_{(TN_1)^\perp} d_p f\, \Pi_{(TN_0)^\perp}(v)\|$. Therefore, for all $p \in A$ and $v \in TM_0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\Pi_{(TN_n)^\perp} d_p f^n(v)\| \leqslant \gamma.$$

In particular, the normal spectrum $S_n(A)$ is bounded above by γ. □

From now on we refer to $\Pi_{(TN_1)^\perp} \circ d_p f \circ \Pi_{(TN_0)^\perp} : (TN_0)^\perp \to (TN_1)^\perp$ at a point $p \in N$ as the *normal derivative of* f *at* p and denote it by $d_p^\perp f$.

Remark 2.7. *If there are critical points, the normal derivative may be singular; in this case, the spectrum need not be bounded from below. For instance, suppose A has a fixed point p at which the normal derivative is singular; then $\lambda_{\delta_p} = -\infty$. However, it is also possible that even if there are critical points for the normal derivative, the spectrum may be bounded from below.*

Theorem 2.8 below shows that non-singularity of the normal derivative is sufficient for the spectrum to be bounded below. This condition is automatically satisfied, for instance, if f is a diffeomorphism.

We define

$$\lambda_{\min} = \inf(S_n(A)) \quad \text{and} \quad \Lambda_{\max} = \sup(S_n(A)).$$

Given the normal bundle $NA = \{(p, v) \in TM : p \in A, v \in (TN_0)^{\perp}\}$ we define the *normal sphere bundle* $SA = \{(p, v) \in NA : \|v\| = 1\}$.

Theorem 2.8. (characterization of spectrum) *Suppose that for all $p \in A$ the normal derivative is injective. Then*

(a) $$\inf_{(p,v)\in SA} \liminf_{n\to\infty} \frac{1}{n} \log \|\Pi_{(TN_n)^{\perp}} d_p f^n(v)\| = \inf_{\mu\in \mathrm{Erg}_f(A)} \lambda_{\perp}^1(\mu) = \lambda_{\min},$$

(b) $$\sup_{(p,v)\in SA} \limsup_{n\to\infty} \frac{1}{n} \log \|\Pi_{(TN_n)^{\perp}} d_p f^n(v)\| = \sup_{\mu\in \mathrm{Erg}_f(A)} \lambda_{\perp}^s(\mu) = \Lambda_{\max}.$$

Remark 2.9. *Note that the projection $\Pi_{(TN_0)^{\perp}}$ is omitted in the statement and proof of theorem 2.8 since $v \in (TN_0)^{\perp}$ throughout.*

Proof. Injectivity of the normal derivative and compactness of A imply that

$$\inf_{\|v\|=1} \|\Pi_{(TN_1)^{\perp}} \circ d_p f(v)\|$$

is uniformly bounded away from zero. Define the induced tangent map $\widehat{Tf} : SA \to SA$ by

$$\widehat{Tf}(p, v) = \left(f(p), \frac{\Pi_{(TN_1)^{\perp}} \circ d_p f(v)}{\|\Pi_{(TN_1)^{\perp}} \circ d_p f(v)\|} \right). \tag{8}$$

The requirement of an injective, or non-singular, normal derivative ensures that this map is well defined for all $(p, v) \in SA$, since the denominator in (8) never vanishes.

Define $\varphi : SA \to \mathbf{R}$ by

$$\varphi(p, v) = \log \|\Pi_{(TN_1)^{\perp}} \circ d_p f(v)\| \tag{9}$$

and note that

$$\sum_{i=0}^{n-1} \varphi\left((\widehat{Tf})^i (p, v) \right) = \log \|\Pi_{(TN_n)^{\perp}} \circ d_p f^n(v)\|. \tag{10}$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\Pi_{(TN_n)^{\perp}} \circ d_p f^n(v)\| = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi\left((\widehat{Tf})^i (p, v) \right). \tag{11}$$

The right-hand side of (11) is a Birkhoff sum. Given (p, v), choose a subsequence $\{n_k\}_{k\geqslant 0}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi\left((\widehat{Tf})^i (p, v) \right) = \lim_{k\to\infty} \frac{1}{n_k - 1} \sum_{i=0}^{n_k-1} \varphi\left((\widehat{Tf})^i (p, v) \right).$$

Since SA is compact, the space of probability measures $\mathcal{M}(SA)$ is compact as well. Therefore, the sequence of probability measures $m_k = \frac{1}{n_k}\sum_{i=0}^{n_k}\delta_{(\widehat{Tf})^i(p,v)}$ has a subsequence converging to a $\widehat{Tf}$-invariant measure $\tilde{m} \in \mathcal{M}_{\widehat{Tf}}(SA)$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi\left((\widehat{Tf})^i(p,v)\right) = \int_{SA}\varphi\,\mathrm{d}\tilde{m}.$$

Thus for any $(p,v)\in SA$

$$\limsup_{n\to\infty}\frac{1}{n}\log\|\Pi_{(TN_n)^\perp}\circ d_pf^n(v)\| \leqslant \sup_{m\in\mathcal{M}_{\widehat{Tf}}(SA)}\int_{SA}\varphi\,\mathrm{d}m. \tag{12}$$

By ergodic decomposition we can replace the right-hand side by

$$\sup_{m\in\mathrm{Erg}_{\widehat{Tf}}(SA)}\int_{SA}\varphi\,\mathrm{d}m.$$

We now relate the integrals $\int_{SA}\varphi dm$ to the Liapunov exponents in A. To do this, note that in the commutative diagram

$$\begin{array}{ccc} SA & \xrightarrow{\widehat{Tf}} & SA \\ \pi_1\downarrow & & \downarrow\pi_1 \\ A & \xrightarrow{f} & A \end{array} \tag{13}$$

where π_1 is projection onto the first factor, an ergodic $\widehat{Tf}$-invariant measure m projects via $\mu = m\circ\pi_1^{-1}$ to an ergodic f-invariant measure on A. Moreover, this projection is onto (consider p generic for μ and any $\|v\|=1$; one can construct an invariant m as above; any ergodic component of this projects onto μ).

Given any ergodic m define $\mu = m\circ\pi_1^{-1}$. For m-almost all $(p,v)\in SA$ we have

$$\lambda(p,v) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varphi((\widehat{Tf})^k(p,v)) = \int_{SA}\varphi\mathrm{d}m.$$

Since this holds for a full μ-measure set of p, Oseledec's theorem says that $\lambda(p,v) = \lambda_\perp^i(\mu)$ for some i. Thus,

$$\sup_{\mu\in\mathrm{Erg}_f(A)}\{\lambda_\perp^1(\mu)\} = \sup_{m\in\mathrm{Erg}_{\widehat{Tf}}(SA)}\int_{SA}\varphi\,\mathrm{d}m.$$

This together with (12) and the definition of $\lambda_\perp^s(\mu)$ shows that

$$\sup_{(p,v)\in SA}\limsup_{n\to\infty}\frac{1}{n}\log\|\Pi_{(TN_n)^\perp}\circ d_pf^n\| = \sup_{\mu\in\mathrm{Erg}_f(A)}\{\lambda_\perp^s(\mu)\}.$$

A similar argument shows that

$$\inf_{(p,v)\in SA}\liminf_{n\to\infty}\frac{1}{n}\log\|\Pi_{(TN_n)^\perp}\circ d_pf^n\| = \inf_{\mu\in\mathrm{Erg}_f(A)}\{\lambda_\perp^1(\mu)\}.$$

An argument analogous to that on Lemma 2.6 shows that the normal spectrum is contained in a compact interval.

Remark 2.10. *If there exists a point $p \in A$ with a dense orbit such the normal Liapunov exponents do not exist, i.e. if the set of limit points of*

$$\frac{1}{n}\log\|d_p f^n(v)\| \tag{14}$$

is non-trivial for some $v \in TN_0$, then it may be shown that normal Liapunov exponents only exist in a set of first Baire category in A; see [14]. However, theorem 2.8 implies that for all $p \in A$ and all non-zero $v \in TM_0$ the set of limit points of (14) is always contained in $[\lambda_{\min}, \Lambda_{\max}]$. *Therefore, the spectrum bounds the possible asymptotic growth rates in the normal direction.*

2.3. Stability indices

Given an ergodic invariant probability measure $\mu \in \mathrm{Erg}_f(A)$, the normal Liapunov exponents $\lambda_\perp^1(\mu) < \ldots < \lambda_\perp^s(\mu)$ exist and are constant in a set B_μ of full μ-measure. The normal stability or instability of this set is determined by the sign of the largest normal Liapunov exponent. We therefore associate to every ergodic μ its *normal stability index* Λ_μ in the following way.

Definition 2.11. *Let μ be an f-invariant ergodic probability measure supported in A, with normal Liapunov exponents $\lambda_\perp^1(\mu) < \ldots < \lambda_\perp^s(\mu)$. The* **normal stability index** Λ_μ *of μ is*

$$\Lambda_\mu = \lambda_\perp^s(\mu).$$

Note that $\Lambda_{\max} = \sup_{\mu\in\mathrm{Erg}_f(A)}\{\Lambda_\mu\}$. The definition is useful because of the following result.

Theorem 2.12. *Suppose A is an asymptotically stable attractor for $f_{|N}$ and the normal derivative of f is injective on A. Then:*

(a) If $\Lambda_{\max} < 0$ then A is an asymptotically stable attractor for f in M.
(b) If $\Lambda_{\max} > 0$ then A is Liapunov-unstable.

The proof of theorem 2.12 is divided into several steps. Firstly, by the asymptotic stability condition, we may take a compact neighbourhood $W \subset N$ of A in the relative topology such that $f(W) \subset W$ and $\bigcap_{n=0}^{\infty} f^n(W) = A$ (see e.g. [28]).

Define the *normal bundle* of W to be

$$TW^\perp = \bigcup_{p\in W}(T_pN)^\perp.$$

An element of $TW^\perp$ is thus (p, v), where $p \in W$, $v \in (T_pN)^\perp$.

Consider the function $g : TW^\perp \to M$ given by

$$g(p, v) = \exp_p(v), \tag{15}$$

where $\exp : TM \to M$ is the exponential map. By the compactness of W, there exists $\delta > 0$ such that $g(p, v)$ is defined for all $(p, v) \in TW^\perp$ if $\|v\| < \delta$. In other words, g is defined in a neighbourhood of the zero-section of $TW^\perp$.

Lemma 2.13. (see e.g [50] vol 1, lemma 19 and theorem 20) *There exists $\epsilon > 0$ such that g is a diffeomorphism between* $\tilde{W}_\epsilon \stackrel{\text{def}}{=} \{(p, v) \in TW^\perp : \|v\| \leqslant \epsilon\}$ *and a compact neighbourhood of A in M.*

Remark 2.14. *By a standard property of the exponential map, $g(p, 0)$ is the identity—that is, for all $(p, 0) \in TN^\perp$, $g(p, 0) = p \in M$.*

Denoting the image of $\tilde{W}_\epsilon$ under g by W_ϵ, we conclude that there is a $\delta > 0$ such that, defining

$$\tilde{f} = g^{-1} \circ f \circ g, \tag{16}$$

the diagram

$$\begin{array}{ccc} \tilde{W}_\epsilon & \xrightarrow{\tilde{f}} & \tilde{W}_\delta \\ g \downarrow & & \downarrow g \\ W_\epsilon & \xrightarrow{f} & W_\delta \end{array} \tag{17}$$

commutes.

Proof of theorem 2.12. ***(a)*** Taking the derivative of (16) at $(p, 0)$ gives $d\tilde{f}_{(p,0)} = d_p f$ where the obvious identifications of tangent spaces are made. Therefore, by induction

$$\Pi_{(TN_n)^\perp} \circ d_{(p,0)} \tilde{f}^n \circ \Pi_{(TN_0)^\perp} = \Pi_{(TN_n)^\perp} \circ d_p f^n \circ \Pi_{(TN_0)^\perp}, \tag{18}$$

that is, f and $\tilde{f}$ have the same normal derivatives. As $g(p, 0)$ is the identity, the normal Liapunov exponents of a point $p \in A$ under f and $\tilde{f}$ coincide. In particular, the normal spectra $S_n(A)$ of A under f and $\tilde{f}$ are equal.

From now on, denote by

$$d_p^\perp \tilde{f} = \Pi_{(TN_1)^\perp} \circ d_{p,0} \tilde{f} \circ \Pi_{(TN_0)^\perp}$$

the normal derivative of $\tilde{f}$ at $(p, 0) \in W$. Let $\epsilon(n) > 0$ and $\delta > 0$ be such that g is a diffeomorphism and diagram (17) commutes. For $0 < \alpha < \min\{\epsilon, \delta\}$, take the compact neighbourhood $\tilde{W}_\alpha$ in $TW^\perp$.

Fix λ such that $\Lambda_{\max} < \lambda < 0$. By compactness of W, continuity of $d_p^\perp \tilde{f}^n$ and theorem 2.8 we conclude that there exists n_0 such that $n \geqslant n_0$ implies, for all $p \in W$, all $v \in (TN_0)^\perp$ with $v \neq 0$,

$$\|d_p^\perp \tilde{f}^n(v)\| < \mathrm{e}^{n\lambda}\|v\|.$$

Choose n such that $\mathrm{e}^{(n-1)\lambda} < \frac{1}{3}$ and denote

$$(p_n, v_n) = \tilde{f}^n(p, v).$$

Thus $p_n = \pi_1 \tilde{f}^n(p, v)$, $\quad v_n = \pi_2 \tilde{f}^n(p, v)$, where π_1, π_2 are the projections on the first and second factors of $TW^\perp$, respectively.

Our choice of $W \subset N$ implies that $\tilde{f}(W) \subset W$, where inclusion is strict. Set $a = d_H(\tilde{f}^n(W), W) > 0$, where d_H is the Hausdorff distance between compact sets in $TW^\perp$. We may take $\alpha > 0$ sufficiently small so that

$$\rho(\tilde{f}^n(p, v), \tilde{f}^n(p, 0)) < a/3 \tag{19}$$

for all $(p, v) \in \tilde{W}_\alpha$, where ρ denotes the Riemannian metric induced on $TW^\perp$ by that of M through the diffeomorphism g, and

$$\rho(\tilde{f}^n(p, v), \tilde{f}(p, 0)) < \mathrm{e}^{-\lambda/2}\|d_p^\perp \tilde{f}^n(v)\|. \tag{20}$$

Condition (19) may be met by uniform continuity of $\tilde{f}^n$, and condition (20) by continuity of $d_p^\perp f$.

With these choices,

$$\begin{aligned}\|v_n\| &\leqslant \rho(\tilde{f}^n(p,v), \tilde{f}^n(p,0)) \\ &< \mathrm{e}^{-\lambda/2}\|d_p^\perp \tilde{f}^n(v)\| \\ &< \mathrm{e}^{-\lambda/2}\mathrm{e}^{n\lambda}\|(v)\| \\ &< \mathrm{e}^{(n-1)\lambda}\|v\| \\ &< \frac{1}{3}\|v\|. \end{aligned} \tag{21}$$

Equation (19) implies that $\tilde{f}^n(\tilde{W}_\alpha) \subset V_{a/2}(\tilde{f}^n(W))$, where $V_\delta(\tilde{f}^n(W))$ is the δ-neighbourhood of $\tilde{f}^n(W)$. In particular this implies $p_n = \pi_1 \tilde{f}^n(p,v) \in W$. These conditions together imply that $(p_n, v_n) = \tilde{f}^n(p,v) \in \tilde{W}_\alpha$ for all $(p,v) \in \tilde{W}_\alpha$, or $\tilde{f}^n(\tilde{W}_\alpha) \subset \tilde{W}_\alpha$. Setting $K_\alpha = \bigcup_{j=0}^{n-1} \tilde{f}^j(\tilde{W}_\alpha)$, we conclude that K_α is compact and forward-invariant. By equation 21 we see that for all $(p,v) \in K_\alpha$, $\|v_n\| \to 0$, which implies that

$$\bigcap_{n\geqslant 0} \tilde{f}^n(K_\alpha) \subset \bigcap_{n\geqslant 0} \tilde{f}^n(W) = A,$$

where the last equality is due to asymptotic stability of A. Therefore A is an asymptotically stable attractor for the map $\tilde{f}$.

Translating these results via the diffeomorphism g into M, we conclude that A is an asymptotically stable attractor for the map $f : M \to M$.

Proof of part (b). Suppose that $\Lambda_{\max} > 0$. Then there exists an ergodic invariant measure μ supported in A with $\Lambda_\mu > 0$ by theorem 2.8. Fix $0 < \lambda < \Lambda_\mu$. By theorem 2.3, for μ-a.a. $p \in A$ there exists a filtration

$$\{0\} = V^0 \subset V^1 \ldots \subset V^l = (TN_0)^\perp$$

of $(TN_0)^\perp$ with

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{(TN_n)^\perp} \circ d_p f^n(v)\| = \Lambda_\mu$$

for $v \in V^l \setminus V^{l-1}$.

As for (a), we consider the commutative diagram (17), the diffeomorphism g and equation (18). Define

$$M_n = \{(p,0) \in W_\epsilon : m \geqslant n \Rightarrow \frac{1}{m} \log \|d_p^\perp \tilde{f}^m(v)\| > \lambda, \ \|v\| = 1, v \in V^l \setminus V^{l-1}\}.$$

By definition, $\mu(M_n) \to 1$ as $n \to \infty$; we take n large enough that $M_n \neq \emptyset$. Therefore, if $p \in M_n$,

$$\|d_p^\perp \tilde{f}^m(v)\| > \mathrm{e}^{m\lambda}\|v\|$$

for $v \in V^l(p) \setminus V^{l-1}(p)$, all $m \geqslant n$.

Using the same notation as in part (a), choose $n > 0$ such that

$$\|d_p^\perp \tilde{f}^n(v)\| > \mathrm{e}^{n\lambda}\|v\|$$

with $\mathrm{e}^{(n-1)\lambda} > 2$ if $v \in V^l \setminus V^{l-1}$. Choose α small enough that

$$\|v_n\| > \mathrm{e}^{-\lambda/2}\|d_p^\perp \tilde{f}^n(v)\|$$

for all $(p,v) \in K_\alpha$. Then

$$\|v_n\| > \mathrm{e}^{-\lambda/2}\mathrm{e}^{n\lambda}\|v\| > 2\|v\|$$

for all $v \in V^l \setminus V^{l-1}$. Thus, for any α, $\bigcap_{n=0}^{\infty} \tilde{f}^{-n}(K_\alpha)$ cannot contain a neighbourhood of A. As the $\{K_\alpha\}_{\alpha>0}$ form a basis of neighbourhoods of W in $TW^\perp$, this implies that there is no neighbourhood V of A such that $\bigcap_{n=0}^{\infty} \tilde{f}^{-n}(V)$ contains a neighbourhood of A. In other words, A is Liapunov unstable in $TW^\perp$. Applying the diffeomorphism g to W_ϵ, we conclude that A is Liapunov unstable in M for the original map f. □

We next give a sufficient condition for the basin of a (Milnor) attractor to be locally riddled.

Given an ergodic measure μ, we denote the set of (measure-theoretical) *generic points* of μ (see e.g. Denker *et al* [14]) by G_μ. That is,

$$G_\mu = \{x \in A : \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \to \mu\}$$

where convergence is in the weak* topology. For any $\alpha > 0$ define

$$G_\alpha = \bigcup_{\substack{\mu \in \mathrm{Erg}_f(A) \\ \Lambda_\mu \geqslant \alpha}} G_\mu.$$

Theorem 2.15. *Suppose A is a Milnor attractor for f, the normal derivative of f is injective on A and $\Lambda_{\max} > 0$. If there exists $\alpha > 0$ such that G_α is dense in A, then A has a locally riddled basin.*

Proof. We set $\beta = \alpha/2$ and argue by contradiction. Suppose A does not have a locally riddled basin. Setting $U(V) = \bigcap_{n=0}^{\infty} f^{-n}(V)$ as in definition 1.6, this means that for every neighbourhood V of A there exist $p \in A$ and $\delta > 0$ such that $\ell(B_\delta(p) \cap U(V)^c) = 0$. As we want to consider arbitrarily small compact neighbourhoods of A in M, we may restrict attention to those contained inside neighbourhoods where g as in lemma 2.13 and in diagram (17) is a diffeomorphism. Zero measure sets, local basin riddling and density of G_α are clearly invariant under the diffeomorphism g^{-1}, so the previous statements translate automatically to the corresponding ones in $\tilde{W}_\epsilon$. Moreover, if $p \in N$ then $g^{-1}(p) = (p, 0) \in TW^\perp$ and α in the statement remains invariant under g.

For convenience we work below with the product metric in $TW^\perp$: for $p \in N$, we set

$$B_\delta(p) = \{(q, v) \in TW^\perp : \max(\rho_N(q, p), \|v\|) < \delta\}.$$

We follow the notations of theorem 2.12: $d_p^\perp \tilde{f} \equiv \Pi_{(TN_1)^\perp} d_{(p,0)} \tilde{f} \Pi_{(TN_0)^\perp}$ is the normal derivative of f at $p \in A$, $(p_n, v_n) = \tilde{f}^n(p, v)$ and π_i is the projection onto the ith factor of $(p, v) \in TW^\perp$, $i = 1, 2$.

Choose $\tilde{W}_\gamma$ sufficiently small that

$$\|v_1\| > \mathrm{e}^{-\beta/4} \|d_p^\perp \tilde{f}\| \; \|v\| \tag{22}$$

and

$$\|d_{p_1}^\perp \tilde{f}\| > \mathrm{e}^{-\beta/4} \|d_{\tilde{f}(p)}^\perp \tilde{f}\| \tag{23}$$

for all (p, v) in $\tilde{W}_\gamma$. This is possible by continuity and non-singularity of $d_p \tilde{f}$ and compactness of $\tilde{W}_\gamma$. Note that $p_1 = \pi_1 \tilde{f}(p, v) \neq \tilde{f}(p, 0)$ in general, as the nonlinear map $\tilde{f}$ does not preserve the fibres over W.

Suppose $p_0 \in A$ and $\delta > 0$ are such that $B_\delta(p_0) \cap U(\tilde{W}_\gamma)^c$ has zero measure. By density of G_α there exists $q \in G_\alpha \cap B_\delta(p_0)$. Then there exists n_0 such that $\|d_q^\perp \tilde{f}^n\| > \mathrm{e}^{n\beta}$ for all $n \geqslant n_0$.

Consider the smallest n satisfying $e^{\frac{n\beta}{2}} > \gamma/\delta$. By continuity of $d_p^\perp \tilde{f}^n$ there exists a compact disk $D_\eta(q) \subset B_\delta(p_0) \cap N$ such that

$$p \in D_\eta(q) \Rightarrow \|d_p^\perp \tilde{f}^n\| > e^{n\beta}. \tag{24}$$

As the $d_p^\perp \tilde{f}^n$ are finite-dimensional linear operators, their norms must be attained by vectors $v(p)$; moreover these vary continuously with p. Thus for each $p \in D_\eta(q)$

$$\|d_p^\perp \tilde{f}^n(v(p))\| > e^{n\beta}\|v(p)\|. \tag{25}$$

Consider the vertical strip $S_{\eta,\delta} = D_\eta(q) \times \{v : \|v\| \leqslant \delta\} \subset B_\delta(p_0)$. Then, for $(p, v) \in S_{\eta,\delta}$

$$\begin{aligned}
\|v_n\| &> e^{-\beta/4}\|d_{p_{n-1}}^\perp \tilde{f}\|\|v_{n-1}\| \\
&> e^{-2\beta/4}\|d_{p_{n-1}}^\perp \tilde{f}\|\|d_{p_{n-2}}^\perp \tilde{f}\|\|v_{n-2}\| \\
&\;\;\vdots \\
&> e^{-n\beta/4}\|d_{p_{n-1}}^\perp \tilde{f}\|\|d_{p_{n-2}}^\perp \tilde{f}\| \cdots \|d_{p_1}^\perp \tilde{f}\|\|d_p^\perp \tilde{f}\|\|v(p)\|
\end{aligned}$$

where inductive use is made of (22). We now apply (23) inductively:

$$\begin{aligned}
\|v_n\| &> e^{-(n+1)\beta/4}\|d_{p_{n-1}}^\perp \tilde{f}\|\|d_{p_{n-2}}^\perp \tilde{f}\| \cdots \|d_{p_2}^\perp \tilde{f}\|\|d_{\tilde{f}(p)}^\perp \tilde{f}\|\|d_p^\perp \tilde{f}\|\|v(p)\| \\
&\;\;\vdots \\
&> e^{-n\beta/2}\|d_{\tilde{f}^{n-1}(p)}^\perp \tilde{f}\|\|d_{\tilde{f}^{n-2}(p)}^\perp \tilde{f}\| \cdots \|d_{\tilde{f}(p)}^\perp \tilde{f}\|\|d_p^\perp \tilde{f}\|\|v(p)\| \\
&\geqslant e^{-n\beta/2}\|d_p^\perp \tilde{f}^n\|\|v(p)\| \\
&> e^{n\beta/2}\|v(p)\| \\
&> \frac{\gamma}{\delta}\|v(p)\|.
\end{aligned}$$

Taking $v(p)$ with $\|v(p)\| = \delta$ for each p, we have $\|\pi_2 \circ \tilde{f}^n(q, v(q))\| > \gamma$. Continuity of $v(p)$ in p and non-singularity of $d^\perp \tilde{f}^n$ imply that $\tilde{f}^n(S_{\eta,\delta}) \cap \tilde{W}_\gamma{}^c$ contains an open set, and has therefore positive measure, contradicting the hypotheses. Therefore A has a locally riddled basin in $TW^\perp$. This statement translates through the diffeomorphism g to the corresponding one in M, proving that A has a locally riddled basin in M. □

2.4. Chaotic saddles

In this section we consider the situation opposite to that of theorem 2.12. We assume that the spectrum lies completely to the right of the origin and conclude, using essentially the same methods, that A is normally repelling: the only points which remain in a small neighbourhood of A for all iterations are those already lying in the submanifold N. In particular, if A is a chaotic attractor for $f_{|N}$, then A is a normally repelling chaotic saddle.

Theorem 2.16. Suppose $f_{|A}$ has an injective normal derivative and $\lambda_{\min} > 0$. Then for any sufficiently small neighbourhood U of A, $f^n(p) \in U$ for all $n \geqslant 0$ if and only if $p \in N$.

Proof. The 'if' part is trivial since A is an asymptotically stable attractor for $f_{|N}$. We concentrate on the 'only if' part.

The notation in this proof will be that of theorems 2.12 and 2.15. As before, the statement in the theorem translates through g^{-1} to $TW^\perp$.

Choose λ such that $0 > \lambda > \lambda_{\min}$. By theorem 2.8, there exists n such that, for all $p \in W$, $m \geqslant n$, $0 \neq v \in (TN_0)^{\perp}$,

$$\|d_p^{\perp} \tilde{f}^m(v)\| > e^{m\lambda}\|v\|. \tag{26}$$

Choose a small enough compact neighbourhood $\tilde{W}_{\gamma}$ of A in $TW^{\perp}$ that $\tilde{f}^n(\tilde{W}_{\gamma})$ remains inside the neighbourhood referred to in lemma 2.13 and

$$\|\pi_2 \circ \tilde{f}^n(p, v)\| > e^{-\lambda/2}\|d_p^{\perp} \tilde{f}^n(v)\|. \tag{27}$$

Let $U = \{(p, v) \in V : \tilde{f}^j(p, v) \in V$ for all $j \geqslant 0\} = \bigcap_{j=0}^{\infty} \tilde{f}^{-j}(V)$. Note that the zero-section of $(TW)^{\perp}$ is contained in U since by hypothesis A is an asymptotically stable attractor for $f_{|N}$. Moreover, U is forward-invariant: $\tilde{f}(U) \subseteq U$.

If $(p, v) \in U$, then by definition $\tilde{f}^j(p, v) \in V$ for all $j \geqslant 0$. But by (26) and (27)

$$\|\pi_2 \tilde{f}^n(p, v)\| > e^{-\lambda/2} e^{n\lambda}\|v\| \tag{28}$$

for all $(p, v) \in V$. Equation (28) shows that if $\|v\| \neq 0$ there exists k such that $\tilde{f}^{kn}(p, v) \notin V$; thus $(p, v) \notin U$. Therefore U coincides with the zero-section W_0 of $W \cap N$ in $(TW)^{\perp}$.

Applying the diffeomorphism g translates this to f in M as the 'only if' part in the statement of this theorem. □

2.5. SBR measures

By assuming more structure than in subsections 2.2–2.4 we can say more about the normal stability of A. We shall be interested in a special type of invariant measure on an attractor—which is sometimes called the *natural*, *Sinai–Bowen–Ruelle* (Ruelle [48]) or *physical* (Eckmann and Ruelle [15]) measure.

Let $f : N \to N$ be a $C^{1+\alpha}$ map. Suppose that A is an asymptotically stable attractor under f, then it is the closure of its unstable manifolds.

Definition 2.17. *An* **SBR measure** *for A is an ergodic invariant measure μ whose support is A and whose conditional measures μ_σ on unstable manifolds W_σ are absolutely continuous with respect to the Riemannian measure induced on these manifolds.*

We call an attractor A supporting an SBR measure an *SBR-attractor*. For a discussion of this definition, see for instance Benedicks and Young [8] or Pugh and Shub [44]. As A is an asymptotically stable attractor for f, it must be the closure of the union of unstable manifolds on N. Suppose that locally A is partitioned measurably

$$A = \bigcup_{\sigma \in \Sigma} W_\sigma,$$

where each W_σ is an open piece of a k-dimensional unstable manifold that carries a conditional measure μ_σ such that, for any Borel measurable set $B \subset A$,

$$\mu_{SBR}(B) = \int \mu_\sigma(B \cap W_\sigma)\, d\mu(W_\sigma).$$

It follows from the work of Pesin [39], Ledrappier and Strelcyn [22], Pugh and Shub [44], that the existence of a measure which is absolutely continuous on unstable manifolds ensures the *absolute continuity of the stable foliation of A*. A family of stable manifolds has this property if for any family of local stable disks D_α^s of dimension $n - k$ in M and any pair of smooth manifolds V_1, V_2 of dimension k which intersect the D_α^s transversely,

$\ell_{V_i}(V_i \cap \bigcup_\alpha D^s_\alpha)$ are both zero or both strictly positive, $i = 1, 2$, where ℓ_{V_i} denotes the Riemannian measure on V_i induced by the Riemannian structure on M.

The importance of SBR measures can be seen from the following corollary (see Pugh and Shub [44]).

Corollary 2.18. *Let U be a neighbourhood of A. Then there is a set $\tilde{U} \subset U$ with positive Riemannian measure such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^j(x)) = \int_A \phi d\mu \tag{29}$$

for all $x \in \tilde{U}$ and all continuous functions $\phi \in C(N, \boldsymbol{R})$.

It is also easy to see (Newhouse [31]) that, if B is an open set with $\mu_{SBR}(B) = \mu_{SBR}(\overline{B})$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_B(f^j(x)) = \mu_{SBR}(B)$ for a.a. x in the basin of A. This means that we can approximate SBR measures and their phase averages by a box-counting procedure.

It must be stressed that non-trivial SBR measures need not, and in many cases do not, exist. Existence of SBR measures has been proved in uniformly hyperbolic systems, such as Axiom A attractors (see Ruelle [45]) and expanding maps of the interval (Lasota and Yorke [21]) or of the circle (Shub and Sullivan [49]). Some results have been proven in the non-uniformly hyperbolic case: these include a set of positive measure in parameter space for the quadratic map of the interval (Misiurewicz [29], Jakobson [17], Benedicks and Carleson [7]). Young [53] has constructed SBR measures for Lozi-type maps; recently Benedicks and Young [8] have done the same for the Hénon map. This has been generalized by Mora and Viana [30] who consider unfoldings of homoclinic tangencies of maps of the plane. A common feature to all known cases is that the only way to prove existence of an SBR measure is by constructing it. However, there is the conviction—supported by numerical evidence (see e.g. Dellnitz *et al.* [13])—that the existence of an SBR measure, or of a weaker version thereof, should in some sense be a common occurrence.

We now state a powerful result of Alexander *et al* [2].

Theorem 2.19. *Let A be an SBR attractor for $f_{|N}$. Suppose $\Lambda_{SBR} < 0$. Then $\ell_M(\mathcal{B}(A)) > 0$. Moreover, if A is either uniformly hyperbolic or μ_{SBR} is absolutely continuous with respect to an n-dimensional Riemannian measure on N then A is an essential attractor.*

Proof. See Alexander *et al* [2]. □

This statement is a consequence of theorem 2.12 in the case where $\Lambda_{\max} < 0$, since A is then an asymptotically stable attractor for f in the global phase space M. The interesting case in theorem 2.19 is therefore when $\Lambda_{\max} > 0 > \Lambda_{SBR}$. By theorem 2.12, if $\Lambda_{\max}$ is positive then A cannot be asymptotically stable—indeed it is Liapunov unstable. This may be pictured in terms of the normal unstable manifolds of A. Let μ be an ergodic measure with $\Lambda_\mu > 0$. Then by a result of Ruelle [47] for μ-a.a. $x \in A$ there exist local unstable manifolds $W^u_{loc}(x)$ not tangent to N, corresponding to the positive normal Liapunov exponents. Thus A is not the closure of the unstable manifolds of its points; these manifolds 'stick out' away from N and so they intersect any neighbourhood of A in M, making A Liapunov unstable. However, the condition $\Lambda_{SBR} < 0$ implies that ℓ_N-a.a. $x \in A$ have empty normal unstable manifolds, and therefore A attracts a set of positive Lebesgue measure. Under the stronger assumptions in the theorem, the measure of this set relative to that of a neighbourhood of A tends to 1 as the neighbourhood shrinks. Therefore A is an essential attractor.

As the examples discussed in section 4 and in [2], [35] show, this is the best one can in general hope for due to the presence of repelling 'tongues' of positive measure in any neighbourhood of a normally unstable ergodic subset of such an attractor.

The proof of theorem 2.19 in [2] may be adapted to yield the following converse. We conjecture that the hypotheses for the following theorem are typically satisfied, but do not have a general proof of this.

Theorem 2.20. *Suppose the attractor A of $f_{|N}$ is a non-trivial SBR attractor. Suppose $\Lambda_{SBR} > 0$, and furthermore all Liapunov exponents are μ_{SBR}-a.e. different from zero and that $\ell_M(\cup_{\mu\neq\mu_{SBR}} G_\mu) = 0$. Then $\ell_M(\mathcal{B}(A)) = 0$; that is, A is a chaotic saddle.*

Proof. Suppose that A has μ_{SBR}-a.e. $k \geqslant 1$ positive and $n-k$ negative tangent Liapunov exponents: this is because $\Lambda_{SBR} > 0$, implying A has at least one normal Liapunov exponent μ_{SBR}-a.e. greater than zero. By ergodicity of μ_{SBR}, there are exactly $m-n$ normal exponents, including multiplicity. Suppose that of these $m-n$ normal exponents d $(\neq 0)$ are positive and $m-n-d$ are negative (by assumption they are all non-zero).

Since μ_{SBR} is absolutely continuous on unstable manifolds in N, we can choose a local unstable manifold $W_\sigma \subset N$ of A such that the conditional measure μ_σ is absolutely continuous with respect to a Riemannian measure on W_σ.

Denote by W'_σ the local unstable manifold of A, of dimension $k+d$, such that

$$W_\sigma = W'_\sigma \cap N.$$

By absolute continuity, μ_σ-a.a. $p \in W_\sigma$ has a local stable manifold D^s_p of dimension $m-(k+d)$ which is transverse to W'_σ. The assumption that non SBR-generic points of the attractor have zero measure implies that we need only consider the stable disks at points that are generic for μ_{SBR}. (We suspect that this hypothesis is unnecessary). Furthermore, by transversality $W'_\sigma \cap D^s_p$ is a single point, namely p. However, we also have $W_\sigma \cap D^s_p = \{p\}$, so that for any μ_σ-measurable $B \subset W_\sigma$,

$$W'_\sigma \cap \bigcup_{p\in B} D^s_p = W_\sigma \cap \bigcup_{p\in B} D^s_p \subset N, \tag{30}$$

so that this intersection is, for all $B \subset W_\sigma$, a subset of the k-dimensional unstable manifold W_σ. Hence, denoting by ℓ'_σ the Riemannian measure induced on the $(k+d)$-dimensional manifold W'_σ, we conclude that

$$\ell'_\sigma\left(W'_\sigma \cap \bigcup_{p\in B} D^s_p\right) = 0. \tag{31}$$

As discussed in [2], for sufficiently small $\delta > 0$ we can find $B_\delta \subset W_\sigma$ with positive $(W_\sigma-)$ Riemannian measure such that the local stable manifold D^s_p makes an angle at least δ with W'_σ and extends at least a distance δ in all $m-(k+d)$ directions with curvature less than $1/\delta$. Moreover, $\mu_\sigma(B_\delta) \to 1$ as $\delta \to 0$.

Next we smoothly foliate a δ^2-tubular neighbourhood $V_{\delta,\sigma}$ about W'_σ with smooth $(k+d)$-dimensional manifolds V_β transverse to $\bigcup_{p\in B_\delta} D^s_p$. The absolute continuity of the stable foliation is a consequence of the fact that all $m-n$ normal exponents are different from zero and of theorem 2 in Pugh and Shub [44]. This fact, together with equation (31) guarantees that

$$\ell_{V_\beta}\left(V_\beta \cap \bigcup_{p\in B_\delta} D^s_p\right) = 0,$$

and consequently the m-dimensional Riemannian measure of $\bigcup_{p\in B_\delta} D^s_p$ is zero. □

2.6. Classification by normal spectrum

For convenience, we collect the results of the previous sections together in the following proposition.

Proposition 2.21. *Suppose $f : M \to M$ is a $C^{1+\alpha}$ map leaving the embedded submanifold N invariant, and that A is an asymptotically stable chaotic attractor for $f_{|N}$. Define $\Lambda_{\max}$, $\lambda_{\min}$ and Λ_{SBR} (if there is an SBR measure supported on A) as above. Then, under $f : M \to M$*

(a) If $\Lambda_{\max} < 0$ then A is an asymptotically stable attractor.
(b) If $\Lambda_{\max} > 0$ then A is Liapunov unstable.
(c) If $\Lambda_{SBR} < 0 < \Lambda_{\max}$ then A is a Milnor (essential) attractor. If in addition there exists $\alpha > 0$ with G_α dense in A, then A has a locally riddled basin.
(d) If $\lambda_{\min} < 0 < \Lambda_{SBR}$, μ_{SBR}-almost all Liapunov exponents are non-zero and $\ell_M(\cup_{\mu\neq\mu_{SBR}} G_\mu) = 0$, then A is a chaotic saddle.
(e) If $0 < \lambda_{\min}$ then A is a normally repelling chaotic saddle.

Proof: This is a collection of the results in theorems 2.12, 2.15, 2.16, 2.19, and 2.20. □

Remark 2.22 (Normal hyperbolicity, perturbations breaking N). *Normal hyperbolicity guarantees persistence of invariant manifolds under small perturbations of the dynamical system [38]. We remark that this study is taking place precisely in the region where $A \subset N$ is not normally hyperbolic. In particular, if A is normally hyperbolic, a riddled basin cannot exist; conversely, if A has a riddled basin then small arbitrary perturbations of f will break up the manifold N. If $\Lambda_{\max} < 0$ or $\lambda_{\min} > 0$ this implies normal hyperbolicity by theorem 2.8.*

3. Normal parameters and normal stability

The previous section is a 'static' classification of normal dynamics using the normal spectrum; until now we have not discussed parameters. The parameter dependence of a chaotic attractor is very delicate for non-uniformly hyperbolic attractors. Rather than considering this question, we concentrate on studying how the transverse behaviour of the attractor changes when the dynamics on the invariant submanifold N are fixed. To do this, we define a *normal parameter* of the system—one that preserves the dynamics on the invariant submanifold but varies it in the rest of the phase space.

Even when the normal dynamics is continuously dependent on a normal parameter, it does not follow that the normal spectrum varies continuously. However, for systems that satisfy a certain technical condition (they map a continuous cone field in the tangent bundle into itself) it is possible to prove continuity of the normal spectrum using a result of Ruelle [46]. Since this condition is open in the space of maps on the tangent bundle, we can expect to observe it.

Using this assumption, we prove in theorem 3.3 that there is an open set in an appropriate function space in which the normal spectrum is continuously dependent on normal parameters. We can therefore discuss the transitions between the set A being an asymptotically stable attractor, a locally riddled basin attractor, a chaotic saddle and a normally repelling chaotic saddle. In particular if N has codimension 1 in M the cone condition is automatically satisfied and these bifurcations occur generically.

Theorem 3.5 suggests that we can expect to observe attractors with locally riddled basins in a persistent way. We characterize a generic loss of stability in proposition 3.9. Liapunov exponents only give a linearised theory of stability; in order to calculate branching behaviour at bifurcations, global stable and unstable manifolds must be investigated. We discuss this and aspects of the dimensions of bifurcating attractors in subsection 3.2.

3.1. Parameter dependence of the normal spectrum

We consider the set of maps of M that are equivalent to f as in subsection 2.1 when restricted to a neighbourhood of A in the invariant submanifold N. To this end, consider a compact neighbourhood $U \subset M$ of A.

Definition 3.1. *Given $h \in C^1(N, N)$ we say $f \in C^1(U, M)$ is an* **extension** *of h if $f_{|U\cap N} = h_{|U\cap N}$. We define the* **set of extensions** *of h in U to be $\mathcal{F}_U(h)$.*

For a discussion of the parameter dependence it is useful to consider paths in $\mathcal{F}_U(h)$ for fixed h and U. In fact, allowing the perturbations to h to be in a general function space would have the effect that the invariant attractor A for h would vary discontinuously (upper-semicontinuously at best) and so the whole structure of $\mathrm{Erg}_f(A)$ would vary discontinuously; this problem is outside the scope of the present paper.

Definition 3.2. *If u is a neighbourhood of A in M, we say a parameter ν of a family of mappings $f_\nu : U \to M$ is a (C^k)-***normal parameter** *if there is an $h \in C^1(N, N)$ such that $f(\cdot, \nu) \in \mathcal{F}_U(h)$ is C^k dependent on ν.*

This means that a normal parameter does not affect the dynamics on N. Normal parameters thus preserve all invariant measures on $A \subset N$, and so $\mathrm{Erg}_{f_\nu}(A)$ is independent of ν. Normal parameters appear naturally if the invariant subspace N is forced by coupling between identical systems (see [4]). An important observation is that tangential Liapunov exponents are independent of normal parameters. This follows directly from formula (6).

Theorem 3.3. *Suppose $h \in C^1(N, N)$ has an asymptotically stable attractor A. Then there is a non-empty open subset $\tilde{\mathcal{F}}_U(h) \subset \mathcal{F}_U(h)$ such that for any C^k-normally parametrised family f_ν in $\tilde{\mathcal{F}}_U(h)$ and any $\mu \in \mathrm{Erg}_h(A)$, the normal Liapunov exponents $\lambda^i_\perp(\mu)$ are C^k functions of ν.*

Before we prove this, we need some more definitions. Recall $\dim N = n$, $\dim M = m$ and define

$$\mathcal{T} = C^0(U \cap N, GL(\mathbf{R}^{m-n})),$$

the Banach space of continuous maps from $U \cap N$ to the space of $(m-n)\times(m-n)$ matrices equipped with the supremum norm (recall that U is compact).

Define a compact neighbourhood $W \subset U \cap N$ of A, a smooth splitting $TW = \{(p, T_pN \oplus (T_pN)^\perp) \;:\; p \in W\}$ and a diffeomorphism $g : (TW)^\perp \to W_\epsilon$ as in subsection 2.1. We define $\tilde{f} = g^{-1} \circ f \circ g$.

For this splitting, there is a natural restriction map $\mathcal{R}\mathcal{F}_U(h) \to \mathcal{T}$ given by

$$(\mathcal{R}f)(p) = \Pi_{(T_{h(p)}N)^\perp} \circ d_{(p,0)} f \circ \Pi_{(T_pN)^\perp} \equiv d^\perp_p f$$

for $p \in N$, where $\mathbf{R}^{m-n}$ is equated with $(T_pN)^\perp$. This map is well defined. It is also surjective; given $(p, v) \in (TW)^\perp$ we define

$$\tilde{f}(p, v) = (h(p), M(p)v)$$

and note that $\mathcal{R}f = M \in GL(\mathbf{R}^{m-n})$ (this map is a skew product that preserves the foliation). $\mathcal{R}$ is continuous with respect to the topologies in $\mathcal{F}_U(h)$ and $\mathcal{T}$. Note that the normal Liapunov exponents of $f \in \mathcal{F}_U(h)$ are exactly the Liapunov exponents of the matrix product defined by $\mathcal{R}f$.

Proof of theorem 3.3: By a result of Ruelle [46], there exists a non-empty open set $\tilde{\mathcal{T}} \subset \mathcal{T}$ (corresponding to linear maps that map a continuous cone bundle into itself) such that the Liapunov exponents are analytic in $\tilde{\mathcal{T}}$. Continuity of $\mathcal{R}$ implies that its pre-image is an open set $\tilde{\mathcal{F}}(h)$. Since $f(.,\nu)$ is assumed to be C^k in ν, the normal Liapunov exponents $\lambda_\perp^i(\mu,\nu)$ will be equally smooth in ν. □

Remark 3.4. *If codim $N = 1$, there is only one normal direction. In this case $\lambda_\mu = \Lambda_\mu$ for all ergodic μ, and there is a unique cone field as in [46] which trivially satisfies the invariance condition. In this case the normal spectrum depends smoothly on normal parameters, i.e. $\tilde{\mathcal{F}}_U(h) = \mathcal{F}_U(h)$. In fact we can in principle compute the normal exponents explicitly; see (39) in section 4 below. However, for codim $N > 1$ the inclusion $\tilde{\mathcal{F}}_U(h) \subset \mathcal{F}_U(h)$ is strict.*

We now show that $\lambda_{\min} < \Lambda_{SBR} < \Lambda_{\max}$ holds generically in $\tilde{\mathcal{F}}_U(h)$ in the C^1 topology if A is a non-trivial (i.e. not a fixed point or periodic orbit) Axiom A attractor.

Theorem 3.5. *Suppose $h \in C^{1+\alpha}(N,N)$ has a non-trivial Axiom A attractor. Then, generically in $\tilde{\mathcal{F}}_U(h)$, Λ_{SBR} is not an extremum of the normal spectrum.*

Proof. We show that generically we have $\Lambda_{SBR} < \Lambda_{\max}$; the proof that $\lambda_{\min} < \Lambda_{SBR}$ follows along similar lines. Continuity of Λ_μ with $f \in \tilde{\mathcal{F}}_U(h)$ for any $\mu \in \mathrm{Erg}_h(A)$ implies that $\Lambda_{SBR} < \Lambda_{\max}$ is satisfied in an open subset of $\tilde{\mathcal{F}}_U$.

To prove density, choose $f \in \tilde{\mathcal{F}}_U(h)$ such that $\Lambda_{SBR} = \Lambda_{\max}$. As A satisfies Axiom A, periodic point measures are dense in $\mathrm{Erg}_f(A)$; see Sygmund [51]. Let x be a periodic point such that

$$\Lambda_\mu(f) > \Lambda_{SBR}(f) - \frac{\epsilon}{3} \tag{32}$$

where μ is the ergodic measure supported on the orbit $\mathcal{O}(x)$ of x.

Define W, $\tilde{W}_\gamma$, $\tilde{f}$ and g as in the proof of theorem 2.12. Let $\tilde{\varphi} = g^{-1} \circ \varphi \circ g$ and recall that the normal Liapunov exponents at $p \in A$ are identical for f and $\tilde{f}$. Define also $\zeta : [0,\infty) \to [0,1]$ a C^1 function such that $\zeta(a) = 1$ for $0 \leqslant a \leqslant 1$, $\zeta(a) = 0$ for $a \geqslant 1$ and $|\zeta'(a)| < 2$. For any $\delta > 0$ with $B_\delta(p,0) \subset \tilde{W}_\gamma$ in the product metric, define $\eta : \tilde{W}_\gamma \to \mathbf{R}$ to be

$$\eta(p,v) = \zeta(\min_{q\in\mathcal{O}(x)}(\rho_N(p,q))/\delta)\zeta(\|v\|/\delta)$$

where ρ_N is the Riemannian distance in N.

Now we can define $\tilde{\psi} : \tilde{W}_\gamma \to \tilde{W}_\gamma$ by

$$\tilde{\psi}(p,v) = (p, \mathrm{e}^{\epsilon\eta(p,v)}v)$$

so that

$$d_{(p,v)}\tilde{\psi} = \begin{pmatrix} \mathbf{1} & \epsilon\frac{\partial\eta}{\partial p}\mathrm{e}^{\epsilon\eta}v \\ 0 & \mathrm{e}^{\epsilon\eta(p,v)}\mathbf{1} + \epsilon\frac{\partial\eta}{\partial v}\mathrm{e}^{\epsilon\eta}v \end{pmatrix}$$

Note that ζ, η and $\tilde{\psi}$ are continuously differentiable and

$$\left\|\frac{\partial\eta}{\partial p}\right\|_0 + \left\|\frac{\partial\eta}{\partial v}\right\|_0 < \frac{4}{\delta}.$$

Define $\tilde{\varphi} : \tilde{W}_\epsilon \to \tilde{W}_\epsilon$ by

$$\tilde{\varphi}(p,v) = \tilde{\psi}\circ\tilde{f}(p,v). \tag{33}$$

For any $p \in A$, $0 \neq v \in (T_pN)^{\perp}$, the normal Liapunov exponents $l_{\perp}(p, v)$ at p for $\tilde{\varphi}$ are:

$$l_{\perp}(p, v) = \lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{(TN_n)^{\perp}} d_{(p,0)}\tilde{\varphi}^n(v)\|_{TM_n}.$$

Since

$$d_{(p,0)}\tilde{\Psi} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathrm{e}^{\epsilon\eta(p,0)}\mathbf{1} \end{pmatrix}$$

it follows that

$$\begin{aligned} l_{\perp}(p, v) &= \lim_{n\to\infty} \frac{1}{n} \log\left(\|d_p^{\perp}\tilde{f}^n(v)\| \prod_{k=0}^{n-1} \mathrm{e}^{\epsilon\eta(f^k(p),0)} \right) \\ &= \lim_{n\to\infty} \frac{1}{n}\left(\log \|\|d_p^{\perp}\tilde{f}^n(v)\| + \sum_{k=0}^{n-1} \epsilon\eta(f^k(p), 0) \right) \\ &= \lambda_{\perp}(p, v) + \epsilon \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \eta(f^k(p), 0), \end{aligned}$$

where $\lambda_{\perp}(p, v)$ are the normal Liapunov exponents at p for f. For $m \in \mathrm{Erg}_f(A)$ it follows from theorem 2.8 that $\lambda_{\perp}$ exists for m-almost all p. On this set the ergodic theorem implies

$$l_{\perp}(p, v) = \lambda_{\perp}(p, v) + \epsilon \int \eta(p, 0)\, \mathrm{d}m(p)$$

for m-almost all $v \in (T_pN)^{\perp}$. In particular, the independence of the integral term on v implies that $\Lambda_m(\tilde{\varphi}) = \Lambda_m(\tilde{f}) + \epsilon \int \eta(p, 0)\, \mathrm{d}m$. Thus

$$\Lambda_{\mu}(\varphi) = \Lambda_{\mu}(f) + \epsilon \tag{34}$$

and

$$\Lambda_{SBR}(\varphi) = \Lambda_{SBR}(f) + \epsilon \int \eta(p, 0)\, d\mu_{SBR}.$$

Since μ_{SBR} is non-trivial we can choose δ such that $\int \eta(p, 0)\, \mathrm{d}\mu_{SBR} < \frac{1}{3}$; thus

$$\Lambda_{SBR}(\varphi) < \Lambda_{SBR}(f) + \frac{\epsilon}{3}. \tag{35}$$

Combining equations (32), (34) and (35) we obtain

$$\frac{\epsilon}{3} + \Lambda_{SBR}(\varphi) < \Lambda_{\mu}(\varphi) \leqslant \Lambda_{\max}(\varphi). \tag{36}$$

Choosing $\epsilon > 0$ such that $\mathrm{e}^{\epsilon} - 1 < 2\epsilon$ (and $\mathrm{e}^{\epsilon} < 2$) we have

$$\begin{aligned} \|d_{(p,v)}\tilde{\Psi} - \mathbf{1}\|_0 &< \|\mathrm{e}^{\epsilon\eta} - 1\|_0 + \epsilon \left\| \frac{\partial\eta}{\partial p} \mathrm{e}^{\epsilon\eta} v \right\|_0 + \epsilon \left\| \frac{\partial\eta}{\partial v} \mathrm{e}^{\epsilon\eta} v \right\|_0 \\ &< 2\epsilon + 2.\epsilon.\frac{4}{\delta}.2.2\delta \\ &= 34\epsilon, \end{aligned}$$

and thus

$$\|d_{(p,v)}\tilde{\varphi} - d_{(p,v)}\tilde{f}\|_0 < 34\epsilon\|d_{(p,v)}\tilde{f}\|_0 < 34\epsilon\|\tilde{f}\|_1.$$

From (33) we have also $\|\tilde{\varphi} - \tilde{f}\|_0 < 2\epsilon\|\tilde{f}\|_1$; therefore

$$\|\tilde{\varphi} - \tilde{f}\|_1 < 36\epsilon\|\tilde{f}\|_1.$$

Thus $\tilde{\varphi}$ is arbitrarily C^1-close to $\tilde{f}$ and $\Lambda_{SBR}(\tilde{\varphi}) < \Lambda_{\max}(\tilde{f})$. This statement translates to M as the density part of the theorem. □

Remark 3.6. *This result is probably true under weaker hypotheses. In particular, we expect that genericity corresponds to codimension infinity in the sense that even in generic k (normal) parameter families in the $C^{1+\alpha}$ topology, we do not expect to see Λ_{SBR} as an extremum of the spectrum.*

Remark 3.7. *If there exists a dense set of periodic points or a set of periodic points with a dense set of preimages, we expect to observe local riddling of basins. This is because the condition*

$$\Lambda_{SBR} < 0 < \Lambda_{\max}$$

is satisfied generically in $\tilde{\mathcal{F}}_U(h)$; then G_α should generically become dense in A prior to Λ_{SBR} crossing zero and we apply theorem 2.15.

We are now in a position to discuss the bifurcation behaviour of A. Note that by proposition 2.21 (subject to fulfilling the technical conditions therein) the names of the bifurcations are descriptive of the change in behaviour on changing the parameter through each bifurcation point.

Definition 3.8. *Let $\nu \in \mathbf{R}$ be a normal parameter for f. We define the following bifurcation points for ν.*

- ν_0 *is a* **point of loss of asymptotic stability** *if $\Lambda_{\max}(\nu) < 0$ for $\nu < \nu_0$ and $\Lambda_{\max}(\nu) > 0$ for $\nu > \nu_0$.*
- ν_0 *is a* **blowout bifurcation point [36]** *if $\Lambda_{SBR}(\nu) < 0$ for $\nu < \nu_0$ and $\Lambda_{SBR}(\nu) > 0$ for $\nu > \nu_0$.*
- ν_0 *is a* **point of bifurcation to normal repulsion** *if $\lambda_{\min}(\nu) < 0$ for $\nu < \nu_0$ and $\lambda_{\min}(\nu) > 0$ for $\nu > \nu_0$.*

Note that if on varying a normal parameter A changes from being an asymptotically stable attractor to a normally repelling chaotic saddle, we are forced in a persistent way (that is, generically in an open subset of $\mathcal{F}_U(h)$) to undergo a sequence of bifurcations:

Proposition 3.9. *Suppose that $f(\cdot, \nu)$ is a smooth path in $\tilde{\mathcal{F}}_U(h)$ and that h has an asymptotically stable Axiom-A attractor A. If for $f(\nu_0)$, the set A is normally hyperbolic and asymptotically stable (i.e. $\Lambda_{\max}(\nu_0) < 0$) and for $f(\nu_1)$ it is a normally hyperbolic, normally repelling chaotic saddle (i.e. $\lambda_{\min}(\nu_1) > 0$) then generically there exist $\nu_0 < \nu_a < \nu_b < \nu_c < \nu_1$ such that*

- *there is a loss of asymptotic stability at ν_a.*
- *there is a blowout bifurcation at ν_b.*
- *there is a bifurcation to normal repulsion at ν_c.*

Proof. This follows from continuity of $\Lambda_{\max}(\nu)$ and $\lambda_{\min}(\nu)$ from theorem 3.3 and the generic property that $\nu_a \neq \nu_b \neq \nu_c$ from theorem 3.5. □

Of course, each of these bifurcations could occur many times; we merely state they occur at least once.

3.2. Global behaviour near bifurcations

In definition 3.8 we characterised the important bifurcations of the normal stability of A in terms of the Liapunov exponents. The branching behaviour near such bifurcations is not determined at linear order. Whether the bifurcations are determined locally or globally depends on the type of invariant measure that becomes unstable.

Bifurcations from a periodic orbit $\{p, f(p), \ldots, f^{n-1}(p)\}$ with $f^n(p) = p$ and $p \in A$ can be dealt with by considering the nth iterate of the map. Each point in the orbit is a fixed point for f^n, and the bifurcations from it are generically determined by quadratic or higher order terms of the Taylor expansion of f at the fixed point. In this sense, the branching behaviour is determined locally.

For a blowout bifurcation of the SBR measure on A, any branching from A will be considerably complicated by the presence of a dense set of unstable manifolds even before bifurcation. We shall discuss and make some conjectures about the blowout bifurcation in section 4.6 and section 5, motivated by numerical experiments in section 4.

4. Applications

4.1. Symmetries and invariant subspaces

Symmetry provides a setting where invariant subspaces and invariant submanifolds arise naturally. Suppose that f commutes with the smooth action of a (compact Lie) group of symmetries Γ on M, and let $\Sigma \leqslant \Gamma$ be a subgroup. Then the fixed-point submanifold

$$N = \mathrm{Fix}(\Sigma) = \{x \in M \;:\; \sigma(x) = x \quad \text{for all } \sigma \in \Sigma\}$$

is invariant under f. The states $x(t)$ that lie in N are those for which, at every instant of time, $x(t)$ is invariant under all elements of Σ. If the dynamics of $A \subset N$ is chaotic, then states of the system that lie in A are 'spatially' ordered (have symmetry group Σ) but are temporally chaotic. If A loses transverse stability, breaking the spatial symmetry but leaving the dynamics chaotic, we have a transition to 'spatio–temporal chaos'. The fact that complicated basin structure can occur here was observed by [40]. An example for Bénard convection in an annular geometry may be found in Caponeri and Ciliberto [11].

Following Melbourne *et al* [26], we define two subgroups of Γ that characterise the symmetry of an attractor A. They are the *symmetry on average*,

$$\Sigma_A = \{\sigma \in \Gamma \mid \sigma A = A\}$$

and the *pointwise symmetry*,

$$T_A = \{\sigma \in \Gamma \mid \sigma x = x \ \forall x \in A\}.$$

Note that T_A is a normal subgroup of Σ_A. For finite groups Γ, Melbourne *et al* [26] and Ashwin and Melbourne [5] have classified the possible subgroups of Γ that can be realized by Σ_A assuming that $T_A = \mathbf{1}$. This has been extended by Melbourne [25] to the case where T_A is non-trivial (note that T_A must be an isotropy subgroup for the action of Γ).

The problems we study relate to the following question. What are the possible generic transitions of symmetry (*symmetry increasing bifurcations* in the terminology of [12]) for one-parameter families of maps with symmetry Γ?

For many groups, the group action will typically stratify the manifold into a hierarchy of fixed point subspaces corresponding to the isotropy types of points on the manifold. This will give rise to a hierarchy of invariant manifolds of differing dimensions, and there is the possibility that loss of transverse stability of an attractor in a low dimensional manifold will proceed by losing stability into progressively higher dimensional invariant subspaces in succession. In this paper we will only consider the case of one invariant subspace. We remark that Aston and Dellnitz [3] have considered such questions for coupled Lorenz systems.

Other reasons for invariant submanifolds There are natural ways other than symmetries for smooth dynamical systems to leave specific submanifolds invariant. Rand *et al* [43] study ecological models in which the subspace N represents zero population for some particular phenotype. Because a species with zero population cannot reproduce, this space must be invariant. Loss of transverse stability of A is now an 'invasion' of the ecology by a new species, triggered by a small perturbation to non-zero population values—possibly through a mutation. Another example is given by Kocarev *et al* [20] who consider linear forcing of one system by another identical system. Although the symmetry is destroyed by the unidirectional coupling, there is still an invariant subspace of synchronous states.

4.2. Numerical examples

We discuss in detail two example families of planar mappings introduced in [4]. Both are extensions of a cubic logistic equation $h : \mathbf{R} \to \mathbf{R}$ defined by

$$h(x) = \frac{3\sqrt{3}}{2}x(x^2-1)$$

to a map of the plane. Let $f_{\alpha,\nu,\epsilon}$ and $g_{\alpha,\nu,\epsilon}$ be three-parameter maps of $\mathbf{R}^2$ to itself that are equivariant under $\mathbf{Z}_2$ generated by $(x_1, x_2) \mapsto (-x_1, x_2)$, and given by (note there was an error in the definition of f in [4]):

$$f_{\alpha,\nu,\epsilon}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{3}}{2}x_1(x_1^2-1)+\epsilon x_1 x_2^2 \\ \nu \mathrm{e}^{-\alpha x_1^2}x_2 + x_2^3 \end{pmatrix} \tag{37}$$

$$g_{\alpha,\nu,\epsilon}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{3}}{2}x_1(x_1^2-1)+\epsilon x_1 x_2^2 \\ \nu \mathrm{e}^{-\alpha x_1^2 - x_2^2}x_2 + \frac{(1-\mathrm{e}^{-x_2^2})}{2}x_2 \end{pmatrix}. \tag{38}$$

The factor $\frac{3\sqrt{3}}{2}$ is such that each of the intervals $[-1, 0]$ and $[0, 1]$ are mapped onto the other in a two-to-one way (except, of course, at critical points). In fact, h has an asymptotically stable attractor $A = [-1, 1] \subset N = \mathbf{R} \times \{0\}$ independently of α, ν and ϵ. All three parameters are normal parameters and these families are in $\tilde{\mathcal{F}}(h)$; their restrictions to N coincide. Moreover, $d_{(x_1,0)}f = d_{(x_1,0)}g$: the derivative maps coincide, and so in particular the normal derivatives are the same. Consequently, the normal spectra of f and g coincide. The linearizations on N are dependent on α and ν but independent of ϵ. The case $\epsilon = 0$ corresponds to the existence of an invariant foliation by vertical lines.

Although these two maps have the same normal spectra, the global dynamics of f and g are quite different. Note that f has a (superstable) attractor at infinity whereas g has a repellor at infinity.

4.3. The spectrum of normal Liapunov exponents

For both families,

$$d_{(x_1,0)}f = d_{(x_1,0)}g = \begin{pmatrix} 3\sqrt{3}(x_1^2 - \frac{1}{2}) & 0 \\ 0 & \nu \mathrm{e}^{-\alpha x_1^2} \end{pmatrix}.$$

and $\Lambda_\mu = \lambda_\mu$ since there is only one normal direction. We can explicitly compute this stability index:

$$\begin{aligned} \lambda_\mu = \Lambda_\mu &= \int_A \log \|\Pi_{(T_{h(x_1,0)}N)^\perp} \circ d_{(x_1,0)}f \circ \Pi_{(T_{(x_1,0)}N)^\perp}\| d\mu(x_1) \\ &= \int_A \log|\nu \mathrm{e}^{-x_1^2\alpha}| d\mu(x_1), \\ &= \log|\nu| - K_\mu \alpha, \end{aligned} \tag{39}$$

where we write

$$K_\mu = \int_A x_1^2 d\mu(x_1).$$

For all invariant measures, K_μ is finite (indeed $K_\mu \leqslant 1$); moreover, it is smoothly dependent on $\alpha \in \mathbf{R}$, $\alpha \neq 0$. We write K_{SBR} for $K_{\mu_{SBR}}$ and K_1 for $\sup_{\mu\in \mathrm{Erg}_f(A)} K_\mu$. Note that for fixed α and ν, the lower extremum of the spectrum, $\lambda_{\min}$, is by (39) equal to $\log|\nu| - K_1\alpha$. Note also that the upper extremum is $\Lambda_{\max} = \log|\nu|$.

It is possible to show that f has an absolutely continuous ergodic measure μ_{SBR} whose support is A: the coordinate change $x = \sin^3(\theta)$ semiconjugates $f_{|A}$ with a piecewise expanding map $F : [0, 2\pi] \to [0, 2\pi]$ whose derivative is strictly greater than unity. By the 'Folklore theorem' of Adler and Flatto [1] (see also Lasota and Yorke [21]), F has a Lebesgue-equivalent ergodic invariant measure; this corresponds to the desired μ_{SBR}. As explained in section 2, normal stability of this measure plays a prominent role as it marks the difference between A being a Milnor attractor or a chaotic saddle in the global phase space. Numerical approximation of the SBR measure from box counting 500 000 iterates in 100 bins gives

$$K_{SBR} = 0.358.$$

Numerical evidence also suggests that the invariant measure on A having the largest value of K (and therefore realizing $\lambda_{\min}$) is supported on the symmetric period-two point at

$$x = \pm\sqrt{1 - \frac{2}{3\sqrt{3}}}$$

implying that

$$K_1 = 1 - \frac{2}{3\sqrt{3}} = 0.615.$$

Figure 1 shows a sequence of pictures of the basin $\mathcal{B}(A)$ in $\mathbf{R}^2$ for $\alpha = 0.7$ and increasing values of ν; figure 2 shows blow-ups of details from this figure.

We now analyse in detail the nature of the invariant set A in the global phase space using the methods developed in section 2. For simplicity we state the results only for positive ν; note, however, that this classification is essentially independent of the sign of ν.

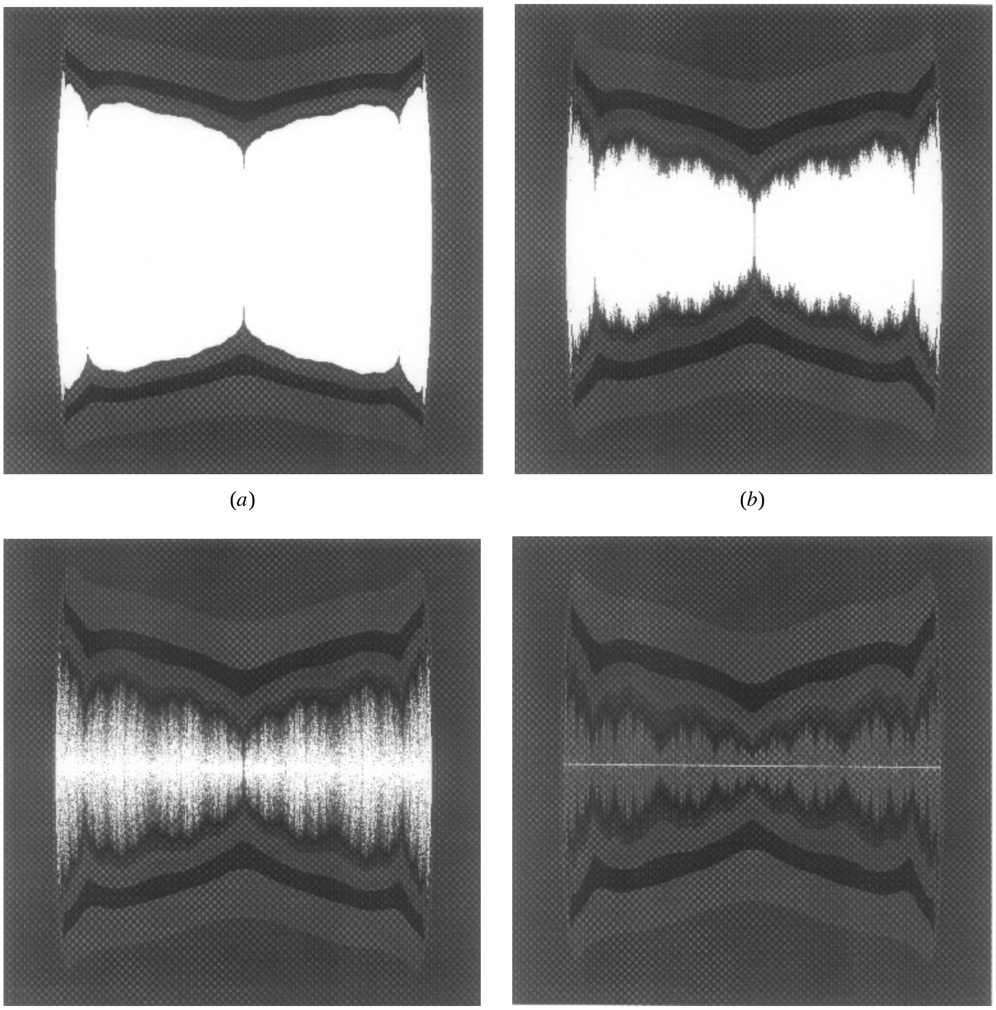
4.4. Global transverse stability for f

Theorem 4.1. *Fix $\epsilon \in \boldsymbol{R}$ and $\alpha > 0$, the behaviour of the map f is as follows:*

(a) For $0 \leqslant \nu < 1$, A is an asymptotically stable attractor.
(b) For $1 < \nu < \mathrm{e}^{K_{SBR}\alpha}$, A is a Milnor attractor whose basin is riddled with that of the attractor at infinity.
(c) For $\mathrm{e}^{K_{SBR}\alpha} < \nu < \mathrm{e}^{K_1\alpha}$, A is a (non-normally repelling) chaotic saddle.
(d) For $\nu > \mathrm{e}^{K_1\alpha}$, A is a normally repelling chaotic saddle.

Proof. (a) It follows from (39) that $\Lambda_\mu \leqslant \log\nu$ for all invariant measures μ. The origin is a fixed point of (37), so $\delta_{(0,0)}$ is an invariant measure for f_A. Indeed, as $\Lambda_{\delta_{(0,0)}} = \log\nu$ from (39), it follows that for all α, ϵ, $\Lambda_{\max} = \log\nu$. Hence for $0 \leqslant \nu < 1$ theorem 2.12 implies that A is an asymptotically stable attractor. This is true for both f and g since, as noted above, their restrictions and linearizations on A coincide.

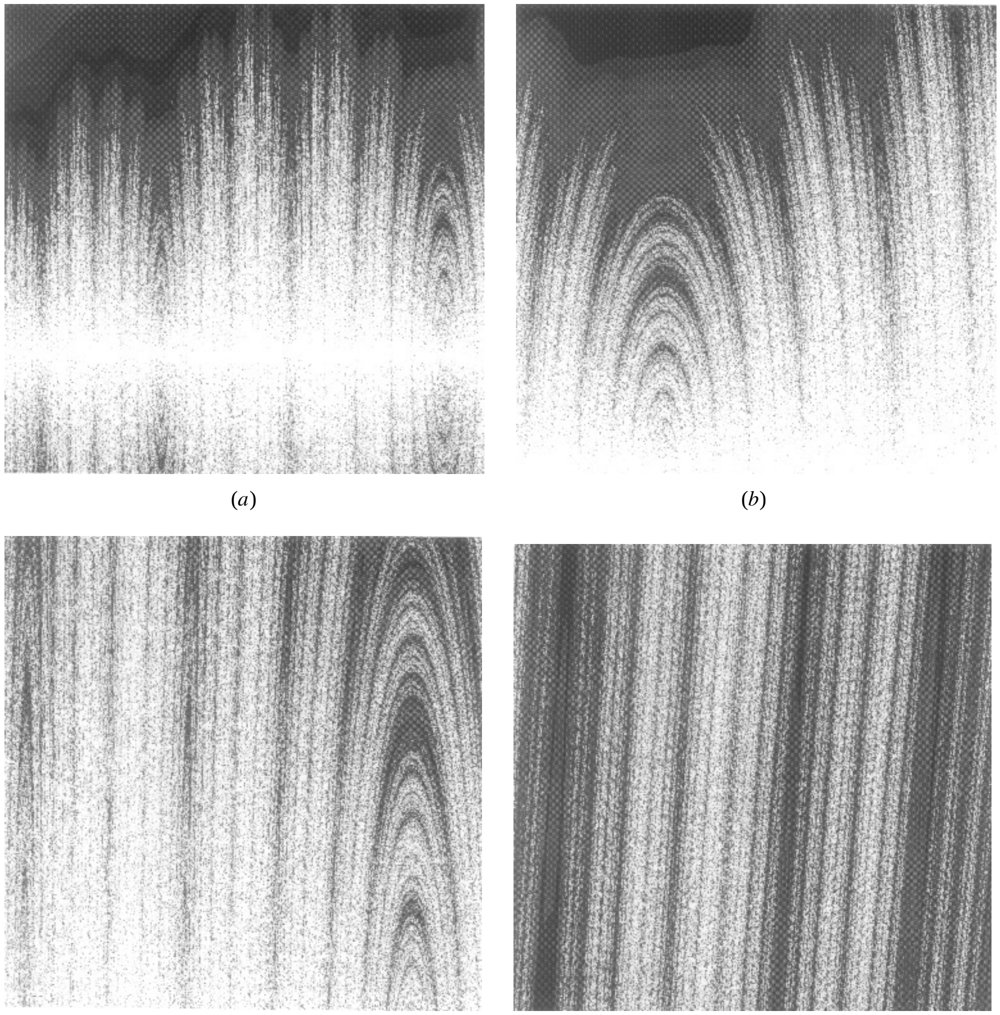
(b) We know from theorem 2.12 that A is Liapunov unstable. As noted above, when $\nu > 1$, the origin is a hyperbolic source with a two-dimensional unstable manifold. The



(*a*) (*b*) (*c*) (*d*)

Figure 1. The basin of attraction $\mathcal{B}(A)$ for $f_{\alpha,\nu,\epsilon}$ with $\alpha = 0.7$, $\epsilon = 0.5$ and various ν. The white area denotes approximately the basin of attraction while the shades of grey indicate the number of iterates required to escape from $\sqrt{x_1^2 + x_2^2} < 10$ (it is left white if it has not escaped by the 128th iterate). The region shown is the rectangle $(x_1, x_2) \in [-1.5, 1.5] \times [-1.1, 1.1]$. (*a*) For $\nu = 0.9$ the attractor is asymptotically stable, though the basin boundary seems to be fractal in nature. (*b*) At $\nu = 1.2$ theorem 4.1 implies that the basin is riddled. This is not apparent in the numerical experiments as the measure of the 'holes' is too small to be seen. (*c*) At $\nu = 1.28$, near the blowout bifurcation there is a complex fractal structure visible in the basin. Figure 2 shows details from this basin. (*d*) At $\nu = 1.48$ the basin has zero measure, but still extends away from the invariant x-axis; A is a non-normally repelling chaotic saddle.

eigendirections associated to this unstable manifold are the coordinate axes. Note that, for $\nu > 1$, f is a local diffeomorphism in a neighbourhood of the origin; it follows from the Hartman–Grobman theorem [37] that there is a neighbourhood of the origin in which the dynamics of f is topologically conjugate to those of its linearization. This linearization



(*a*) (*b*) (*c*) (*d*)

Figure 2. Details of figure 1(*c*) are shown for (x_1, x_2) in the following regions: (*a*) $[0.048, 0.623] \times [-0.104, 0.317]$; (*b*) $[0.204, 0.306] \times [-0.005, 0.239]$; (*c*) $[0.150, 0.250] \times [0.039, 0.134]$; (*d*) $[0.200, 0.210] \times [0.150, 0.160]$.

admits a one-parameter family of codimension-one local invariant submanifolds given by

$$W_k^{loc} = \{(x, y) \in U : |x| = k|y|^a\}, \tag{40}$$

where U is the neighbourhood of the origin in the Hartman–Grobman theorem and $a = \frac{\log \nu}{\log(\frac{3\sqrt{3}}{2})}$. So, for $1 < \nu < \frac{3\sqrt{3}}{2}$, the situation is as depicted in figure 3.

Extending this family of local submanifolds by topological continuation (that is, $W_k = \bigcup_{n\geqslant 0} f^n(W_k^{loc})$) produces a foliation of the two-dimensional unstable manifold of the origin by invariant submanifolds.

We next observe that the line $x_1 = 0$ is invariant for the dynamics. Moreover, as the only fixed point is the origin, it follows that W_0 is the whole y-axis, which is in the basin of ∞. It is easy to check that the complement of the strip $|y| \leqslant 1$ is in the basin of ∞ for

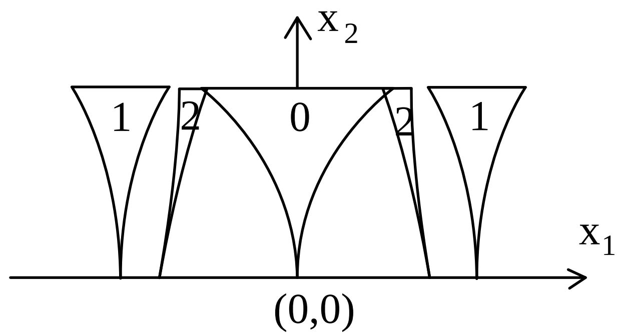


Figure 3. Local unstable manifolds of the origin; a 'tongue' of instability consisting of points that are uniformly repelled away to infinity is denoted by 0. The density of pre-images of the origin in A implies that the pre-images of the tongues form an open dense subset in any neighbourhood of any point on the attractor. The tongues marked 1 and 2 show schematically the first and second pre-images of 0. The measure of the union of them is positive but may be arbitrarily small near the x_1 axis.

all $\nu > 0$; it follows by continuity that there is a neighbourhood of 0 in the variable k such that the submanifolds W_k cross the lines $|y| = 1$ and are thus in the basin of ∞. We thus get the global picture depicted in figure 1(c).

Therefore, for $\nu > 1$ there is a cusp-shaped open set near the origin which is repelled away from A and attracted towards ∞, which we call a 'repulsive tongue' T_0; see remark 4.2.

Prei-mages of the fixed point 0 are dense in A ([27]) and so

$$T = \bigcup_{n \geqslant 0} f^{-n}(T_0)$$

is an open set in $\mathbf{R}^2$ whose boundary is dense in A and which consists of points that are repelled away to ∞. In other words, A is densely filled of points which are the bases of repulsive tongues. Therefore the basin $\mathcal{B}(A)$ is riddled with that of the attractor at infinity. It can be shown that, if we take a small neighbourhood U of A, the relative (Lebesgue) measure of the tongues T (i.e. $\frac{\ell(T \cap U)}{\ell(U)}$, where ℓ is a two-dimensional Lebesgue measure) converges to 0 with $\ell(U)$. This implies that A is an essential attractor.

(c) From (39) it follows that, for $\nu > \mathrm{e}^{K_{SBR}\alpha}$, Λ_{SBR} is positive. It thus follows from theorem 2.20 that $\mathcal{B}(A)$ has zero measure and therefore A is a chaotic saddle.

(d) By theorem 2.16, if $\lambda_{\min} = \inf_{\mu \in \mathrm{Erg}_f(A)} \lambda_\mu > 0$, there exists a neighbourhood V of A such that, if $|y| \neq 0$, then some iterate of (x_1, x_2) leaves V. The set A is then a normally repelling chaotic saddle. □

Remark 4.2. *The essential point in part (2) is that the pre-images of the fixed point 0 are dense, and therefore the generic set* G_μ *for the ergodic measure* $\mu = \delta_0$ *is dense. We can then appeal to theorem 2.15 to prove local riddling. However, in this case we were able to prove the stronger statement that the basin is riddled. The tongues are thus the nonlinear counterpart of the open sets constructed in theorem 2.15.*

For both f and g, the loss of asymptotic stability, blowout bifurcation and the bifurcation to normal repulsion take place on the curves $\nu = 1$, $\nu = \mathrm{e}^{K_{SBR}\alpha}$ and $\nu = \mathrm{e}^{K_1\alpha}$, respectively.

This is shown in the (α, ν) plane in figure 4; the lower and upper lines correspond to loss of asymptotic stability and bifurcation to normal repulsion, respectively. The curve marked by the crosses is the blowout bifurcation.

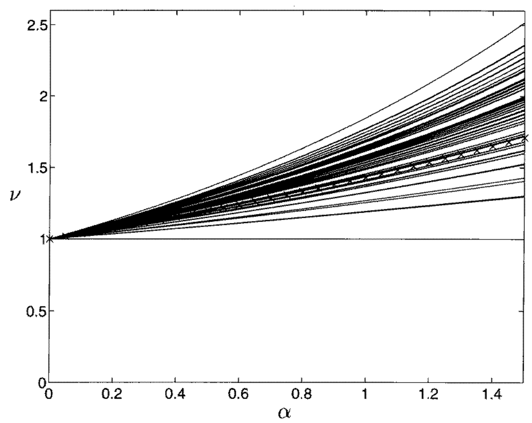


Figure 4. Codimension-one bifurcations of f and g in the plane of normal parameters (α, ν) (independent of ϵ). The line $\nu = 1$ corresponds to loss of asymptotic stability as the fixed point at the origin loses stability; the line indicated by crosses marks the blowout bifurcation (loss of transverse stability of the natural measure) and the uppermost line shows the bifurcation to normal repulsion corresponding to loss of normal stability of a symmetric period-two orbit. Transverse bifurcations of all periodic orbits up to period seven are shown.

We note that periodic points in A become normally unstable through subcritical pitchfork bifurcations, i.e. the bifurcate from saddles to unstable nodes surrounded by two saddles.

4.5. *Global transverse stability for g*

The case for g is somewhat more complicated to handle. Figure 5 shows a sequence of attractors for the map g, also at $\epsilon = 0.5$ and $\alpha = 0.7$.

As before, for $0 \leqslant \nu < 1$, A is an asymptotically stable attractor. For $\nu > 1$, g has a locally riddled basin attractor. Numerical experiments suggest that, at least for ϵ small enough, there are no attractors away from the invariant subspace; in this case $\mathcal{B}(A)$ is open but locally riddled. We summarize this in the following conjecture.

Conjecture 1. *For any $\alpha > 0$ and small enough $|\epsilon|$ the behaviour of the map g is as follows:*

(a) For $1 < \nu < \mathrm{e}^{K_{SBR}\alpha}$, A is a Milnor attractor and $\mathcal{B}(A)$ is open.
(b) For a set of $\mathrm{e}^{K_{SBR}\alpha} < \nu$ with positive Lebesgue density at $\mathrm{e}^{K_{SBR}\alpha}$, there is a family of attractors A_ν containing A, supporting an SBR measure μ_ν that converge in the weak* topology to the SBR measure of $g_{|N}$ as ν tends to $\mathrm{e}^{K_{SBR}\alpha}$ from above.

We remark that in the case $\epsilon = 0$, the property $A \subset A_\nu$ of (b) has been demonstrated in [6].

4.6. *Criticality of the blowout bifurcation*

The evidence for the maps f and g suggests that it is possible to define ‘criticality’ for blowout bifurcations, analogous to that for bifurcation of fixed points in invariant subspaces. For example, steady-state bifurcations with $\mathbf{Z}_2$ symmetry are generically

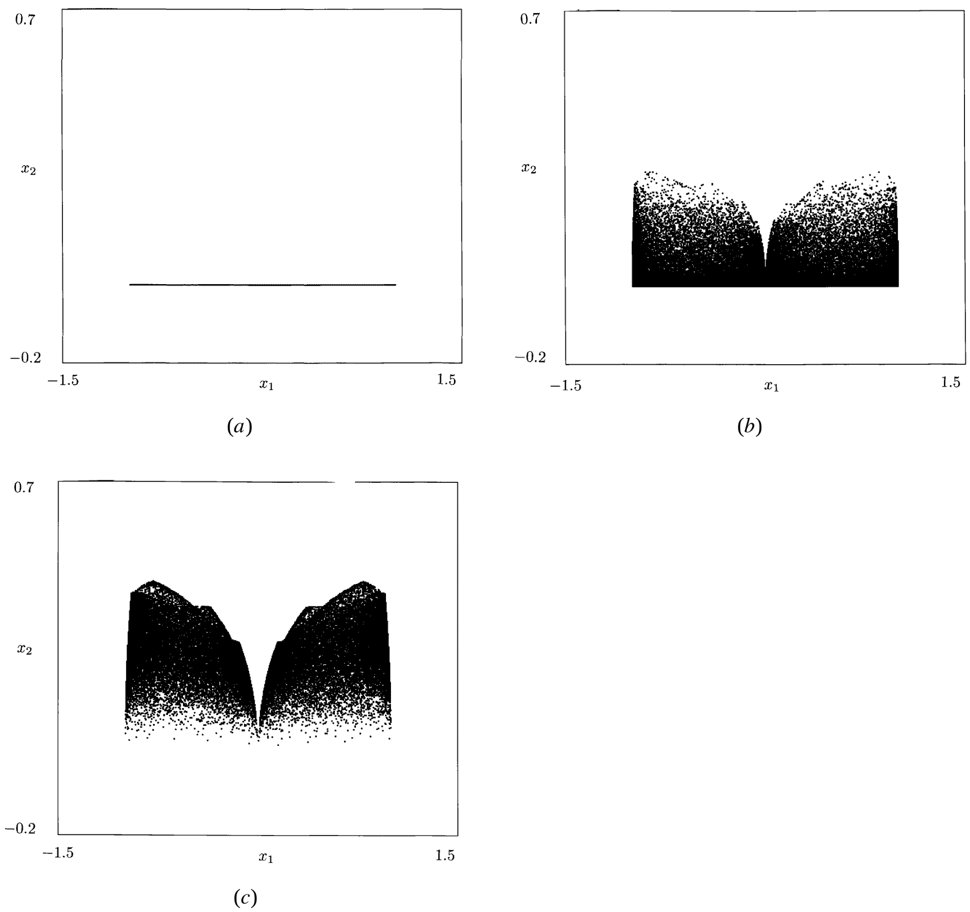


Figure 5. Attractors for $g_{\alpha,\nu,\epsilon}$ with $\alpha = 0.7$, $\epsilon = 0.5$ and increasing ν in the region $(x_1, x_2) \in [-1.5, 1.5] \times [-0.2, 0.7]$. Any transients were allowed to die away before these points were plotted. (*a*) $\nu = 1.2$; an attractor with a locally riddled basin exists. (*b*) $\nu = 1.3$; the attractor has undergone a blowout to create on–off intermittent behaviour of an attractor with no symmetry; the invariant measure is concentrated strongly near the x_1–axis (100,000 iterates). (*c*) $\nu = 1.4$; the deviations of the trajectory away from the invariant subspace are more frequent (50,000 iterates).

pitchfork bifurcations that are either subcritical or supercritical. This criticality corresponds to the hysteretic and non-hysteretic scenarios observed by Ott *et al* [36].

For f, we note that there is an unstable invariant set A_ν, namely the boundary dividing the basins of A and the attractor at infinity for $\nu < \nu_0$, and this is destroyed on passing through $\nu = \nu_0$. In this sense, f exhibits a *subcritical* blowout bifurcation on passing through ν_0.

For g, there is numerical evidence of a family of attractors $\{A_\nu \ : \ \nu > \nu_0\}$; these correspond to on-off intermittent attractors; see [6] for a further discussion of this. Conjecture 1 (b) corresponds to a possible definition of a *supercritical* blowout bifurcation on passing through ν_0. Figure 6 shows examples of unstable manifolds from the fixed points that have bifurcated from the origin.

Thus there appears to be an ‘important’ family of invariant sets that appear to branch

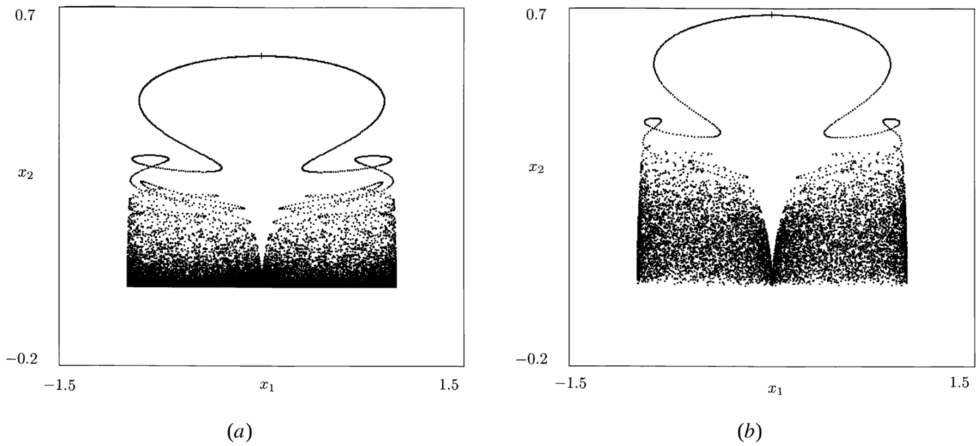


Figure 6. The unstable manifolds of fixed points for the map g at the corresponding parameter values of the attractors shown in figure 5(*a*) and (*b*). (*a*) $\nu = 1.2$ and the manifolds appear to be dense in a neighbourhood of the attractor (in the x_1-axis) with the locally riddled basin. (*b*) $\nu = 1.3$; the manifold has topologically changed little, but the attractor now includes a sizeable proportion of this unstable manifold.

from A to $\nu < \nu_0$ in the subcritical case or to $\nu > \nu_0$ in the supercritical case. For the supercritical case, we observe that these invariant sets are attractors.

Note that for g, all periodic points in A are observed to undergo supercritical pitchfork bifurcations on varying ν, whereas those for f are all subcritical; thus we might expect that all bifurcations of A for g are supercritical while for f they are subcritical. However, this need not generally be the case. We emphasise that as yet we have no proof that criticality for the blowout bifurcation is well-defined.

In contrast to the situation for a steady-state bifurcation where the bifurcation is determined by dynamics on any small neighbourhood around the fixed point, for blowout bifurcations we need to consider the dynamics in a neighbourhood of $\tilde{A}$ to discover the criticality of the bifurcation. Thus 'higher-order normal Liapunov exponents' (relative to higher-order normal derivatives) for the attractor A will not determine criticality; the dynamics far from A is immediately important.

Remark 4.3 (Implications for observables). *Suppose we have an observable $\phi : M \to \boldsymbol{R}$ that measures the distance of a trajectory from the invariant manifold, $\phi(p) = 0$ for all $p \in N$. This might be, for instance, a detective (Dellnitz et al [13]) for the $\boldsymbol{Z}_2$ symmetry for the map g. For trajectories that do not remain within N, we can say something about the symmetry of the attractor by looking at the long-term behaviour of ϕ evaluated on the trajectory. Typically, we examine the ergodic average of ϕ,*

$$\phi_{\mathrm{Erg}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(x_k)$$

where x_k is a trajectory of the system. If $\phi_{\mathrm{Erg}}^{(\nu)} \to 0$ as $\nu \to \nu_0$ (a blowout bifurcation point), this suggests that the bifurcation is supercritical. If we were to compute

$$\phi_{sup} = \limsup_{n\to\infty} \phi(x_n)$$

instead, this would exhibit a discontinuous jump to zero on passing through the blowout bifurcation point and would not distinguish between criticalities.

This contrasts the discontinuous behaviour of the family of attractors (topologically defined) and the continuous behaviour of the natural invariant measures (measure-theoretically defined).

By theorem 2.19 we know that in the locally riddled basin case, a positive measure set of initial points have trajectories with $\phi_{sup} = 0$. We remark that it is not known whether there may also be a positive measure set of initial points whose trajectories have $\phi_{\mathrm{Erg}} = 0$ but $\phi_{sup} > 0$. This would be analogous to the situation for an attracting heteroclinic cycle; the ergodic averages correspond to some convex combination of delta functions supported on the fixed points, while the ω-limit set includes the connections between them.

Remark 4.4 (Symmetry of attractors). *We mention that for g, the attractors created at the blowout bifurcation appear to change symmetry in the following way:*

(a) For $0 \leqslant \nu < \mathrm{e}^{K_{SBR}\alpha}$, there is an attractor A with $T_A = \Sigma_A = \boldsymbol{Z}_2$.
(b) For $\nu < \mathrm{e}^{K_{SBR}\alpha}$, there are two conjugate attractors A_ν with $T_{A_\nu} = \Sigma_{A_\nu} = \boldsymbol{1}$. (see also Ashwin [6] for a discussion of this).

5. Discussion

We have presented a characterization of the normal dynamics of an attractor in an invariant subspace by considering the spectrum of normal Liapunov exponents. We discuss parameter dependence and present numerical examples of applications of this theory.

Our approach differs from that of Alexander *et al* [2], Ott *et al* [35, 36] in that we consider the spectrum of normal Liapunov exponents rather than deviations of Liapunov exponents for some natural measure. Although the two approaches are closely related, by looking at the Liapunov exponents of all invariant measures supported on the attractor, we can get a picture of where regions of instability or riddled basins come from.

We briefly mentioned that near the blowout bifurcation there will be no normal hyperbolicity and therefore the attractors do not persist on some perturbed manifold of the same dimension as the invariant manifold. Numerical experiments indicate that instead, on introducing perturbations that destroy the invariant subspace, we will have an attractor of higher dimensionality than that of the invariant manifold. This is suggested by the Kaplan–Yorke conjecture [19] which states that the Hausdorff dimension of an attractor is (usually) equal to the Liapunov dimension. Because a Liapunov exponent crosses zero at a blowout bifurcation, the Liapunov dimension can exceed the dimension of the submanifold. On perturbing to break invariance of the submanifold, we expect the Kaplan–Yorke formula to hold; this will give a discontinuous change in the dimension of the attractor.

Noise As with perturbations that break the invariance of the submanifold, addition of noise that does not preserve this manifold should give rise to a big jump in the size of the attractor near blowout bifurcation (see Ashwin *et al* [4]. In particular, the addition of noise to a map $f : M \to M$ with attractor A in an invariant submanifold N, $f(N) \subset N$ will have the following effects (see also [16], [34], [35], [36], [42]):

(i) If A is a Milnor attractor whose basin is riddled with that of an asymptotically stable attractor C, then adding low-level noise will cause all trajectories in a neighbourhood of A to converge to C almost surely.

(ii) If A is a Milnor attractor whose basin is locally riddled but open, then adding low-level noise will cause all trajectories in a neighbourhood of A to recurrently explore a neighbourhood of $\tilde{A}$, the union of all unstable manifolds of points in A. We call this behaviour the *bubbling* of the attractor, and as such it resembles on–off intermittency. This sort of behaviour has been observed by Platt *et al* [42] to cause a discontinuous change in the parameter where the blowout bifurcation appears.

In both cases, A is no longer an attractor after noise is added. In the first case it disappears; in the second we expect to see intermittent excursions transversely away from A in a manner similar to on–off intermittency [41], with the difference that this is now a noise-driven phenomenon. Note that there have been experimental observations of on-off intermittency and an observation of a scaling law for the distribution of laminar phases [16]. We expect that there will be scaling properties of the invariant measure for the noisy problem that distinguishes it from noisy forcing of an asymptotically stable attractor (see [42]).

The work here suggests many paths for generalizations and clarifications. We have considered only maps, but the systems could be flows or diffeomorphisms. For instance, Kan has found an open set of diffeomorphisms of a three-dimensional manifold with two attractors whose basins are dense in the whole manifold [18]. The case of invariant manifolds of codimension higher than one should pose many more questions. Finding ways to characterize global stability would also be of interest; there could be crises where the unstable manifold from a point on A hits another invariant set. Another interesting question is the behaviour on varying parameters that are not normal parameters. Analogous questions for Hamiltonian systems deserve attention.

6. Acknowledgements

We thank Claudio Arezzo, Andreas Hamm, Greg King, Andrea Loi, Mario Micallef, David Mond, Mark Muldoon, Mark Pollicott, David Rand and especially Peter Walters for many helpful conversations. Thanks to Steve Rumsby for help with the colour pictures. JB acknowledges support from JNICT/CIENCIA and Fundação Calouste Gulbenkian. PA acknowledges support by the Nonlinear Initiative of the SERC of the United Kingdom and European Union HCM grant ERBCHBGCT930503. We thank the referees for providing very helpful comments and advice.

References

[1] Adler R and Flatto L 1991 Geodesic flows, interval maps and symbolic dynamics *Bull. Am. Math. Soc.* **25** 229–334

[2] Alexander J C, Kan I, Yorke J A and Zhiping You 1992 Riddled basins *Int. J. Bif. Chaos* **2** 795–813

[3] Aston P J and Dellnitz M 1995 Symmetry breaking bifurcation of chaotic attractors *Int. J. Bif. Chaos* **5** 1643–1676

[4] Ashwin P, Buescu J and Stewart I N 1994 Bubbling of attractors and synchronisation of oscillators *Phys. Lett.* A **193** 126–139

[5] Ashwin P and Melbourne I 1994 Symmetry groups of attractors *Arch. Rat. Mech. Anal.* **126** 59–78

[6] Ashwin P 1995 Attractors stuck on to invariant subspaces *Phys. Lett.* **209A** 338-344

[7] Benedicks M and Carleson L 1985 On iterations of $1 - ax^2$ on $[-1, 1]$ *Ann. Math.* (2) **122** 1–25

[8] Benedicks M and L.-Young S 1993 SBR measures for certain Hénon maps *Inv. Math.* **112** 541–76

[9] Buescu J 1995 Exotic Attractors *PhD Thesis* Mathematics Institute, University of Warwick

[10] Buescu J and Stewart I N 1994 Liapunov Stability and Adding Machines *Ergod. Theor. Dynam. Sys.* **15** 1–20

[11] Caponeri M and Ciliberto S 1992 Thermodynamic aspects of the transition to spatio-temporal chaos *Physica* D **58** 365–383

[12] Chossat P and Golubitsky M 1988 Symmetry-increasing bifurcation of chaotic attractors *Physica* D **32** 423–436

[13] Dellnitz M, Golubitsky M and Nicol M 1994 Symmetry of attractors and the Karhunen–Loève decomposition *App. Math. Sci.* **100** 73–109 (Berlin: Springer)

[14] Denker M, Grillenberger C and Sigmund K 1976 Ergodic theory in Compact Spaces *Lecture Notes in Mathematics* **527** (Berlin: Springer)

[15] Eckmann J P and Ruelle D 1985 Ergodic Theory of Chaos and Strange Attractors *Rev. Mod. Phys.* **57** 617—656

[16] Hammer P W, Platt N, Hammel S M, Heagy J F and Lee B D 1994 Experimental observation of on-off intermittency *Phys. Rev. Lett.* **73** 1095–1098

[17] Jakobson M 1981 Absolutely continuous invariant measures for one parameter families of one-dimensional maps *Comm. Math. Phys.* **81** 39–88

[18] Kan I 1994 Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin *Bull. Am. Math. Soc.* **31** 68–74

[19] Kaplan J L and Yorke J A 1979 Chaotic behaviour of multidimensional difference equations Functional differential equations and approximations of fixed points (ed H-O Peitgen and H-O Walther) *Springer Lecture Notes in Maths* **730**

[20] Kocarev L, Shang A and Chua L O 1993 Transitions in dynamical regimes by driving: a unified method of control and synchronisation of chaos *Intl. J. Bif. Chaos* **3** 479–483

[21] Lasota A and Yorke J A 1973 On the existence of invariant measures for piecewise monotonic transformations *Trans. Am. Math. Soc.* **186** 481–488

[22] Ledrappier F and Strelcyn J M 1982 A proof of the estimation from below in the Pesin entropy formula *Erg. Theor. Dynam. Sys.* **2** 203–219

[23] Mañé R 1987 *Ergodic Theory and Differentiable Dynamics* (Berlin: Springer)

[24] Melbourne I 1991 An example of a non-asymptotically stable attractor *Nonlinearity* **4** 835–844

[25] Melbourne I 1995 Generalizations of a result on symmetry groups of attractors *Normal forms and homoclinic chaos* ed W F Langford and W Nagata *Fields Institute Communications, AMS*

[26] Melbourne I, Dellnitz M and Golubitsky M 1993 The Structure of Symmetric Attractors *Arch. Rational. Mech. Anal.* **123** 75—98

[27] de Melo W and van Strien S 1994 *One-dimensional dynamics* (Berlin: Springer)

[28] Milnor J 1985 On the concept of attractor. *Commun. Math. Phys.* **99** 177–195, 1985; Comments *Commun. Math. Phys.* **102** 517–519

[29] Misiurewicz M 1981 Absolutely continuous measures for certain maps of the interval *Inst. Haut. Étud. Sci. Pub. Math.* **53** 17–51

[30] Mora L and Viana M 1993 Abundance of strange attractors *Acta Math.* **171** 1–71

[31] Newhouse S 1980 Lectures on dynamical systems *Dynamical Systems* CIME lectures, Bressanone (Basel: Birkhäuser)

[32] Nusse H and Yorke J A 1991 Analysis of a procedure for finding numerical trajectories close to chaotic saddle hyperbolic sets *Ergod. Theor. Dynam. Sys.* **11** 189—208

[33] Oseledec V I 1968 A multiplicative ergodic theorem: Liapunov characteristic numbers for dynamical systems *Trans. Moscow Math. Soc.* **19** 197–231

[34] Ott E, Sommerer J C, Alexander J C, Kan I and Yorke J A 1993 Scaling behaviour of chaotic systems with riddled basins *Phys. Rev. Lett.* **71** 4134–4137

[35] Ott E, Sommerer J C, Alexander J C, Kan I and Yorke J A 1994 A transition to chaotic attractors with riddled basins *Physica* D **76** 384–410

[36] Ott E, Sommerer J C 1994 Blowout bifurcations: the occurrence of riddled basins and on-off intermittency *Phys. Lett.* A **188** 39–47

[37] Palis J and de Melo W 1982 *Geometric Theory of Dynamical Systems—an Introduction* (Berlin: Springer)

[38] Palis J and Takens F 1993 Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations *Cambridge studies in advanced maths* **35** (Cambridge: Cambridge Uuniversity Press)

[39] Pesin Y B. 1978 Characteristic Liapunov exponents and smooth ergodic theory *Russian Math. Surv.* **32** 55–114

[40] Pikovsky A S and Grassberger P 1991 Symmetry breaking of coupled chaotic attractors *J. Phys. A: Math. Gen.* **24** 4587–4597

[41] Platt N, Spiegel E A and Tresser C 1993 On-off intermittency; a mechanism for bursting *Phys. Rev. Lett.* **70** 279–282

[42] Platt N, Hammel S M and Heagy J F 1994 Effects of additive noise on on-off intermittency *Phys. Rev. Lett.* **72** 3498–3501

[43] Rand D A, Wilson H and McGlade J 1994 Dynamics and evolution: evolutionarily stable attractors invasion exponents and phenotype dynamics. *Phil. Trans. R. Soc.* B, **343** 261–283

[44] Pugh C and Shub M 1989 Ergodic attractors *Trans. Am. Math. Soc.* **312** 1–54

[45] Ruelle D 1976 A measure associated with Axiom A attractors *Am. J. Math.* **98** 619–654

[46] Ruelle D 1979 Analyticity properties of the characteristic exponents of random matrix products *Adv. Math.* **32** 68–80

[47] Ruelle D 1979 Ergodic theory of differentiable dynamical systems *Publ. Math. I.H.E.S.* **50** 27–58

[48] Ruelle D 1989 *Chaotic evolution and strange attractors* (Lezione Lincee) (Cambridge: Cambridge University Press)

[49] Shub M and Sullivan D 1985 Expanding endomorphisms of the circle revisited *Ergod. Theor. Dynam. Sys.* **5** 285 – 289

[50] Spivak M 1970 *A comprehensive introduction to differential geometry* (Houston, TX: Publish or Perish)

[51] Sygmund K 1970 Generic properties of invariant measures for Axiom A diffeomorphisms *Inv. Math.* **11** 99–109

[52] Walters P 1982 *An Introduction to Ergodic Theory* (Berlin: Springer)

[53] Young L-S 1985 Bowen–Ruelle measures for certain piecewise hyperbolic maps *Trans. Am. Math. Soc.* **287** 41–48