A novel dynamic programming algorithm for track-before-detect in radar systems

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Abstract

In this paper we present a novel procedure for multi-frame detection in radar systems. The proposed architecture consists of a pre-processing stage, which extracts a set of candidate alarms (or plots) from the raw data measurements (e.g., this can be the Detection and Plot-Extraction stage of common radar systems), and a track-before-detect (TBD) processor, which jointly elaborates observations from multiple scans (or frames) and confirms reliable plots. A computationally efficient dynamic programming algorithm for the TBD processor is derived, which does not require a discretization of the state space and operates directly on the input plot-lists. Finally, a simple algorithm to solve possible data association problems arising at the track-formation step is given, and a thorough complexity and performance analysis is provided, showing that large detection gains with respect to the standard radar processing are achievable with negligible complexity increase.

Index Terms

Multi-frame detection (MFD), track-before-detect (TBD), dynamic programming, radar systems, Detector, Plot Extractor, tracking.

I. INTRODUCTION

The classical approach to radar detection and tracking follows the scheme reported in Fig. 1. At each scan (or frame), the raw data from the sensor undergo a processing chain in the Detection and

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Plot-Extraction block: the output measurements, called plots, are the result of a clustering operation to merge signal returns which appear to come from the same object, a constant false alarm rate filtering to mitigate clutter, and a thresholding operation retaining only “significant” (i.e., of sufficient strength) plots. If necessary, target tracking procedures, such as Kalman filtering or multiple hypothesis tracking, may be implemented in order to estimate targets’ trajectories on the basis of surviving plots [1].

The traditional approach shows some limitations in remote radar surveillance, where the signal amplitude is weak compared to the background noise, and, in general, in the detection of dim, fluctuating targets. Multi-frame detection (MFD) is a method to improve the detection of weak targets by integrating their signal returns over multiple consecutive scans. MFD is challenging in the presence of target motion, as track-before-detect (TBD) techniques are required to correctly integrate the echoes along the unknown target trajectory. Two main approaches to the MFD problem have been considered in the literature. In an *atomistic* approach [2]–[4], the integration simply takes place in the sensor measurement space: here the goal is to obtain more reliable detections to be sent to the subsequent, distinct tracking stage, which combines the positions (or the estimated track segments) into long continuous tracks. In a *holistic* approach [5]–[7], instead, the detection and tracking stages are fully merged: energy integration takes place in the target state space, and the estimated trajectory is returned at the same time as detection is declared.\(^1\)

Starting from early studies on this topic [5], [9]–[12], a number of applications and improvements have been proposed with reference to both passive [6]–[8], [13]–[17] and active sensors [18]–[29], possibly accounting for some prior information on the target motion, and/or considering an unknown number of targets, and/or adopting sequential data processing [30]–[39].

State-of-the-art TBD strategies, however, hardly lead to real-time implementable schemes in the presence of high-mobility targets when the number of sensor resolution elements is large, even resorting

\(^1\)As pointed out in [8], TBD should be more properly defined as track-before-declare in this case, since target is tracked before declaring it to be a valid target. The term “detect,” instead, may generate confusion, in that it could be referred to target detections (i.e., measurements, alarms).
to dynamic programming algorithms, such as the Viterbi algorithm [40]. The main reason for such an un-affordable complexity is that all of the observations are retained at each epoch and processed. Another drawback of procedures elaborating raw data measurements is that the presence of many weak returns could result in a lower measurements’ accuracy. To overcome complexity limitations, the authors of [20], [25], [41] envisioned the adoption of a pre-processing stage. In [20], only the measurements whose signal amplitude exceeds a primary threshold are elaborated in the subsequent stage based on the Hough transform. In [41], the input data undergoes a one-bit quantization, and the multi-scan processing is simply a 3-D matched filter. In [25], instead, a multiple-bit quantization is considered, and the subsequent stage implements the Viterbi algorithm. None of these studies, however, take into account the fact that, at each epoch, the set of candidate detections may shrink or expand based on the significance of the received returns.

In this paper, we follow an atomistic approach to MFD and, starting from preliminary results obtained in [42]–[45], we propose a two-stage detection architecture. The first stage is the classical Detector and Plot-Extractor, but the threshold is lowered in order to obtain a richer set of candidate plots. The second stage, instead, is a TBD processor which exploits the space-time correlation among the candidate plots taken at different scans to confirm or delete them.\(^2\) The main contribution is the derivation of a novel dynamic programming algorithm to compute the test statistics needed in the second stage. The algorithm operates directly on the candidate plots and efficiently takes into account the fact that their number is, in general, much smaller than that of resolution elements. A complexity analysis is provided, stating the conditions under which the proposed procedure outperforms its direct competitor, i.e., the Viterbi algorithm. Finally, a simple algorithm to solve possible data association problems arising at the track-formation step is given, and a thorough performance assessment is undertaken to elicit the trade-offs among the primary threshold, the achievable performance, and the computational complexity.

The reminder of the paper is organized as follows. In the next section, the two-stage detection architecture is described. In Sec. III the novel TBD algorithm is presented, and its performance is analyzed in Sec. IV. Finally, concluding remarks are given in Sec. V.

\(^2\)Here the main performance measure is the probability of detection, the estimated trajectory being a side-result of the multi-frame processing. The track initiation and maintenance processes are handled by the subsequent tracking stage, which combines the positions (or the estimated track segments) into long continuous tracks.
II. TWO-STAGE MULTI-FRAME DETECTION

The proposed two-stage approach follows the scheme outlined in Fig. 2. At each scan $n$, the Detection and Plot-Extraction stage receives the raw data collected by the sensor and produces a list of candidate plots (or alarms). The processing typically concerns a detection, a clustering, and an extraction step, and operates with a primary threshold $\gamma_1$. The $k$-th plot at scan $n$ is the 5-dimensional vector

$$s_{k,n} = \left( t_{k,n}, r_{k,n}, \theta_{k,n}, A_{k,n}, N_{k,n} \right)$$

(1)

where $t_{k,n}$ is the time instant at which the alarm has been taken, $r_{k,n}$ is the range measurement, $\theta_{k,n}$ is the azimuth measurement, $A_{k,n}$ is the amplitude of the received signal, and $N_{k,n}$ is the power of the disturbance (thermal noise plus clutter). These are the typical measurements taken by a long-range surveillance radar system, but additional measurements, such as elevation angle and/or range-rate, can be taken into account. We also assume that the radar is ground-based, but the discussion can be easily generalized to ship- and air-borne radars by compensating the (known) platform motion.

The plots at scan $n$, whose number is denoted $D_n$, are organized in a plot-list, which is the $D_n \times 5$ matrix defined, for $D_n \neq 0$, as

$$S_n = \begin{bmatrix} s_{1,n} \\ \vdots \\ s_{D_n,n} \end{bmatrix}.$$

The plot-lists corresponding to the current and the previous $L - 1$ scans are the input of the second stage. After examining the correlation among the alarms in $\{S_{\ell}\}_{\ell=n-L+1}^n$, the second stage confirms or deletes each alarm contained in the current plot-list $S_n$ through a secondary threshold $\gamma_2$.

This operating scheme is extremely flexible and versatile, and it subsumes both the traditional scheme in Fig. 1, if the memory of the second stage is set equal to $L = 1$, and MFD with raw input data [2]–[4].

Figure 2. Operating scheme of the proposed detection architecture: the primary threshold is lowered, and an Along-Track Integration procedure confirms reliable detections.
[25], if the primary threshold $\gamma_1 = -\infty$. In its typical operating mode, the first stage adopts a primary threshold lower than that used in the traditional scheme in Fig. 1, and this causes an increment in both the probability of detection (PD), which is the probability that an alarm is true, and the false alarm rate (FAR), which is the average number of false alarms in a minute. The goal of the second stage is to restore the FAR to the level originally granted by the traditional scheme through a higher threshold value while maintaining part (if not all) of the gain in terms of PD, or, alternatively, to obtain a gain in terms of FAR maintaining the same PD as the traditional scheme.

In the following, the probability that an alarm is false will be referred to as PFA, and the subscripts “in/out” will be added to PD, PFA, and FAR to specify whether these quantities refer to the plot-list at the input or at the output of the second stage, respectively. In the reminder of the section we present the test statistics employed in the second stage and the target kinematic constraints involved, while in Sec. III we illustrate in detail the whole processing chain carried out by the second stage.

A. Test statistics

To simplify exposition, let us assume that $S_L$ is the current plot-list, so that the observations taken at scans $\ell = 1, \ldots, L$ are jointly processed. The task of the second stage is to confirm or delete $s_{k,L}$ for $k = 1, \ldots, D_L$. The trajectory of a prospective target from scan 1 to $L$ can be specified by an $L$-dimensional vector, say $\nu = (\nu_1, \ldots, \nu_L)$, with $\nu_\ell \in \{0, 1, \ldots, D_\ell\}$ for $\ell = 1, \ldots, L$. Specifically, $\nu_\ell = k$ means that the target is observed at scan $\ell$, and the corresponding alarm is $s_{k,\ell}$, while $\nu_\ell = 0$ that there is a missing observation at scan $\ell$. The sequence of plots indexed by $\nu$ is $\{s_{\nu_\ell,\ell} : \ell = 1, \ldots, L \text{ and } \nu_\ell \neq 0\}$.

Define, for $\ell = 1, \ldots, L$,

$$z_{k,\ell} = \begin{cases} A_{k,\ell}^2/N_{k,\ell}, & \text{if } k \in \{1, \ldots, D_\ell\} \\ \eta, & \text{if } k = 0 \end{cases}$$

(2)

so that, if $k \neq 0$, $z_{k,\ell}$ represents the normalized strength of the signal return for plot $k$ at scan $\ell$, while, if $k = 0$, the parameter $\eta$, which is to be set at the design stage, accounts for the missing observation. Then $\sum_{\ell=1}^{L} z_{\nu_\ell,\ell}$ is a meaningful decision statistic, since it is related to the overall energy back-scattered by the target during its motion if a target with trajectory $\nu$ is actually present, while it contains only disturbance if all plots indexed by $\nu$ are false alarms. The uncertainty as to the target trajectory can be removed by resorting to a maximization process, so that the decision statistics to be computed for the plots contained in $S_L$ are

$$\max_{\nu \in \mathcal{R}_{k,L}} \sum_{\ell=1}^{L} z_{\nu_\ell,\ell}, \quad k = 1, \ldots, D_L$$

(3)
Figure 3. Mean radial and tangential velocities associated to the plots taken at time $t_1$ and $t_2$.

where $\mathcal{R}_{k,L}$ is the set of $L$-dimensional vectors indexing the admissible trajectories ending in $s_{k,L}$ at scan $L$. See the Appendix for a derivation of the test statistic.

B. Track constraints

Every real target must comply with some physical constraints on its kinematics, which limit the number of candidate trajectories in the set $\mathcal{R}_{k,L}$. To be more specific, $\mathcal{R}_{k,L}$ is composed of all such trajectories $\nu$ such that $\nu_L = k$, and $\{s_{\nu,\ell} : \ell \in \{1,\ldots,L\}$ and $\nu_\ell \neq 0\}$ are compatible (i.e., satisfy the constraints). The constraints considered here are on the maximum target speed, say $v_{\text{max}}$, and possibly on the maximum target acceleration, say $a_{\text{max}}$.

Consider two plots corresponding to two successive scans, whose measurements are $(t_1 r_1 \theta_1)$ and $(t_2 r_2 \theta_2)$, with $t_1 < t_2$. Then it is not difficult to verify (see also Fig. 3) that the radial and tangential mean velocities for $(t_2 r_2 \theta_2)$ are

$$v_{r,2} = \frac{r_2 - r_1 \cos(\theta_2 - \theta_1)}{t_2 - t_1}$$

$$v_{t,2} = \frac{r_1 \sin(\theta_2 - \theta_1)}{t_2 - t_1}.$$  \hspace{1cm} (4a)

If there were no uncertainty on these velocities, then the velocity constraint would be satisfied if $v_{r,2}^2 + v_{t,2}^2 < v_{\text{max}}^2$. However, range and azimuth measurements are affected by errors, which propagates into the evaluation of velocity (and acceleration). Let $\sigma_r$ and $\sigma_\theta$ be the standard deviations of the range and azimuth errors, respectively, and assume that these errors are zero-mean; then the standard deviations of

$$\sigma_{r,2} = \sqrt{\sigma_r^2 + \left(\frac{v_{r,1} \sigma_{\theta,1}}{t_2 - t_1}\right)^2}$$

$$\sigma_{t,2} = \sqrt{\sigma_\theta^2 + \left(\frac{v_{t,1} \sigma_r}{t_2 - t_1}\right)^2}.$$  \hspace{1cm} (4b)
the errors on the velocities in (4) are approximately equal to

\[ \sigma_{v_r,2} = \frac{\sqrt{1 + \cos^2(\theta_2 - \theta_1)} \sigma_r^2 + 2r_1^2 \sin^2(\theta_2 - \theta_1) \sigma_\theta^2}{t_2 - t_1} \]

\[ \sigma_{v_t,2} = \frac{\sin^2(\theta_2 - \theta_1) \sigma_r^2 + 2r_1^2 \cos^2(\theta_2 - \theta_1) \sigma_\theta^2}{t_2 - t_1} \]

and the velocity constraint becomes

\[ \left[ \left( |v_{r,2}| - \beta \sigma_{v_r,2} \right) \right]^2 + \left[ \left( |v_{t,2}| - \beta \sigma_{v_t,2} \right) \right]^2 < v_{\text{max}}^2 \]

where \( x^+ = \max\{x, 0\} \), and \( \beta \) accounts for a given percentage of the errors.\(^4\)

If measurements \((t_2, r_2, \theta_2)\) and \((t_1, r_1, \theta_1)\) pass the velocity check, and if there is a previous measurement \((t_0, r_0, \theta_0)\) in the trajectory, with \( t_0 < t_1 \), then the target acceleration can be also checked by comparing the mean velocities in (4) with the mean velocities for \((t_1, r_1, \theta_1)\), say \( v_{r,1} \) and \( v_{t,1} \), obtained from the measurements \((t_1, r_1, \theta_1)\) and \((t_0, r_0, \theta_0)\) (see Fig. 3). Hence, the radial and tangential accelerations for \((t_2, r_2, \theta_2)\) are

\[ a_{r,2} = \frac{v_{r,2} - \left( v_{r,1} \cos(\theta_2 - \theta_1) + v_{t,1} \sin(\theta_2 - \theta_1) \right)}{t_2 - t_1} \]

\[ = \frac{r_2 - r_1 \cos(\theta_2 - \theta_1) - r_1 \cos(\theta_2 - \theta_1) - r_0 \cos(\theta_2 - \theta_0)}{(t_2 - t_1)^2 (t_2 - t_1)(t_1 - t_0)} \]

\[ a_{t,2} = \frac{v_{t,2} - \left( -v_{r,1} \sin(\theta_2 - \theta_1) + v_{t,1} \cos(\theta_2 - \theta_1) \right)}{t_2 - t_1} \]

\[ = \frac{r_1 \sin(\theta_2 - \theta_1)}{(t_2 - t_1)^2} - \frac{r_0 \sin(\theta_2 - \theta_0)}{(t_2 - t_1)(t_1 - t_0)}. \]

Taking into account the errors on the radial and tangential acceleration, whose approximated expressions of the standard deviations are

\[ \sigma_{a_{r,2}} = \frac{1}{t_2 - t_1} \left\{ \frac{1}{(t_2 - t_1)^2} + \frac{\cos^2(\theta_2 - \theta_1)}{(t_2 - t_1)^2} \left( \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \right)^2 + \frac{\cos^2(\theta_2 - \theta_0)}{(t_1 - t_0)^2} \right\} \]

\[ + \left[ \left( r_1 \sin(\theta_2 - \theta_1) \left( \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \right) - \frac{r_0 \sin(\theta_2 - \theta_0)}{t_1 - t_0} \right)^2 \right] \]

\[ + r_1^2 \sin^2(\theta_2 - \theta_1) \left( \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \right)^2 + \frac{r_0^2 \sin^2(\theta_2 - \theta_0)}{(t_1 - t_0)^2} \right\}^{1/2} \]

\(^3\)Recall that the standard deviation of a non-linear function \( f \) of the uncorrelated random variables \( x_1, \ldots, x_n \) with standard deviations \( \sigma_{x_1}, \ldots, \sigma_{x_2} \) is approximately equal to \( \sigma_f = \left( \sum_{i=1}^n (\partial f/\partial x_i)^2 \sigma_{x_i}^2 \right)^{1/2} \) [46].

\(^4\)E.g., in the Gaussian case, about 95.5% and 99.7% of the error values are within \( \beta = 2 \) and \( \beta = 3 \) standard deviations from the mean, respectively. If the distribution of the errors is not known, then Chebyshev’s inequality can be used: e.g., the amount of data within \( \beta = 2 \) or \( \beta = 3 \) standard deviations from the mean is always at least 75% or 89%, respectively.
The acceleration constraint becomes

\[
\sigma_{a_{1,2}} = \frac{1}{t_2 - t_1} \left\{ \frac{\sin^2(\theta_2 - \theta_1)}{(t_2 - t_1) + \frac{1}{t_1 - t_0}} + \frac{\sin^2(\theta_2 - \theta_0)}{(t_1 - t_0)^2} \right\}^{1/2}
\]

We finally consider an additional constraint on the maximum number of consecutive misses in the candidate trajectories. This ensures that trajectories with large “holes,” are not examined: this is desirable since consecutive plots too spaced away in time are scarcely correlated. Clearly, consecutive misses at the beginning of the trajectory should be allowed, since they account for newly born targets, i.e., targets that enter the scene during the \(L\) elaborated scans. Therefore, denoting \(z(v)\) the maximum number of consecutive zeros after the first non-zero entry of the vector \(v\), we require that all \(v \in R_{k,L}\) satisfy \(z(v) \leq P - 1\), with \(P \in \{1, \ldots, L - 1\}\).

### III. Algorithm Description

The proposed TBD processor in Fig. 2 consists of four distinct blocks, as shown in Fig. 4:

i) *Track Formation*, which computes the test statistics in (3) accounting for the measurement accuracy and the constraints of Sec. II-B;

ii) *Track Pruning*, which solves the data-association problem arising when multiple estimated trajectories share a common root;
iii) *Plot Confirmation*, which compares the decision statistics with the threshold $\gamma_2$ and confirm or delete the plots accordingly;

iv) *Track Smoothing* (optional), which improves the measurement accuracy of the confirmed plots.

In the following each block is described in detail.

### A. Track Formation

In most radar applications, the cardinality of the set of physically-admissible candidate trajectories is very large, and a brute-force, exhaustive search is not feasible, as it would be exponentially complex in the number of integrated scans. A possible way to reduce complexity from exponential to linear in the number of integrated scans is to discretize the covered area and resort to the Viterbi algorithm [4]–[7], [25]. However, this approach can be still too demanding, since the number of resolution elements can be very large (of the order of $10^5$ or more in many applications). The track formation algorithm presented next avoids discretization of the covered area and allows to compute the statistics in (3) with a complexity that is affordable in most applications.

Let $\mathcal{R}_{k,\ell}$ be the set of $\ell$-dimensional vectors indexing the admissible (i.e., satisfying the constraints of Sec. II-B) trajectories ending in $s_{k,\ell}$ at scan $\ell$, and define

$$
\tau_{k,\ell} = \arg \max_{\nu_\ell \in \mathcal{R}_{k,\ell}} \sum_{p=1}^{\ell} u_{\nu_\ell,p},
$$

$$
F_{k,\ell} = \max_{\nu_\ell \in \mathcal{R}_{k,\ell}} \sum_{p=1}^{\ell} z_{\nu_\ell,p}
$$

for $k = 1, \ldots, D_\ell$, and $\ell = 1, \ldots, L$, so that the test statistics in (3) are just $\{F_{k,L}\}_{k=1}^{D_L}$. Also, let $\mathcal{M}_{k,\ell}$ denote the set of indexes addressing all past alarms compatible (i.e., satisfying the constraints of Sec. II-B) with alarm $k$ at scan $\ell$, i.e.,

$$
\mathcal{M}_{k,\ell} = \left\{(j,p) : p \in \{ \max\{1, \ell - P\}, \ldots, \ell - 1\}, j \in \{1, \ldots, D_p\}, \text{ and } s_{k,\ell} \right\}.
$$

Then, Algorithm 1 computes (approximately or exactly, whether the constraint on the acceleration in Sec. II-B is considered or not, respectively) $\{\tau_{k,L}, F_{k,L}\}$, for $k = 1, \ldots, D_L$.

It is worthwhile giving some comments on the recursive step (lines 5–16) of Algorithm 1. In order to compute $F_{k,\ell}$ the algorithm searches in the set $\mathcal{M}_{k,\ell}$ for the best past alarm that can be linked with alarm $s_{k,\ell}$. If no past alarm is compatible ($\mathcal{M}_{k,\ell} = \emptyset$), then $F_{k,\ell}$ is initialized with the current measurement.
Algorithm 1 Computes $\{\tau_{k,L}, F_{k,L}\}_{k=1}^{D_L}$

1. for $k = 1, \ldots, D_1$ do
2. $F_{k,1} = A_{k,1}^2 / N_{k,1}$
3. $\tau_{k,1} = k$
4. end for
5. for $\ell = 2, \ldots, L$ do
6. for $k = 1, \ldots, D_\ell$ do
7. if $\mathcal{M}_{k,\ell} \neq \emptyset$ then
8. $(h, m) = \arg \max_{(j,p) \in \mathcal{M}_{k,\ell}} F_{j,p}$
9. $F_{k,\ell} = F_{h,m} + (\ell - m - 1)\eta + A_{k,\ell}^2 / N_{k,\ell}$
10. $\tau_{k,\ell} = (\tau_{h,m} 0 \cdots 0 k)$
11. else
12. $F_{k,\ell} = (\ell - 1)\eta + A_{k,\ell}^2 / N_{k,\ell}$
13. $\tau_{k,\ell} = (0 \cdots 0 k)$
14. end if
15. end for
16. end for

$z_{k,\ell}$ incremented by $(\ell - 1)\eta$, and the corresponding trajectory has $\ell - 1$ trailing zeros and $k$ as the last entry (lines 12 and 13). Otherwise the best admissible past alarm $(s_{h,m},$ form line 8) is linked with $s_{k,\ell}$: $F_{k,\ell}$ is computed by adding the statistic $z_{k,\ell}$ to the largest previous metric, stored in $F_{h,m}$, incremented by $(\ell - m - 1)\eta$ (to account for $\ell - m - 1$ misses, see line 9), and the corresponding trajectory $\tau_{k,\ell}$ is updated accordingly (line 10).

Once the track formation algorithm is terminated, plot $s_{k,L}$, along its statistic $F_{k,L}$ and its estimated trajectory $\tau_{k,L}$, is sent to the Track Pruning stage.

B. Track Pruning

A data association problem may arise after computing the statistics in (3). Indeed, several estimated trajectories may share a common root, and true target echoes may be responsible not only for the confirmation of the true alarms they caused, but also of false alarms in their proximity. This ambiguity is solved by the Track Pruning stage, which executes Algorithm 2, where $(v)_\ell$ denotes the $\ell$-th entry of
Algorithm 2 Prunes \( \{ \tau_{k,L} \}_{k=1}^{D_L} \) and recomputes \( \{ F_{k,L} \}_{k=1}^{D_L} \)

1. \( \mathcal{W} = \{1, \ldots, D_L \} \)
2. while \( \mathcal{W} \neq \emptyset \) do
3. \( q = \arg \max_{w \in \mathcal{W}} F_{w,L} \)
4. for \( p \in \mathcal{W} : p \neq q \) do
5. \( \ell = 1 \)
6. while \( (\tau_{p,L})_{\ell} = (\tau_{q,L})_{\ell} \) do
7. \( (\tau_{p,L})_{\ell} = 0 \)
8. \( \ell = \ell + 1 \)
9. end while
10. if # of non-zero entries of \( \tau_{p,L} < Q \) then
11. \( \tau_{p,L} = (0 \cdots 0 p) \)
12. end if
13. \( F_{p,L} = \sum_{\ell=1}^{L} z(\tau_{p,L})_{\ell,\ell} \)
14. end for
15. \( \mathcal{W} = \mathcal{W} \setminus \{q\} \)
16. end while

the vector \( v \).\(^5\) The algorithm assigns the common root only to the trajectory with the largest test statistic (line 3), and all other trajectories are pruned accordingly (lines 5–9). Moreover, if the number of non-zero entries of a trajectory is smaller than a specified minimum value, say \( Q \), then the trajectory is considered unreliable, and only the final plot is maintained (line 11). Finally, all test statistics corresponding to the new shortened trajectories are recomputed (line 13).

Once the algorithm is terminated, plot \( s_{k,L} \) along with the associated pruned trajectory \( \tau_{k,L} \) and test statistic \( F_{k,L} \) is sent to the Plot confirmation stage.

\(^5\)This algorithm is an extension of the procedure introduced in [35], which has been preferred among the many proposed in the past as it offers a good compromise between complexity and performance.
C. Plot Confirmation

Each plot in the current plot-list $S_L$ is confirmed or deleted by comparing the corresponding decision statistic with the secondary threshold $\gamma_2$, i.e.,

$$F_{k,L} \begin{cases} \geq \gamma_2 & \Rightarrow \text{confirm } s_{k,L} \\ < \gamma_2 & \Rightarrow \text{delete } s_{k,L} \end{cases}$$

for all $k \in \{1, \ldots, D_L\}$. Confirmed plots and their associated trajectories are sent, if needed, to the Track Smoothing stage.

D. Track Smoothing

Standard linear regression (or quadratic, if large maneuvers are expected) can be applied to confirmed plots bearing an estimated trajectory to improve the accuracy of range and azimuth measurements. Notice that, as a side result, the regression can also give information about velocity.

E. Complexity analysis

The computational complexity of the scheme in Fig. 4 is ruled by the complexity of Algorithm 1, which is a function of the number of integrated scans and of the number of plots per scan. The innermost loop of the algorithm requires to evaluate the set $M_{k,\ell}$, which amounts to check the kinematic constraints between $s_{k,\ell}$ and the tracklet indexed by $\tau_{j,p}$, for all $p = \max\{1, \ell - P\}, \ldots, \ell - 1$, and $j = 1, \ldots, D_p$. Therefore, the number of operations required in Algorithm 1 is in the order of

$$\sum_{\ell=2}^{L} D_\ell \sum_{p=\max\{1,\ell-P\}}^{\ell-1} D_p.$$ 

Notice now that $\{D_1, \ldots, D_L\}$ can be assumed to be a sequence of independent and identically distributed random variables. Specifically, denoting $N_r$ and $N_a$ the number of resolution elements in range and azimuth, respectively, and $K \in \{0, 1, \ldots, N_rN_a\}$ the number of targets present in the scene, then each $D_\ell$ can be modeled as the sum of two independent Binomial random variables with parameters $(N_rN_a - K, PFA_{in})$ and $(K, PD_{in})$. Thus, the average number of required operations is on the order of

$$\mathbb{E} \left[ \sum_{\ell=2}^{L} D_\ell \sum_{p=\max\{1,\ell-P\}}^{\ell-1} D_p \right] = \sum_{\ell=2}^{L} \sum_{p=\max\{1,\ell-P\}}^{\ell-1} \left( (N_rN_a - K) PFA_{in} + KPD_{in} \right)^2 \approx P \left( L - \frac{P + 1}{2} \right) \left[ (N_rN_a - K) PFA_{in} + KPD_{in} \right]^2$$

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where $\mathbb{E}$ denotes statistical expectation. Hence, the average complexity of Algorithm 1 when $K$ targets are present in the scene isootnote{Let $f$ and $g$ be positive real functions defined on the integers, then $f(n) = \mathcal{O}(g(n))$ means that there exists positive constants $c$ and $m$, such that $f(n) \leq cg(n)$, for all $n \geq m$.}

$$\mathcal{O}\left( LP \left[ (N_r N_a - K) \text{PFA}_{\text{in}} + K \text{PD}_{\text{in}} \right]^2 \right)$$

i.e., linear in the number of integrated scans and quadratic in the average number of alarms in the input plot-list.

Recall now that the complexity of the Viterbi-based solution is $\mathcal{O}(L N_r N_a Q)$, $Q$ denoting the number of possible state transitions in a scan, (e.g., see [40], [47]) whereby the proposed procedure is preferable if the average number of alarms per scan grows at a rate smaller than $\sqrt{N_r N_a Q/P}$, and this condition can be met adjusting $\gamma_1$ accordingly.

IV. NUMERICAL RESULTS

The performance measures used to test the proposed scheme in Fig. 2 are PD and FAR at both the input and the output of the second stage, and the subscripts “in” and “out” will be used to avoid confusion.

The formal definition of $\text{PD}_{\text{out}}$ is slightly different from that of $\text{PD}_{\text{in}}$. The latter has a “local” meaning and is defined as the probability that an alarm in the input plot-list is true. The former, instead, has a “global” meaning: specifically, since every alarm in the output plot-list bears attached a trajectory, $\text{PD}_{\text{out}}$ is the probability that the trajectory corresponding to an alarm in the output plot-list is true, i.e., caused by a target actually present in the scene. The root mean square error (RMSE) on the estimation of the target position is also considered, and, again, a subscript is added to address the input and the output. It is defined as

$$\text{RMSE} = \sqrt{\mathbb{E}[e^2(s_{k,L}) | H_1]}$$

where $H_1$ is the event that $s_{k,L}$ is confirmed by the second stage, and that its trajectory is true, and $e(s_{k,L})$ is the Euclidean distance between the true and the estimated target position.

We discuss a numerical example, where a Swerling I fluctuation model is assumed. Each variable $A_{k,\ell}^2/N_{k,\ell}$ is exponentially distributed, with location parameter $\gamma_1$, and scale parameter $1$ or $(1 + \text{SDR})^{-1}$, whether it comes from a false or true alarm. The constant $\eta$, has been set equal to zero. Range and azimuth measurements are affected by errors, which are independent, zero mean, Gaussian random variables, with standard deviations $\sigma_r = 20$ m, and $\sigma_\theta = 0.5^\circ$, respectively, independent of the SDR. Targets follow a
constant acceleration model (commonly used to account maneuvers [48]), wherein initial position, initial velocity, and acceleration are randomly generated at each run. The scan period is 1 s, and the search area is ±60° and 40 to 140 km, where $K = 20$ are present, and this value is unknown. With reference to Fig. 4, the radar is set so as to detect targets with velocities up to $v_{\text{max}}$, specified in each experiment, and accelerations up to $a_{\text{max}} = 20 \text{ m/s}^2$. The maximum number of consecutive misses in the Track Formation stage has been set to $P = 4$ and the minimum number of plots required by the Track Pruning stage in each trajectory to $Q = 3$, which is the minimum value to guarantee that the acceleration constraint can be checked. As to the Track Smoothing stage, a standard linear regression is adopted.

In the first set of figures, $v_{\text{max}}$ is 100 m/s. Fig. 5 reports $\text{PD}_{\text{in}}$ and $\text{PD}_{\text{out}}$ versus $\text{FAR}_{\text{in}}$ for various SDR’s when $L = 1$ and $L = 10$ scans are integrated, while Fig. 6 shows the corresponding RMSE on the estimated target position. Recall that, for $L = 1$, the scheme in Fig. 2 reduces to the traditional detector in Fig. 1. When $L = 10$, virtually all the detection gain obtained by lowering the threshold $\gamma_1$ in the
Figure 6. Root mean square error on the estimated target position at the input and output of the second stage versus the input false alarm rate for different values of SDR. The search area is ±60° and 40 to 140 km.

first stage is maintained for SDR ≥ 12 dB. When SDR = 9.30 dB most of this gain is still preserved: notice in particular that, lowering γ_1 so as to have FAR_in = 1700 per minute, a detection gain of 100% is possible with respect to the traditional detection scheme in Fig. 1, boosting PD from 0.2 to 0.4. As to the RMSE, a larger accuracy in the position measurements is possible at the output of the second stage for all the considered interval of FAR_in, and this accuracy improves as FAR_in is increased. Observe that, when FAR_in = 1 per minute, PD_out = PD_in, since the second stage cannot confirm more plots than those present in the input plot-list; however, RMSE_out is lower than RMSE_in, as the track information can be exploited to refine position estimates.

Fig. 7 shows PD_in and PD_out versus L for different values of SDR when FAR_in = 10^3 per min, and Fig. 8 reports the corresponding RMSE on the estimated target position. It can be seen that the PD_out rapidly reaches its maximum value, and that for L = 10 (which corresponds to an observation window of 10 s) PD_out saturates in all the considered SDR’s. As to RMSE_out, instead, it decreases monotonically,
since longer trajectories give rise to position estimates with smaller errors. Observe that both $\text{PD}_{\text{out}}$ and $\text{RMSE}_{\text{out}}$ remain equal to their corresponding input values for $L < 3$: this happens since the Track Pruning stage in Fig. 4 shrinks the trajectories with less than $Q = 3$ alarms to the final plot, so that the second stage does not act if less than 3 scans are integrated.

Fig. 9 shows $\text{PD}_{\text{out}}$ versus $\text{FAR}_{\text{out}}$—i.e., the receiver operating characteristic (ROC)—for $\text{SDR} = 12.0$ dB when $L = 1$ and $L = 10$ scans are integrated. $\text{FAR}_{\text{out}}$ is reported in logarithmic scale, and different values of $\text{FAR}_{\text{in}}$ are considered. Observe that, when $L = 10$, the $\text{PD}_{\text{out}}$ curve for a given $\text{FAR}_{\text{in}}$ level ends when $\text{FAR}_{\text{out}} = \text{FAR}_{\text{in}}$, since this is the largest admissible FAR. Moreover, these curves have a smaller slope than PD for the traditional scheme ($L = 1$), which means that the proposed two-stage scheme is less sensitive to threshold’s variations: to give an example, in the range $1 \leq \text{FAR} \leq 10^3$, while PD of the traditional scheme increase from 0.35 to 0.61, $\text{PD}_{\text{out}}$ for $L = 10$ is always around 0.6 for $\text{FAR}_{\text{in}} = 10^3$. It is also interesting to notice that, when $\text{FAR}_{\text{in}}$ varies, the ROC curves of the proposed
Figure 8. Root mean square error on the estimated target position at the input and output of the second stage versus the number of integrates scans for different values of SDR. The search area is ±60° and 40 to 140 km.

two-stage scheme describe a region, which lies above the ROC curve of the traditional scheme: therefore, a better performance can be achieved and a larger freedom is given at the system design stage.

In the second set of figures, $v_{max}$ is increased up to approximately Mach-2. Figs. 10 and 11 show PD and RMSE versus FAR$_{in}$, respectively, for different values of SDR when $L = 1$ and $L = 10$ scans are integrated, and $v_{max} = 300$ m/s, while Figs. 12 and 13 consider the same scene when $v_{max} = 600$ m/s. A comparison with Figs. 5 and 6 shows that almost no loss is incurred when FAR$_{in} \leq 10^3$, so that large gains with respect to the traditional scheme ($L = 1$) are possible even in the detection of fast, maneuvering targets. When FAR$_{in}$ is increased to values larger than $10^3$, the detection gain remain almost unchanged, but the accuracy in the position estimation degrades, and, when $v_{max} = 600$ m/s, it becomes poorer than that of the traditional scheme. We remark, however, that these values of FAR are not of interest, as the computational complexity needed to run Algorithm 1 becomes comparable to that of the Viterbi algorithm and, then, unaffordable.
Figure 9. Probability of detection versus false alarm rate at the output of the second stage for different values of FAR$_{in}$ when SDR = 12.0 dB. The search area is ±60° and 40 to 140 km.

In Fig. 14 we report a short data snapshot of 120 scans (i.e., 2 minutes) for SDR = 12.0 dB, FAR$_{in}$ = $10^3$ per min, and $v_{\text{max}}$ = 600 m/s. The black dots represent the alarm’s positions, while the red straight lines indicate the true target trajectories. Plot (a) reports the input of the second stage, while Plots (b) and (c) refer to the outputs when $L = 1$ and $L = 10$, respectively. It is seen that the proposed two-stage scheme is able to confirms 1030 alarms, as opposite to the 619 alarms of the traditional scheme, and all these alarms are correctly located along the true target trajectories.

Finally, for comparison purposes, we analyze the case where the primary threshold $\gamma_1 \rightarrow -\infty$, and the detection architecture in Fig. 2 reduces to the detectors operating on raw input data considered in [2]–[4], [25]. To limit the computational burden, we restrict the search area to ±3° and 98 to 102 km, where one target with velocity up to $v_{\text{max}}$ = 100 m/s is simulated. The scan period is 1 s, FAR$_{\text{out}}$ = 1 per minute, and all other parameters remains unchanged.

Fig. 15 reports PD$_{\text{out}}$ versus SDR for different values of FAR$_{in}$ when $L = 1$ and $L = 10$ scans are
integrated, while Fig. 16 shows the corresponding RMSE’s on the estimated target position. Similarly to the previous plots, PD_{out} increases monotonically with FAR_{in}, showing here a gap of about 10 dB at PD_{out} = 0.5 between the traditional scheme in Fig. 1 (L = 1) and MFD with raw data (L = 10 and γ_1 = −∞). Any point between these two extrema can be achieved with proper selection of γ_1 in the proposed scheme in Fig. 2, and complexity is traded for performance. As to RMSE_{out}, it is decreasing with SDR, but no strict ordering can be observed among different values of FAR_{in}. Furthermore, it can be noticed that in the small SDR regime the detection gain with respect to the traditional scheme in Fig. 1 comes at the price of worse accuracy in the position estimation. In Figs. 17 and 18, instead, we analyze...
Figure 11. Root mean square error on the estimated target position at the input and output of the second stage versus the input false alarm rate for different values of SDR. The search area is ±60° and 40 to 140 km.

the performance of the procedure in terms of track estimation and range-rate estimation, respectively,\textsuperscript{7} and RMSE\textsubscript{out} is reported versus SDR for different values of FAR\textsubscript{in} and for \( L = 10 \). Again, there is no strict ordering among the different levels of FAR\textsubscript{in}, but it can be observed that FAR\textsubscript{in} = 10 (which is equivalent to FAR\textsubscript{in} = 5 \cdot 10^3 in the ±60° and 40 to 140 km scenario) delivers, with negligible computational complexity increase with respect to the traditional scheme, a gain in the detection probability, a higher accuracy in the position estimation over a wide range of SDR’s (corresponding to PD\textsubscript{out} \geq 5 \cdot 10^{-2}), and a range-rate information (otherwise not available).

\textsuperscript{7}Track estimates are side results of the energy integration process, and they can be exploited to extract range-rate information. We remark here that they are not long continuous tracks in terms of Cartesian position and velocity, as it happens in standard tracking, but small tracklets in term of range-azimuth measurements only, which can be possibly used by the following tracking stage to form long continuous tracks.
Figure 12. Detection probability at the input and output of the second stage versus the input false alarm rate for different values of SDR. The search area is ±60° and 40 to 140 km.

V. CONCLUSION

In this work we have proposed and analyzed a two-stage architecture for target detection in radar systems, wherein a TBD processor operates on a set of candidate plots provided by the Detector and Plot-Extractor. A novel dynamic programming algorithm, which does not require a discretization of the state space, has been derived for plot validation. The complexity analysis reported in Sec. III-E has shown that in standard radar scenarios the proposed algorithm has a computational complexity which can be much lower than that of a multi-frame detection procedure based on the Viterbi algorithm (hardly amenable to a real-time implementation when the number of resolution elements is large). Additionally, the numerical analysis in Sec. IV has shown that the proposed two-stage detection procedure guarantees a large detection gain with respect to the standard radar processing operating at the same level of false alarm rate.
Figure 13. Root mean square error on the estimated target position at the input and output of the second stage versus the input false alarm rate for different values of SDR. The search area is ±60° and 40 to 140 km.

APPENDIX

Let $f_0$ and $f_1$ be the densities of $A_{k,\ell}^2/N_{k,\ell}$ whenever $s_{k,\ell}$ is a false or a true alarm, respectively, and denote $\Lambda = \ln(f_1/f_0)$. Assume that $A_{k,\ell}^2/N_{k,\ell} > \eta$ with probability one, as a result of the first stage thresholding. Then the density of $z_{k,\ell}$ in (2) is

\begin{align*}
g_1(z) &= (1 - \text{PD}_{\text{in}}) \mathbb{1}_{\{z=\eta\}} + \text{PD}_{\text{in}}f_1(z) \\
g_0(z) &= (1 - \text{PFA}_{\text{in}}) \mathbb{1}_{\{z=\eta\}} + \text{PFA}_{\text{in}}f_0(z)
\end{align*}

whether it is a true alarm or not, respectively, where $\mathbb{1}_B$ is the indicator function of the event $B$. Let $\nu$ be the trajectory of the target, and assume scan-to-scan independence conditioned on the absence or

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8It is a mixed (discrete/continuous) random variables, and the density is computed with respect to the measure defined as the sum of the Dirac measure centered in $\eta$ and the Lebesgue measure.
presence of a target with this trajectory, then the log-likelihood ratio of \( \{ z_{\nu,\ell} \}_{\ell=1}^L \) is
\[
\sum_{\ell=1}^L \left[ \left( \ln \frac{\text{PD}_{\text{in}}}{1 - \text{PFA}_{\text{in}}} + \Lambda( z_{\nu,\ell} ) \right) \mathbb{1}_{\{ \nu_{\ell} \neq 0 \}} + \ln \frac{1 - \text{PD}_{\text{in}}}{1 - \text{PFA}_{\text{in}}} \mathbb{1}_{\{ \nu_{\ell} = 0 \}} \right]
\]
\[
= \sum_{\ell=1}^L \left[ \Lambda( z_{\nu,\ell} ) \mathbb{1}_{\{ \nu_{\ell} \neq 0 \}} + \kappa \mathbb{1}_{\{ \nu_{\ell} = 0 \}} \right] + L \ln \frac{\text{PD}_{\text{in}}}{1 - \text{PFA}_{\text{in}}}
\]
where \( \kappa = \ln \frac{1/\text{PD}_{\text{in}} - 1}{1/\text{PFA}_{\text{in}} - 1} \). The uncertainty so as to \( \nu \) can be removed by maximizing over the set of admissible trajectories \( R_{k,L} \), and the generalized likelihood ratio test (see [49]) amounts to compare with a threshold the statistic
\[
\max_{\nu \in R_{k,L}} \sum_{\ell=1}^L \left[ \Lambda( A_{k,\ell}^2 / N_{k,\ell} ) \mathbb{1}_{\{ \nu_{\ell} \neq 0 \}} + \kappa' \mathbb{1}_{\{ \nu_{\ell} = 0 \}} \right]
\]
where it has been exploited the fact that \( z_{\nu,\ell} = A_{k,\ell}^2 / N_{k,\ell} \) if \( \nu_{\ell} \neq 0 \).

Observe that if \( \Lambda \) is an affine, increasing function, i.e., \( \Lambda(z) = az + b \), \( a > 0 \), then (5) is equivalent to
\[
\max_{\nu \in R_{k,L}} \sum_{\ell=1}^L \left( A_{k,\ell}^2 / N_{k,\ell} \mathbb{1}_{\{ \nu_{\ell} \neq 0 \}} + \kappa' \mathbb{1}_{\{ \nu_{\ell} = 0 \}} \right)
\]
Figure 15. Detection probability at the output of the second stage versus SDR for different values of FAR_{in}. The search area is ±3° and 98 to 102 km.

which corresponds to (3) if \eta \text{ is set equal to } \kappa' = (\kappa - b)/a. E.g., if \ f_0 \text{ and } f_1 \text{ are densities of exponential distributions with parameters } \lambda_0 \text{ and } \lambda_1 \text{, respectively, with location parameter } \gamma_1 \text{ and scale parameter } \lambda_0 > \lambda_1 \text{, respectively, with } \lambda_0 > \lambda_1 \text{ (i.e., } f_i(z) = \lambda_i e^{-\lambda_i(z-\gamma_i)} 1_{\{z\geq\gamma_i\}}, i = 0, 1, \text{ which is a common model for radar measurements), then } \Lambda \text{ is affine and increasing.}

REFERENCES

Figure 16. Root mean square error on the estimated target position at the output of the second stage versus SDR for different values of FAR$_{in}$. The search area is $\pm 3^\circ$ and 98 to 102 km.


Figure 17. Root mean square error on the estimated target trajectory at the output of the second stage versus SDR for different values of $\text{FAR}_{\text{in}}$. The search area is $\pm 3^\circ$ and 98 to 102 km.

Figure 18. Root mean square error on the estimated target range-rate at the output of the second stage versus SDR for different values of $\text{FAR}_{\text{in}}$. The search area is $\pm 3^\circ$ and 98 to 102 km.


