Generalized Frequency-Interval Balanced Model Reduction Method

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Abstract—In this paper, a new method for model reduction of bilinear systems is presented. The method is developed in particular for many applications in which one is interested to approximate a system in a given frequency-interval. To this end, new generalized frequency-interval gramians are introduced for bilinear systems. It is shown that these gramians are the solutions to the so-called frequency-interval generalized Lyapunov equations. Algorithms are proposed to solve such equations iteratively. The method is further illustrated with the help of an illustrative example. The numerical results show that the method is more accurate than its previous counterpart which is based on the ordinary gramians.

I. INTRODUCTION

The order of today’s dynamic models are increasing rapidly as a result of the ever increasing demand for detailed mathematical modeling of the complex systems and advanced engineering processes. The simulation, analysis and control synthesis for the systems of high orders are difficult and costly and sometimes even impossible. To cope with these problems, over the past, there has been increasing interest in model reduction methods. These methods reduce the order of dynamical models while approximating the input-output behavior and preserving the important features of the system [1].

The model reduction techniques can be divided into two broad categories: singular value decomposition (SVD) based methods and the moment matching based techniques. The SVD-based methods have a guaranteed upper bound for the approximation error and they usually preserve the stability of the original model in the reduction process. The moment matching based methods are usually more efficient computationally, but they have no guaranteed error bound. The stability of the reduced order model is not guaranteed when these methods apply [1]. Most of the studies related to model reduction presented so far have been devoted to linear case and just few methods have been proposed for nonlinear cases. However, in many applications, linear models are often insufficient to describe the behavior of the processes. On the other hand, due to the complexity of nonlinear systems, methods for analyzing nonlinear systems or synthesizing their controllers are not as well-developed as their linear counterparts. In between the spectrum of different models from linear model to highly nonlinear models, the bilinear models often offer an adequately accurate model [2].

Bilinear systems are important class of nonlinear systems with a lot of practical applications. Bilinear systems enjoy well-established theories and find applications in the variety of fields to describe the processes ranging from electrical networks, hydraulic systems to heat transfer, and chemical processes. Moreover, many highly nonlinear systems may be modeled as bilinear systems with appropriate state feedback or can be approximated as bilinear systems in the so-called bilinearization process see e.g. [3]-[4]. A new method for model reduction of bilinear systems is presented in this paper. The method is devised in particular for many applications in which one is interested to approximate a system within a given frequency-bound. To this end, new generalized frequency-interval gramians are introduced for bilinear systems. It is shown that these gramians are the solutions to the so-called frequency-interval generalized Lyapunov equations. We propose algorithms to solve such equations iteratively. The method is further illustrated with the help of an illustrative example.

The notation used in this paper is as follows: $M^*$ denotes transpose of matrix if $M \in R^{n \times m}$ and complex conjugate transpose if $M \in C^{n \times m}$. The $\otimes$ stands for the Kronecker Product. The standard notation $> , \geq (<, \leq)$ is used to denote the positive (negative) definite and semidefinite ordering of matrices.

II. FREQUENCY-INTERVAL BALANCED TRUNCATION FOR
LINEAR SYSTEMS

The balanced model reduction introduced in [10] is one of the most common model reduction schemes. To apply balanced truncation, the system is first represented in a basis where the states which are difficult to reach are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the controllability and the observability gramians, which are solutions to the controllability and the observability Lyapunov equations. Then, the reduced model is obtained by truncating the states which are related to the set of the least diagonal elements of the balanced gramians. Balanced model reduction method is modified and developed from different viewpoints [1]. The frequency-interval balanced truncation is among the methods which improves the accuracy of the ordinary balanced truncation.

The method was developed in particular for the applications in which one is interested to approximate a system in a given frequency-interval. This method was first proposed in [5] and has received a lot of attention over the last few decades [5]. In [5], it was suggested that the frequency-interval balanced realization can be obtained based on the frequency-interval gramians. For a dynamic system with the minimal realization:

\[ G(s) := (A, B, C, D) \] (1)
where \( G(s) \) is the transfer matrix with the state-space representation:

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in \mathbb{R}^n
\]

\[
y(t) = Cx(t) + Du(t).
\]

The ordinary controllability gramian \( P \) and the ordinary observability gramians \( Q \) are given by \([10],[11]\):

\[
P = \int_{0}^{\infty} e^{A\tau} BB^* e^{A^*\tau} d\tau,
\]

\[
Q = \int_{0}^{\infty} e^{A^*\tau} C^* Ce^{A\tau} d\tau.
\]

For the frequency-interval \([\omega_1, \omega_2]\), the frequency-interval gramians are defined as \([5]\):

\[
P(\omega_1, \omega_2) = P(\omega_2) - P(\omega_1),
\]

\[
Q(\omega_1, \omega_2) = Q(\omega_2) - Q(\omega_1),
\]

where:

\[
Q(\omega) := \frac{1}{2\pi} \int_{-\omega}^{\omega} (-Ij\theta - A^*)^{-1} C^* C(Ij\theta - A)^{-1}\theta d\theta,
\]

\[
P(\omega) := \frac{1}{2\pi} \int_{-\omega}^{\omega} (Ij\theta - A)^{-1} BB^* (Ij\theta - A^*)^{-1}\theta d\theta.
\]

These gramians are the solutions of the following Lyapunov equations \([5]\):

\[
AP(\omega_1, \omega_2) + P(\omega_1, \omega_2)A^* + W_c(\omega_1, \omega_2) = 0,
\]

\[
A^*Q(\omega_1, \omega_2) + Q(\omega_1, \omega_2)A + W_o(\omega_1, \omega_2) = 0.
\]

where:

\[
S(\omega) := \frac{1}{2\pi} \int_{-\omega}^{\omega} (Ij\theta - A)^{-1}\theta d\theta,
\]

\[
W_c(\omega) = S(\omega)BB^* + BB^*S^*(\omega),
\]

\[
W_o(\omega) = C^*CS(\omega) + S^*(\omega)C^*C,
\]

\[
W_c(\omega_1, \omega_2) = W_c(\omega_2) - W_c(\omega_1),
\]

\[
W_o(\omega_1, \omega_2) = W_o(\omega_2) - W_o(\omega_1).
\]

The frequency-interval balanced realization is obtained by balancing the frequency-interval gramians \( P(\omega_1, \omega_2) \) and \( Q(\omega_1, \omega_2) \). The reduced model is obtained by truncating the states which are associated to the least diagonal elements of the balanced frequency-interval gramians. The linear frequency-interval balanced truncation is modified over the years in the literature \([1],[6],[7],[8]\). However, all modifications and developments are based on the frequency-interval gramians.

### III. Balanced Model Reduction for Bilinear Systems

Let \( \Sigma \) be a bilinear dynamical systems which is described by:

\[
\Sigma : \begin{cases}
\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} N_j x(t)u_j(t) + Bu(t), \\
y(t) = Cx(t).
\end{cases}
\]

where \( x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^m, \ y(t) \in \mathbb{R}^m \).

The controllability gramian for this system is defined as \([11]-[15]\):

\[
P := \sum_{i=1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} P_i P_i^* dt_1...dt_i,
\]

where:

\[
P_i(t_1) = e^{At_i}\ B
\]

\[
P_i(t_1,...,t_i) = e^{At_i} \begin{bmatrix} N_1 P_{i-1} & N_2 P_{i-1} & \cdots & N_m P_{i-1} \end{bmatrix}
\]

and the observability gramian is defined as \([11]-[15]\):

\[
Q := \sum_{i=1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} Q_i^* Q_i dt_1...dt_i,
\]

where:

\[
Q_i(t_1) = Ce^{At_i}
\]

\[
Q_i(t_1,...,t_i) = \begin{bmatrix} Q_{i-1} N_1 & Q_{i-1} N_2 & \cdots & Q_{i-1} N_m \end{bmatrix} e^{At_i}.
\]

If \( A \) is stable, the gramians are given by the solutions of the generalized Lyapunov equations \([14],[16]\):

\[
AP + PA^* + \sum_{j=1}^{m} N_j PN_j^* + BB^* = 0,
\]

\[
A^*Q + AQ + \sum_{j=1}^{m} N_j^* QN_j + C^*C = 0.
\]

The balanced reduction algorithm for bilinear systems is very similar to that of linear system i. e. the reduced model is obtained by balancing the gramians \( P \) and \( Q \) and truncating the states which are associated to the least diagonal elements of the balanced gramians.

### IV. Frequency-Interval Model Reduction for Bilinear Systems

The frequency-interval bilinear model reduction is built upon the notion of the frequency-interval generalized gramians. For the limited frequency-interval \( \Omega = [\gamma_1, \gamma_2] \), we define the frequency-interval generalized controllability gramian \( P(\Omega) \) as follows:

\[
P(\Omega) := P(\gamma_2) - P(\gamma_1),
\]
where:
\[ P(\omega) := \sum_{i=1}^{\infty} \frac{1}{(2\pi)^i} \int_{-\omega}^{+\omega} \cdots \int_{-\omega}^{+\omega} P_i(\omega_1, \ldots, \omega_i) P_i(\omega_1, \ldots, \omega_i) \, d\omega_1 \cdots d\omega_i \] (18)

\[ P_1(\omega_1) = P(\omega) \text{ and } Q(\omega) \text{ which are defined in (18) and (21) are the solution to the following generalized Lyapunov equations:} \]

\[ \text{Proof:} \]

In (23), let \( \omega \to \infty \), we have:

\[ P(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (Ij\omega - A)^{-1}M(-Ij\omega - A^*)^{-1} \, d\omega. \] (28)

Using the Parseval’s theorem:

\[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} (Ij\omega - A)^{-1}M(-Ij\omega - A^*)^{-1} \, d\omega = \int_{0}^{+\infty} e^{At}Me^{A^*t}dt \]

Therefore [9]:

\[ AP(\infty) + P(\infty)A^* + M = 0 \] (30)

Plugging this into (28), we conclude:

\[ P(\omega) = \frac{1}{2\pi} \int_{-\omega}^{+\omega} P(\infty)(-j\omega I - A^*)^{-1} + (j\omega I - A)^{-1} P(\infty) \, d\omega, \] (31)

Hence:

\[ P(\omega) = P(\infty)S(\omega) + S(\omega)P(\infty) \] (32)

From definitions for \( S(\omega) \) and \( P(\infty) \) and the fact that we have:

\[ (j\omega_1 I - A)^{-1}(j\omega_2 I - A)^{-1} = (j\omega_2 I - A)^{-1}(j\omega_1 I - A)^{-1}, \] (33)

for all \( \omega_1, \omega_2 \in R \), we have:

\[ S(\omega)P(\infty) = \frac{1}{2\pi} \int_{-\omega}^{+\omega} S(\omega)M + MS(\omega)(-j\mu I - A^*)^{-1} \, d\mu \] (34)

Plugging this into (32)

\[ P(\omega) = \frac{1}{2\pi} \int_{-\omega}^{+\omega} S(\omega)M + MS(\omega)(-j\mu I - A^*)^{-1} \, d\mu \] (35)

Since \( A \) is stable from this we can conclude that \( P(\omega) \) is the solution to:

\[ AP(\omega) + P(\omega)A^* + S(\omega)M + MS^*(\omega) = 0, \]

the proof for the \( Q(\omega) \) is similar and therefore omitted.

\[ \square \]

In the following, with the help of this lemma we show that \( P(\omega) \) is the solution to a generalized Lyapunov equation.

**Theorem 1.** The gramian \( P(\omega) \) and \( Q(\omega) \) which are defined in (18) and (21) are the solution to the following generalized Lyapunov equations:
\[ AP(\omega) + P(\omega)A^* + S(\omega)(\sum_{j=1}^{m} N_j P(\omega) N_j^*) \\
+ \left( \sum_{j=1}^{m} N_j^* Q(\omega) N_j \right) S(\omega) + S(\omega) BB^* + BB^* S^*(\omega) = 0 \]  
(36)

\[ A^* Q(\omega) + Q(\omega) A + S^*(\omega)(\sum_{j=1}^{m} N_j^* Q(\omega) N_j) \\
+ \left( \sum_{j=1}^{m} N_j^* Q(\omega) N_j \right) S(\omega) + C^* C = 0 \]  
(37)

**Proof:**

Let:

\[ \tilde{P}_1(\omega) = \frac{1}{2\pi} \int_{-\omega}^{+\omega} P_1(\omega_1) P_1^*(\omega_1) d\omega_1 \]  
(38)

\[ \vdots \]

\[ \tilde{P}_1(\omega) = \frac{1}{(2\pi)^2} \int_{-\omega}^{+\omega} \int_{-\omega}^{+\omega} P_1(\omega_1, \omega_2) P_1^*(\omega_1, \omega_2) d\omega_1 d\omega_2 \]  
(39)

we have:

\[ P(\omega) = \sum_{i=1}^{\infty} \tilde{P}_i(\omega) \]  
(40)

Using Lemma 1 with \( M = BB^* \), it is clear that \( P_1(\omega) \) is the solution to:

\[ A \tilde{P}_1(\omega) + \tilde{P}_1(\omega) A^* + S(\omega) BB^* + BB^* S^*(\omega) = 0 \]  
(41)

For \( \tilde{P}_2(\omega) \), we have:

\[ \tilde{P}_2(\omega) = \frac{1}{(2\pi)^2} \int_{-\omega}^{+\omega} \int_{-\omega}^{+\omega} P_2(\omega_1, \omega_2) P_2^*(\omega_1, \omega_2) d\omega_1 d\omega_2 \]  
(42)

\[ = \frac{1}{(2\pi)^2} \int_{-\omega}^{+\omega} \int_{-\omega}^{+\omega} (j \omega_2 I - A)^{-1} \left[ \begin{array}{c} P_1^* N_1 \\ \vdots \\ P_m^* N_m \end{array} \right] (-j \omega_2 I - A^*)^{-1} d\omega_1 d\omega_2 \]  
(43)

Here with \( M = \sum_{j=1}^{m} N_j \tilde{P}_1(\omega) N_j^* \), Lemma 1 applies and consequently \( \tilde{P}_2(\omega) \) is the solution to:

\[ A \tilde{P}_2(\omega) + \tilde{P}_2(\omega) A^* + S(\omega) \sum_{j=1}^{m} N_j \tilde{P}_1(\omega) N_j^* + \sum_{j=1}^{m} N_j \tilde{P}_1(\omega) N_j^* S^*(\omega) = 0 \]  
(44)

Following the same procedure, according to Lemma 1 \( \tilde{P}_i(\omega) \) will be the solution to:

\[ A \tilde{P}_i(\omega) + \tilde{P}_i(\omega) A^* + S(\omega) \sum_{j=1}^{m} N_j \tilde{P}_{i-1}(\omega) N_j^* + \sum_{j=1}^{m} N_j \tilde{P}_{i-1}(\omega) N_j^* S^*(\omega) = 0 \]  
(45)

We add up equations (47), (41) and by using (40) we get:

\[ AP(\omega) + P(\omega) A^* + S(\omega) \sum_{j=1}^{m} N_j P(\omega) N_j^* \\
+ \left( \sum_{j=1}^{m} N_j^* Q(\omega) N_j \right) S(\omega) + S(\omega) BB^* + BB^* S^*(\omega) = 0 \]  
(46)

The proof for \( Q(\omega) \) is similar and hence omitted.

The generalized Lyapunov equations (36) can be solved iteratively. The controllability gramian \( P(\omega) \) is obtained by:

\[ P(\omega) = \lim_{i \to \infty} \tilde{P}_i(\omega) \]  
(48)

where:

\[ A \tilde{P}_i(\omega) + \tilde{P}_i(\omega) A^* + S(\omega) BB^* + BB^* S^*(\omega) = 0, \]  
(49)

The observability gramian is dually obtained by:

\[ Q(\omega) = \lim_{i \to \infty} \tilde{Q}_i(\omega) \]  
(50)

where:

\[ A^* \tilde{Q}_i(\omega) + \tilde{Q}_i(\omega) A + C^* C = 0 \]  
(51)

Theorem 1 can be used to compute the frequency-interval gramians. However, analogous to linear case, a generalized
Lyapunov equations in terms of $P(\Omega)$ and $Q(\Omega)$ can obtained as following:

$$AP(\Omega) + P(\Omega)A^* + S(\gamma_2)(\sum_{j=1}^{m} N_j P(\Omega) N_j^*) + (\sum_{j=1}^{m} N_j P(\Omega) N_j^*) S(\gamma_2) + W_c(\gamma) = 0, \quad \text{(52)}$$

where:

$$W_c(\gamma_1, \gamma_2) = S(\gamma_2)(\sum_{j=1}^{m} N_j P(\gamma_1) N_j^*) + (\sum_{j=1}^{m} N_j P(\gamma_1) N_j^*)S(\gamma_2) + S(\gamma_2)BB^* - S(\gamma_1)BB^* + BB^*S(\gamma_2) - BB^*S(\gamma_1). \quad \text{(53)}$$

and

$$A^*Q(\Omega) + Q(\Omega)A + (\sum_{j=1}^{m} N_j^* Q(\Omega) N_j)S(\gamma_2) + S(\gamma_2)BB^* - S(\gamma_1)BB^* + BB^*S(\gamma_2) - BB^*S(\gamma_1) = 0, \quad \text{(54)}$$

where:

$$W_o(\gamma_1, \gamma_2) = (\sum_{j=1}^{m} N_j^* Q(\gamma_1) N_j) S(\gamma_2) + S(\gamma_2)(\sum_{j=1}^{m} N_j^* Q(\gamma_1) N_j) + C^*CS(\gamma_2) - C^*CS(\gamma_1) + S(\gamma_2)C^*C - S(\gamma_1)C^*C. \quad \text{(55)}$$

Similar to linear case, using (52) or (36) to obtain the frequency-interval gramians for model reduction might be conservative and therefore some modifications are proposed. In what follows these modifications are presented and one can think of this as the extension of the modified frequency-limited model reduction of linear system to bilinear systems.

In the modified version we need to decompose $W_c(\gamma_1, \gamma_2)$ and $W_o(\gamma_1, \gamma_2)$ such that:

$$W_c(\gamma_1, \gamma_2) := MAM^* = M \text{ diag} (\lambda_1, \ldots, \lambda_n) M^*, \quad W_o(\omega_1, \omega_2) := N\Delta N^* = N \text{ diag} (\delta_1, \ldots, \delta_n) N^*, \quad \text{(56)}$$

$$MM^* = NN^* = I_n, \quad |\lambda_i| \geq |\lambda_{i+1}| \geq 0.$$

Since $W_c(\gamma_1, \gamma_2)$ and $W_o(\gamma_1, \gamma_2)$ are symmetric such decomposition exists. Let:

$$\hat{B} := M \text{ diag} (|\lambda_1|^{1/2}, \ldots, |\lambda_n|^{1/2}, 0, \ldots, 0) \quad \hat{C} := \text{ diag} (|\delta_1|^{1/2}, \ldots, |\delta_n|^{1/2}, 0, \ldots, 0) N^* \quad \text{(57)}$$

where:

$$\xi = \text{ rank} (W_c(\gamma_1, \gamma_2)) \quad \rho = \text{ rank} (W_o(\gamma_1, \gamma_2))$$

The modified gramians satisfy the following generalized Lyapunov equations:

$$A\hat{P}(\Omega) + \hat{P}(\Omega)A^* + S(\gamma_2)(\sum_{j=1}^{m} N_j \hat{P}(\Omega) N_j^*) + (\sum_{j=1}^{m} N_j \hat{P}(\Omega) N_j^*) S(\gamma_2) + \hat{B}\hat{B}^* = 0, \quad \text{(58)}$$

where:

$$A\hat{P}_i(\Omega) + \hat{P}_i(\Omega)A^* + \hat{B}\hat{B}^* = 0, \quad A\hat{P}_i(\Omega) + \hat{P}_i(\Omega)A^* + S(\gamma_2)(\sum_{j=1}^{m} N_j \hat{P}_{i-1}(\Omega) N_j^*) + (\sum_{j=1}^{m} N_j \hat{P}_{i-1}(\Omega) N_j^*) S(\gamma_2) + \hat{B}\hat{B}^* = 0, \quad i = 2, 3, \ldots \quad \text{(60)}$$

The frequency-interval observability gramian $Q(\Omega)$ can be similarly obtained.

In order to reduce the system over $Q(\Omega) = [\gamma_1, \gamma_2]$, we need to transform the system into the frequency-interval balanced realization. In the frequency-interval balanced realization, the frequency-interval controllability gramian of the system and the frequency-interval observability gramian are equal and diagonal with decreasing diagonal elements. To transform the system into frequency-interval balanced realization one way is to use a method similar to the one proposed by Laub [17]. Plugging the frequency-interval generalised gramians into the well-known Laub algorithm, the result will be the balancing coordinate transformation which transforms the system into the frequency-interval balanced realization. The matrices in this realization can be partitioned and truncated and in such a way the reduced order model can be obtained. Alternatively, one can use Schur method and square root algorithms [1], [18] to find suitable projection. The Schur method and square root algorithms provide the projection matrices to apply balanced reduction without balanced transformation. These method can also be used for modified frequency-interval model reduction. For the modified technique we need to use $P(\Omega)$ and $Q(\Omega)$ in the algorithm instead of $P(\Omega)$ and $Q(\Omega)$.

### V. Illustrative Example

In this section, the purposed method is used to reduce a bilinear system. The method is compared with its previous counterpart. Consider a SISO bilinear system which is described by (12) with the following matrices:
The ordinary balanced truncation and the frequency-interval balanced reduction are used to reduce the model into second order models. The results of the frequency-interval balanced reduction within $\Omega = [0, 0.2]$ is the following second order system:

$$A = \begin{bmatrix}
-0.81 & 0.47 & -0.43 & 1.6 & 0.26 & -0.4 & 0.52 \\
-0.61 & -1.9 & 0.8 & -1.6 & 2.0 & 0.98 & -0.9 \\
0.5 & -1.2 & -2.1 & -1.6 & -1.1 & 0.14 & -0.87 \\
-0.24 & -0.081 & 1.6 & -3.6 & -1.3 & 1.7 & -2.6 \\
1.3 & -0.96 & -1.3 & -0.57 & 3.7 & -1.2 & -1.3 \\
-0.16 & 1.5 & -0.99 & 1.5 & 0.61 & -2.2 & -3.3 \\
\end{bmatrix},$$

$$N = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix},$$

$$C = \begin{bmatrix}
-0.804 & 0 & 0.835 & -0.244 & 0.216 & -1.17 & -1.15 \\
\end{bmatrix}. $$

$$B = \begin{bmatrix}
0 & 0 & -0.196 & 1.42 & 0.292 & 0.158 & 1.59 \\
\end{bmatrix}. $$

The ordinary balanced truncation reduces the system into the following second order model:

$$\hat{A} = \begin{bmatrix}
-1.2 & 0.36 \\
0.422 & -0.485 \\
\end{bmatrix}, \quad \hat{B} = \begin{bmatrix}
0.337 \\
-0.122 \\
\end{bmatrix}$$

$$\hat{N} = \begin{bmatrix}
-1.0 \\
0.000000000000000069 \end{bmatrix},$$

$$\hat{C} = \begin{bmatrix}
-2.56 & 2.1 \end{bmatrix}. $$

The frequency-interval balanced reduction results in a more accurate approximation than the ordinary balanced truncation. This is shown in Figure 1, in which the step responses of the original system and the reduced models are depicted.

VI. CONCLUSIONS

In this paper, a new method for model reduction of bilinear systems is presented. The proposed technique is from the family of gramian-based model reduction methods. The frequency-interval generalized gramians are introduced and used in the reduction procedure. The frequency-interval generalized gramians are the solutions to the generalized Lyapunov equations. Algorithms for solving these equations are proposed. The method is further illustrated with the help of an illustrative examples. The numerical results show that the method is more accurate than its previous counterpart which is based on the ordinary gramians.

REFERENCES


Fig. 1. Step responses of the original system (solid line), the balanced truncated reduced model(dotted) and the $[0, 0.2]$ frequency-interval balanced reduced model (dash-dotted).